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# ON A CLASS OF GROUPS WITH LAGRANGIAN FACTOR GROUPS

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**Abstract:** We classify the 2-dual minimal non supersoluble groups, whose factor groups satisfy the converse of the Lagrange's theorem. From this classification we deduce that there is no upper bound for the 3-rank, as for the 2-rank, of a group with lagrangian factor groups. We conjecture that the groups with lagrangian factor groups are $p$-supersoluble, for each prime $p \neq 2, 3$.

## 1 – Introduction

A lagrangian group is a finite group which satisfies the converse of the Lagrange's theorem, that is a group $G$ possessing, for every divisor of $|G|$, a subgroup of that order. It is immediate to verify that the property is inherited neither by subgroups nor by factor groups. In 1939 O. Ore ([4]) showed that the inheritance by subgroups and factor groups is equivalent to supersolubility. In the following year G. Zappa ([6]) improved the Ore's result, by showing that the groups with lagrangian subgroups are the supersoluble groups. With regard to the groups with lagrangian factor groups, it is immediate to verify that they are not necessarily supersoluble. Nevertheless, many classes of groups with lagrangian factor groups are classes of supersoluble groups, as shown in 1974 in [2] and in 1984 in [1]. The results of [1] we refer to are based on the classification — contained in the same work — of the non supersoluble lagrangian groups, whose proper factor groups are supersoluble. From such classification the following propositions directly derive.

**Proposition A.** *There is no upper bound for the 2-rank of a group with lagrangian factor groups.*

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Moreover:

**Proposition B.** *If $G$ is a non supersoluble lagrangian group, with supersoluble proper factor groups, then $\pi(G) = \{2, 3\}$.*

**Proposition C.** *The 2-rank of a non supersoluble lagrangian group, with supersoluble proper factor groups, is a power of 2 and its 3-rank is equal to 1. Moreover: for every integer $m \geq 1$, there is, up to isomorphisms, only one lagrangian group, with supersoluble proper factor groups, whose 2-rank is $2^m$.*

This paper provides some new examples of non supersoluble groups with lagrangian factor groups; precisely gives a classification of the 2-dual minimal non supersoluble([1]) groups with lagrangian factor groups. Such classification directly leads to following propositions, which generalize the preceding A, B and C propositions.

**Proposition A′.** *There is no upper bound for the $p$-rank of a group with lagrangian factor groups, if $p = 2$ or 3.*

Moreover:

**Proposition B′.** *If $G$ is a $\widehat{S}_2$-group with lagrangian factor groups, then $|\pi(G)| \leq 3$ and $\{2, 3\} \subseteq \pi(G)$. ($S$ is the class of supersoluble groups.)*

**Proposition C′.** *The 2-rank of a $\widehat{S}_n$-group ($n = 1$ or 2) with lagrangian factor groups is a power of 2 and its 3-rank is a power of 3.*

*Moreover: for every ineger $m \geq 1$, there is, up to isomorphisms, only one $\widehat{S}_2$-group, with lagrangian factor groups, whose 2-rank is $2^m$ and whose 3-rank is $3^{2^{m-1}}$.*

**Conjecture.** Do the groups with lagrangian factor groups be $p$-supersoluble, for every prime $p$ different from 2 and 3?

The author conjectures that the factor groups of a group $G$ are lagrangian if and only if there exists a supersoluble immersion of $G_{\{2,3\}}$ in $G$ and the factor groups of $G_{\{2,3\}}$ are lagrangian. The sufficient condition has been proved in [1].

---

([1]) If $P$ is a class of groups, the class $\widehat{P}_n$ ($n \in \mathbb{N}_0$) of the groups $n$-dual minimal non $P$-groups, is defined inductively as follows:

$$\widehat{P}_0 = P\,; \quad G \in \widehat{P}_n \ \text{ iff } G \notin \bigcup_{i=0}^{n-1} \widehat{P}_i \quad \text{and} \quad G/N \in \bigcup_{i=0}^{n-1} \widehat{P}_i \ \text{ for every } 1 \neq N \triangleleft G\,.$$

## 2 – Some classes of $\widehat{S}_2$-groups with lagrangian factor groups

With reference to [1], the following notations are used.

$S(m)$, $m \geq 1$, denotes the subgroup of $GL(2^m, 2)$:

$$S(m) = \left\langle c_\ell, b_h \mid \ell = 1, ..., 2^{m-1};\ h = 0, ..., m-1 \right\rangle ,$$

where: $\langle c_\ell |\ \ell = 1, ..., 2^{m-1}\rangle$ is the matrix group of degree $2^m$ consisting of all block diagonal matrices $[A^{\alpha_i}\ \delta_{i,j}]_{i,j=1,...,2^{m-1}}$, with $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$; $b_0$ is the block diagonal matrix $[B\ \delta_{i,j}]_{i,j=1,...,2^{m-1}}$, with $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $b_h$ is the permutation matrix $[I_2\ \delta_{i,\sigma_h(j)}]_{i,j=1,...,2^{m-1}}$, where $\sigma_h$ is the permutation of degree $2^{m-1}$:

$$\begin{aligned} \sigma_h = (1, 1+2^{h-1}) \cdots (2^{h-1}, 2^h)\, (2^h+1, 2^h+1+2^{h-1}) \cdots (2^h+2^{h-1}, 2^{2h}) \cdots \\ \cdots (2^{m-1}-2^h+1, 2^{m-1}-2^{h-1}+1) \cdots (2^{m-1}-2^{h-1}, 2^{m-1}) \end{aligned}$$

$(h = 1, ..., m-1)$, (i.e., $\langle b_h |\ h = 1, ..., m-1\rangle$ is an elementary abelian regular permutation group of degree $2^{m-1}$). In [1] the holomorph of the additive group of $V(2^m, 2)$ by $S(m)$ has been denoted with $\Sigma(m)$. We can obviously assume that:

$$c_\ell = [A^{\alpha(i,\ell)}\ \delta_{i,j}]_{i,j=1,...,2^{m-1}} ,$$

with $\alpha(1, \ell) = 1$, for each $\ell = 1, ..., 2^{m-1}$, and $\alpha(\sigma_h(i), \ell) = 2^{\gamma_{\ell,h}}\, \alpha(i, \ell)$ $(i, \ell = 1, ..., 2^{m-1};\ h = 1, ..., m-1)$, where

$$\ell = \sum_{h=0}^{m-1} \gamma_{\ell,h}\, 2^{m-1-h} \qquad (\ell = 1, ..., 2^{m-1})$$

is the binary representation of $\ell$. It follows $\alpha(i, \ell) \equiv \pm 1 \pmod 3$, hence, because $A^3 = I_2$, we can assume $\alpha(i, \ell) = \pm 1$ for each $i, \ell = 1, ..., 2^{m-1}$, and we have the following presentation for $\Sigma(m)$.

**2.1.** $\Sigma(m) = \left\langle x_{2i-1}, x_{2i}, c_\ell, b_h \mid i, \ell = 1, ..., 2^{m-1};\ h = 0, ..., m-1 \right\rangle$.
Relations:

$$\Delta_0 \begin{cases} x_{2i-1}^2 = x_{2i}^2 = [x_u, x_v] = 1 & (i = 1, ..., 2^{m-1};\ u, v = 1, ..., 2^m) \\ c_\ell^3 = b_h^2 = [c_{\ell_1}, c_{\ell_2}] = [b_{h_1}, b_{h_2}] = 1 & (\ell, \ell_1, \ell_2 = 1, ..., 2^{m-1};\ h, h_1, h_2 = 0, ..., m-1) \\ b_0^{-1} c_\ell\, b_0 = c_\ell^{-1};\ b_h^{-1} c_\ell\, b_h = c_\ell^{2^{\gamma_{\ell,h}}} & (\ell = 1, ..., 2^{m-1};\ h = 1, ..., m-1) \ , \end{cases}$$

$$\Delta_1 \begin{cases} c_\ell^{-1}\, x_{2i-1}\, c_\ell = x_{2i-1}^{\delta_{-1,\alpha(i,\ell)}}\, x_{2i} \\ c_\ell^{-1}\, x_{2i}\, c_\ell = x_{2i-1}\, x_{2i}^{\delta_{1,\alpha(i,\ell)}} \end{cases} \quad (i = 1, ..., 2^{m-1}) \ ,$$

$$\Delta_2 \Big\{ b_0^{-1}\, x_{2i-1}\, b_0 = x_{2i}; \quad b_0^{-1}\, x_{2i}\, b_0 = x_{2i-1} \qquad (i = 1, ..., 2^{m-1}) \ ,$$

$$\Delta_3 \Big\{ b_h^{-1}\, x_{2i-1}\, b_h = x_{2\sigma_h(i)-1}; \; b_h^{-1} x_{2i}\, b_h = x_{2\sigma_h(i)} \quad (i=1, ..., 2^{m-1}; \; h=1, ..., m-1).$$

In order to simplify the description, we report the main result of [1].

**2.2 Theorem** ([1]). *The non supersoluble lagrangian groups, with supersoluble proper factors groups, are, up to isomorphisms, the groups $\Sigma(m)$.*

**2.3 The groups** $\Sigma^*(m)$. Let $m$ be an integer $\geq 1$. If $N$ is an elementary abelian group of order $2^{2^m}$, with the identification $\operatorname{Aut} N = GL(2^m, 2)$, let $\chi : \Sigma(m) \to \operatorname{Aut} N$ be the product of the canonical homomorphism $\Sigma(m) \to \Sigma(m)/O_2(\Sigma(m)) \simeq S(m)$ and the embedding of $S(m)$ in $GL(2^m, 2)$. We shall refer to a group extension of $N$ by $\Sigma(m)$ with coupling $\chi$ as a group of type $\Sigma^*(m)$. In particular, we denote the split extension with $\overline{\Sigma}(m)$. It is immediate to verify that $\overline{\Sigma}(m)$ is a $\widehat{S}_2$-group with lagrangian factor groups. With the following proposition we provide the structure of the groups $\Sigma^*(m)$, non isomorphic to $\overline{\Sigma}(m)$, and show that they are $\widehat{S}_2$-groups with lagrangian factor groups.

**Proposition 2.3.1.** *Let $G$ be a group of type $\Sigma^*(m)$ and $G \neq \overline{\Sigma}(m)$. Then:*

- **i**) *$G = K \triangleright O_2(G)$, with $K \simeq S(m)$ and $O_2(G) = F(G)$ abelian of type $(2^2, ..., 2^2)$ and order $2^{2^{m+1}}$;*
- **ii**) *$\Phi(G) = \Phi(O_2(G))$;*
- **iii**) *$K\,\Phi(G) \simeq G/\Phi(G) \simeq \Sigma(m)$;*
- **iv**) *$G$ is a $\widehat{S}_2$-group with lagrangian factor groups.*

**Proof:** As $G$ is of type $\Sigma^*(m)$, $G/N \simeq \Sigma(m)$ where $N$ is a minimal normal subgroup of $G$ of order $2^{2^m}$; we have then $N \leq \Phi(G)$ because $G$ is not a split extension of $N$. It follows that $\Phi(G) = N$ and $F(G) = O_2(G)$, because $\Phi(\Sigma(m)) = 1$ and $F(\Sigma(m)) = O_2(\Sigma(m))$. On the other hand, since $G/N \simeq \Sigma(m)$, $N = \Phi(G)$ is minimal normal in $G$ and $C_G(N) = O_2(G)$. That being stated, assume $G = MO_2(G)$, with $M \cap O_2(G) = \Phi(G)$. We have that $\Phi(G) = N$ is a minimal normal subgroup of $M$ and $C_M(N) = N$. Hence $M = K \triangleright \Phi(G)$

(see, for instance, [3], II Satz 3.3); it follows $G = K \triangleright O_2(G)$, $K \simeq S(m)$ and $K\,\Phi(G) \simeq \Sigma(m)$. Using all what we have obtained, it is easy to verify that $G$ is a $\widehat{S}_2$-group with lagrangian factor groups, if we notice that $\Phi(G)$ is the only minimal normal subgroup of $G$.

Suppose that $O_2(G)$ is elementary abelian. Since $O_2(K) = 1$ and $|\pi(K)| = 2$, $O_2(G)$ is a completely reducible $GF(2)[K]$-module (see [5], Lemma 1.5) and thus the contradiction that $G$ splits on $\Phi(G)$ is reached.

It follows, recalling that $\Phi(G)$ is the only minimal normal subgroup of $G$ and $O_2(G)/\Phi(G)$ is minimal normal in $G/\Phi(G)$, that $\Phi(G) = \Phi(O_2(G)) = O_2(G)^2$; therefore, being $K\,\Phi(G) \simeq \Sigma(m)$, if $O_2(G)$ is abelian, $O_2(G)$ is of type $(2^2, ..., 2^2)$. In order to show that $O_2(G)$ is abelian, we assume, with reference to presentation 2.1 of $\Sigma(m) \simeq G/\Phi(G)$:

$$O_2(G) = \left\langle x_{2i-1}, x_{2i} \mid i = 1, ..., 2^{m-1} \right\rangle, \qquad x_u^2, [x_u, x_v] \in \Phi(G) \quad (u, v = 1, ..., 2^m) \ ,$$

$$K = \left\langle c_\ell, b_h \mid \ell = 1, ..., 2^{m-1};\ h = 0, ..., m-1 \right\rangle \ ,$$

with relations:

$$\Delta_1' \begin{cases} c_\ell^{-1}\, x_{2i-1}\, c_\ell = x_{2i-1}^{\delta_{-1,\alpha(i,\ell)}}\, x_{2i}\, y_{2i-1,\ell} \\ c_\ell^{-1}\, x_{2i}\, c_\ell = x_{2i-1}\, x_{2i}^{\delta_{1,\alpha(i,\ell)}}\, y_{2i,\ell} \end{cases} \quad (i, \ell = 1, ..., 2^{m-1}) \ ,$$

$$\Delta_2' \begin{cases} b_0^{-1}\, x_{2i-1}\, b_0 = x_{2i}\, z_{2i-1} \\ b_0^{-1}\, x_{2i}\, b_0 = x_{2i-1}\, z_{2i} \end{cases} \quad (i = 1, ..., 2^{m-1}) \ ,$$

$$\Delta_3' \begin{cases} b_h^{-1}\, x_{2i-1}\, b_h = x_{2\sigma_h(i)-1}\, w_{2i-1,h} \\ b_h^{-1}\, x_{2i}\, b_h = x_{2\sigma_h(i)}\, w_{2i,h} \end{cases} \quad (i = 1, ..., 2^{m-1};\ h = 1, ..., m-1) \ ,$$

where:

$$y_{u,\ell}, z_u, w_{u,h} \in \Phi(G) \quad (u = 1, ..., 2^m;\ \ell = 1, ..., 2^{m-1};\ h = 1, ..., m-1) \ .$$

From $\Delta_r'$ $(r = 1, 2, 3)$ it immediately follows that the subgroup $\langle [x_{2i-1}, x_{2i}] | \; i = 1, ..., 2^{m-1} \rangle$ is normal in $G$ and is centralized by $\langle c_\ell, b_0 | \; \ell = 1, ..., 2^{m-1} \rangle$. We have then $[x_{2i-1}, x_{2i}] = 1$, for each $i = 1, ..., 2^{m-1}$, because $C_G(\Phi(G)) = O_2(G)$. Thus, if $m = 1$, $O_2(G)$ is abelian.

Let then $m \geq 2$. Notice first that $\alpha(i, 2^{m-1}) = 1$, for each $i = 1, ..., 2^{m-1}$. From $\Delta_1'$, with $c = c_{2^{m-1}}$, we get, since $O_2(G)$ has class at most 2, that:

$$\begin{aligned} & c^{-1}[x_{2i-1}, x_{2j-1}]\, c = [x_{2i}, x_{2j}] \ , \\ (2.3.1.1) \qquad & c^{-1}[x_{2i}, x_{2j}]\, c = [x_{2i-1}, x_{2j-1}]\, [x_{2i-1}, x_{2j}]\, [x_{2i}, x_{2j-1}]\, [x_{2i}, x_{2j}] \ , \\ & c^{-1}[x_{2i-1}, x_{2j}]\, c = [x_{2i}, x_{2j-1}]\, [x_{2i}, x_{2j}] \end{aligned}$$

$(i,j=1,...,2^{m-1})$. From here we get:

$$c^{-1}[x_{2i-1},x_{2j}]\,[x_{2i},x_{2j-1}]\,c=[x_{2i-1},x_{2j}]\,[x_{2i},x_{2j-1}]\ ,$$

from which:

$$[x_{2i-1},x_{2j}]=[x_{2i},x_{2j-1}]\qquad (i,j=1,...,2^{m-1})\ , \tag{2.3.1.2}$$

because $C_N(c)=1$, since $\langle c\rangle \triangleleft K$ and $K$ acts irreducibly on $N$. On the other hand, as the matrix $[\alpha(i,\ell)]$ is non singular, for each pair of different indexes $i$ and $j$ between 1 and $2^{m-1}$, there exists $\ell\in\{1,...,2^{m-1}\}$ such that $\alpha(i,\ell)=-\alpha(j,\ell)$. In this case we get from $\Delta_1'$ and (2.3.1.2) that:

$$\begin{aligned}
&c_\ell^{-1}[x_{2i},x_{2j-1}]\,c_\ell=\\
&\qquad=[x_{2i-1},x_{2j-1}]^{\delta_{-1,\alpha(j,\ell)}}\,[x_{2i},x_{2j}]^{\delta_{1,\alpha(i,\ell)}}\,[x_{2i-1},x_{2j}]^{1+\delta_{1,\alpha(i,\ell)}\delta_{-1,\alpha(j,\ell)}}=\\
&=c_\ell^{-1}[x_{2i-1},x_{2j}]\,c_\ell=\\
&\qquad=[x_{2i-1},x_{2j-1}]^{\delta_{-1,\alpha(i,\ell)}}\,[x_{2i},x_{2j}]^{\delta_{1,\alpha(j,\ell)}}\,[x_{2i-1},x_{2j}]^{1+\delta_{-1,\alpha(i,\ell)}\delta_{1,\alpha(j,\ell)}}\ ,
\end{aligned}$$

from which:

$$[x_{2i-1},x_{2j-1}]\,[x_{2i},x_{2j}]=[x_{2i-1},x_{2j}]\qquad (i,j=1,...,2^{m-1})\ . \tag{2.3.1.3}$$

On the other hand, we get from $\Delta_1'$:

$$\begin{aligned}
c_\ell^{-1}[x_{2i-1},x_{2j-1}]\,c_\ell&=[x_{2i-1},x_{2j}]^{\delta_{-1,\alpha(i,\ell)}+\delta_{-1,\alpha(j,\ell)}}\,[x_{2i},x_{2j}]=\\
&=[x_{2i-1},x_{2j}]\,[x_{2i},x_{2j}]
\end{aligned}$$

and therefore, from (2.3.1.3), we get:

$$c_\ell^{-1}[x_{2i-1},x_{2j-1}]\,c_\ell=[x_{2i-1},x_{2j-1}]\ .$$

It follows, since $C_N(c_\ell)=1$, $[x_{2i-1},x_{2j-1}]=1$ $(i,j=1,...,2^{m-1})$, from which and (2.3.1.1) we get:

$$[x_{2i},x_{2j-1}]=[x_{2i-1},x_{2j}]=[x_{2i},x_{2j}]=1$$

and thus $O_2(G)$ is abelian. ∎

It follows easily from the previous proposition, by routine calculations, that there is (up to isomorphism) only one group of type $\Sigma^*(m)$, non-isomorphic to $\overline{\Sigma}(m)$; precisely the following group $\widehat{\Sigma}(m)$

$$\widehat{\Sigma}(m)=\left\langle x_{2i-1},x_{2i},c_j,b_h\mid i,j=1,...,2^{m-1};\ h=0,...,m-1\right\rangle\ .$$

Relations:

$$
\begin{array}{ll}
x_{2i-1}^4 = x_{2i}^4 = [x_u, x_v] = 1 & (i=1,...,2^{m-1};\ u,v=1,...,2^m)\ , \\
c_j^3 = b_h^2 = [c_{j_1}, c_{j_2}] = [b_{h_1}, b_{h_2}] = 1 & (j,j_1,j_2=1,...,2^{m-1};\ h,h_1,h_2=0,...,m-1)\ , \\
b_0^{-1} c_j b_0 = c_j^{-1},\quad b_h^{-1} c_j b_h = c_{\sigma_h(j)} & (j=1,...,2^{m-1};\ h=1,...,m-1)\ , \\
c_j^{-1} x_{2i-1} c_j = x_{2i-1},\quad c_j^{-1} x_{2i} c_j = x_{2i} & (i \neq j = 1,...,2^{m-1})\ , \\
c_i^{-1} x_{2i-1} c_i = x_{2i},\quad c_i^{-1} x_{2i} c_i = x_{2i-1}^{-1} x_{2i}^{-1} & (i=1,...,2^{m-1})\ , \\
b_0^{-1} x_{2i-1} b_0 = x_{2i},\quad b_0^{-1} x_{2i} b_0 = x_{2i-1} & (i=1,...,2^{m-1})\ , \\
b_h^{-1} x_{2i-1} b_h = x_{2\sigma_h(i)-1},\quad b_h^{-1} x_{2i} b_h = x_{2\sigma_h(i)} & (i=1,...,2^{m-1},\ h=1,...,m-1)\ .
\end{array}
$$

**2.4 The groups $\Omega(m)$.** Let $m$ be an integer $\geq 1$. From the presentation 2.1 of $\Sigma(m)$ we get easily, for any $c \in C = \langle c_\ell |\ \ell = 1, ..., 2^{m-1}\rangle$, that:

$$
\left.
\begin{array}{l}
c\, x_{2i-1}\, c^{-1} = x_{2i-1}^{\lambda(i,c)}\, x_{2i}^{\mu(i,c)} \\
c\, x_{2i}\, c^{-1} = x_{2i-1}^{\mu(i,c)}\, x_{2i}^{\lambda(i,c)+\mu(i,c)}
\end{array}
\right. \qquad (i = 1, ..., 2^{m-1})\ ,
$$

where $\lambda(i, c)$ and $\mu(i, c)$ are integers modulo 2 $((\lambda(i, c), \mu(i, c)) \neq (0, 0))$.

Considered the regular $GF(3)[C]$-module $N$, the following positions give $N$ a structure or $GF(3)[\Sigma(m)]$-module irreducible, on which $\Sigma(m)$ acts faithfully:

$$
\left.
\begin{array}{l}
c^{x_{2i-1}} = (-1)^{\lambda(i,c)+\mu(i,c)}\, c \\
c^{x_{2i}} = (-1)^{\lambda(i,c)}\, c
\end{array}
\right. \qquad (i = 1, ..., 2^{m-1})\ ,
$$

$$
\left.
\begin{array}{l}
c^a = c\, a \\
c^{b_h} = b_h^{-1}\, c\, b_h
\end{array}
\right. \qquad (a \in C;\ h = 0, ..., m-1)\ ,
$$

where $c \in C$, being $C$ the canonical base of $N = GF(3)[C]$. The holomorph of the additive group of $N$ by $\Sigma(m)$ is denoted with $\Omega(m)$.

**Proposition 2.4.1.** *$\Omega(m)$ is a $\widehat{S}_2$-group with lagrangian factor groups.*

**Proof:** For each $t = 1, ..., 2^{m-1}$ and for each of the $\binom{2^{m-1}}{t}$ subsets $I_{t,k} = \{\ell_1, ..., \ell_t\}$ of $\{1, ..., 2^{m-1}\}$ of order $t$, we consider the subset of $C$:

$$
C_{t,k} = \left\{ c_{\ell_1}^{\beta_1} \cdots c_{\ell_t}^{\beta_t} \mid \beta_1, ..., \beta_t = \pm 1 \right\} \qquad (k = 1, ..., \tbinom{2^{m-1}}{t}))\ .
$$

Assuming $D = \langle x_{2i-1}, x_{2i}, b_h |\ i = 1, ..., 2^{m-1};\ h = 0, ..., m-1 \rangle$ and $W = GF(3)1$, we have obviously that $\sum_{c\,C_{t,k}} W^c$ is a $D$-submodule of $N$, hence $\Omega(m)$ has subgroups of order $3^n \cdot 2^{2^m+m}$, where $n = \sum_{t=0}^{2m-1} r_t\, 2^t$ with $0 \le r_t \le \binom{2^{m-1}}{t}$. From this result the statement easily follows. ∎

## 3 – Classification of the $\widehat{S}_2$-groups with lagrangian factor groups

**Proposition 3.1.** *Let $G$ be a $\widehat{S}_2$-group with lagrangian factor groups. If $\Phi(G) \neq 1$, then $\Phi(G)$ is minimal normal in $G$ and $G/\Phi(G) \simeq \Sigma(m)$ (for some $m \geq 1$).*

**Proof:** Let $N$ be a minimal normal subgroup of $G$ with $N \le \Phi(G)$. Assume that $N \neq \Phi(G)$. Since the proper factor groups of $G/N$ are supersoluble, we have that $G/\Phi(G) \simeq \frac{G/N}{\Phi(G)/N}$ is supersoluble and therefore $G$ is supersoluble, which contradicts the hypothesis. The statement $G/\Phi(G) \simeq \Sigma(m)$ follows from Theorem 2.2. ∎

**Proposition 3.2.** *Let $G$ be a $\widehat{S}_2$-group with lagrangian factor groups, with $\Phi(G) \neq 1$. If $G$ has a minimal normal subgroup different from $\Phi(G)$, then $|\Phi(G)|$ is prime (= 2 or 3).*

**Proof:** Let $N$ be a minimal normal subgroup of $G$, different from $\Phi(G)$, then assuming $G = K \triangleright N$ ($K$ maximal subgroup of $G$), we have, by Proposition 3.1, $\Sigma(m) \simeq G/\Phi(G) = K/\Phi(G) \triangleright \frac{\Phi(G) \triangleright N}{\Phi(G)}$ and therefore $K/\Phi(G) \simeq S(m)$; it follows, as obviously $\Phi(G) \le \Phi(K)$, that $K$ is supersoluble and so, since $\Phi(G)$ is minimal normal in $G$, $|\Phi(G)|$ is prime. ∎

**Proposition 3.3.** *Let $G$ be a $\widehat{S}_2$-group with lagrangian factor groups, with $\Phi(G) \neq 1$. If $\Phi(G)$ is the only minimal normal subgroup of $G$ and $|\Phi(G)|$ is not prime, then $G$ is a group of type $\Sigma^*(m)$ (non isomorphic to $\overline{\Sigma}(m)$).*

**Proof:** Since $G/\Phi(G) \simeq \Sigma(m)$ and $\Phi(G)$ is the only minimal normal subgroup of $G$, as $O_2(\Sigma(m)) = F(\Sigma(m))$, we have that $\Phi(G) \le O_2(G) = F(G)$ and $O_2(G) \le C_G(\Phi(G))$. That being stated, by arguing as in the proof of the Proposition 2.3.1, we get $G = K \triangleright O_2(G)$, with $K \simeq S(m)$ and $\Phi(G) = \Phi(O_2(G)) = O_2(G)^2$; therefore, in order to complete the proof, we need only to show that

$K\,\Phi(G) \simeq \Sigma(m)$. We can assume, as in the proof of the Proposition 2.3.1:

$$O_2(G) = \left\langle x_{2i-1}, x_{2i} \mid i=1,...,2^{m-1} \right\rangle ,$$

$$K = \left\langle c_\ell, b_h \mid \ell=1,...,2^{m-1};\ h=0,...,m-1 \right\rangle ,$$

with the relations $\Delta_1'$, $\Delta_2'$ and $\Delta_3'$. From these it immediately follows that $K\,O_2(G)^2$ $(=\ K\,\Phi(G))$ is a holomorphic image of $\Sigma(m)$, hence, obviously, $K\,\Phi(G) \simeq \Sigma(m)$. ∎

**Proposition 3.4.** *Let $G$ be a $\widehat{S}_2$-group with lagrangian factor groups, with $\Phi(G)=1$. Then one of the following conditions holds:*

**i)** *$G$ is a (split) extension of a group of prime order by a group $\Sigma(m)$;*

**ii)** *$G$ is isomorhic to $\overline{\Sigma}(m)$ (for some $m$);*

**iii)** *$G$ is isomorphic to $\Omega(m)$ (for some $m$).*

**Proof:** Let $N$ be a minimal normal subgroup of $G$, such that the proper factor groups of $G/N$ are supersoluble and $G/N$ is not supersoluble. We have (Theorem 2.2) that $G/N \simeq \Sigma(m)$ (for some $m \geq 1$) and so $G = K \triangleright N$, with $K \simeq \Sigma(m)$. If $|N|$ is not prime, i.e., the condition i) does not hold, we have, since $G$ is lagrangian, $N \leq O_2(G)$ or $N \leq O_3(G)$. Assuming $|N| = p^n$ ($p=2$ or 3 and $n>1$), we examine the two cases separately.

**1st case:** $p=2$.

Since $O_2(K) \neq 1$, we have $N < O_2(G) = O_2(K)\,N$ and therefore, since $N \cap Z(O_2(G)) \neq 1$, we get $O_2(G) = O_2(K) \times N$; from this, since $O_2(K)$ is minimal normal in $K$, $O_2(K)$ is minimal normal in $G$, hence, since $G/O_2(K)$ is non supersoluble, we have $G/O_2(G) \simeq \Sigma(m')$ $(2^{m'} = n)$, and then, obviously, $m = m'$ and so $G \simeq \overline{\Sigma}(m)$.

**2nd case:** $p=3$.

We have, obviously, that $C_G(N) = N$, therefore, with the identification $K = \Sigma(m)$, $N$ is an irreducible $GF(3)[\Sigma(m)]$-module, on which $\Sigma(m)$ acts faithfully. That being stated, assume $O_2(\Sigma(m)) = D$. We have, by Clifford's theorem, that, as $D$ is (non cyclic) elementary abelian, $N$ is imprimitive and that the isotypic $D$-submodules of $N$ are the irreducible $D$-submodules and thus have dimension 1.

If $U = GF(3)\,u$ is an irreducible $D$-submodule, we can assume:

$$u^{x_{2i-1}} = u^{x_{2i}} = (-1)^{\varepsilon_i}\,u \qquad (i=1,...,2^{m-1}) \ .$$

It follows that $U^{b_0}$ is $D$-isomorphic to $U$, and so, since $U^{b_0}$ and $U$ are $D$-isotypic, $U^{b_0} = U$. On the other hand, with $h \in \{1,...,m-1\}$ $(m \geq 2)$, we can assume, as to suitable cases $\{y_{2i-1}, y_{2i} |\ i = 1, ..., 2^{m-1}\}$ of $D$:

$$u^{y_{2i-1}} = u^{y_{2i}} = u^{b_h y_{2i-1} b_h^{-1}} = u^{b_h y_{2i} b_h^{-1}} .$$

It follows, as previously, $U^{b_h} = U$, and so $U^b = U$ for any $b \in \langle b_h |\ h = 0, ..., m-1\rangle$. We have then that $N$ is an irreducible $DC$-module, where $C = \langle c_\ell |\ \ell = 1, ..., 2^{m-1}\rangle$. It follows, since $N$ is imprimitive, that the action of $C$ on the irreducible $D$-submodules of $N$ is transitive, and therefore, as $C$ is abelian, $N$ is a regular $C$-module, that is, up to isomorphisms, $N = GF(3)[C]$. After these preliminary remarks, as in $K = \Sigma(m)$ we have (see 2.4)

$$\begin{aligned} c\, x_{2i-1}\, c^{-1} &= x_{2i-1}^{\lambda(i,c)}\, x_{2i}^{\mu(i,c)} \\ c\, x_{2i}\, c^{-1} &= x_{2i-1}^{\mu(i,c)}\, x_{2i}^{\lambda(i,c)+\mu(i,c)} \end{aligned} \qquad (i = 1, ..., 2^{m-1};\ c \in C) \ , \tag{3.4.1}$$

we can assume in $G$, with the identification $N = GF(3)[C]$:

$$\begin{aligned} c^{x_{2i-1}} &= (-1)^{f(i,c)}\, c \\ c^{x_{2i}} &= (-1)^{g(i,c)}\, c \end{aligned} \qquad (i = 1, ..., 2^{m-1};\ c \in GF(3)[C] = N) \ ,$$

where $f(i,c)$ and $g(i,c)$ are integers modulo 2. From 2.1 and (3.4.1) we easily get:

$$\begin{aligned} f(i,c) &= \lambda(i,c) + \mu(i,c) \ , \\ g(i,c) &= \lambda(i,c) \ . \end{aligned}$$

Hence we get $G \simeq \Omega(m)$. ∎

From the previously proved propositions follows the theorem, which provides the aforesaid classification of the $\widehat{S}_2$-groups with lagrangian factor groups.

**Theorem 3.5.** *A group $G$ is a $\widehat{S}_2$-group with lagrangian factor groups, if and only if one of the following conditions holds:*

**i**) *$G$ is an extension of a group of prime order by a group $\Sigma(m)$;*

**ii**) *$G$ is isomorphic to $\overline{\Sigma}(m)$;*

**iii**) *$G$ is isomorphic to $\widehat{\Sigma}(m)$;*

**iv**) *$G$ is isomorphic to $\Omega(m)$.*

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