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STABILIZATION OF THE SCHRÖDINGER EQUATION

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Abstract: We study the stabilization problem for Schrödinger equation in a bounded domain in two different situations. First, the boundary stabilization problem is considered. Dissipative boundary conditions are introduced. By using multiplier techniques and constructing energy functionals well adapted to the system, the exponential decay in H^1 is proved. On the other hand, the internal stabilization problem is considered. When the damping term is effective on a neighborhood of the boundary, the exponential decay in L^2 is proved by multiplier techniques. These results extend to Schrödinger equation recent results on the stabilizability of wave and plate equations.

1 – Introduction and main results

This paper is devoted to study the stabilization problem for Schrödinger equation.

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with boundary $\Gamma = \partial\Omega$ of class C^3.

It is well known that $L^2(\Omega)$ and $H^1(\Omega)$-norms of solutions of Schrödinger equation

$$\begin{cases} i\,\varphi_t + \Delta\varphi = 0 & \text{in } \Omega \times (0,\infty) \\ \varphi = 0 & \text{on } \Gamma \times (0,\infty) \\ \varphi(0) = \varphi & \text{in } \Omega \end{cases} \tag{1.1}$$

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are conserved, i.e.

$$\|\varphi(t)\|_{L^2(\Omega)} = \|\varphi^0\|_{L^2(\Omega)}; \quad \|\nabla\varphi(t)\|_{(L^2(\Omega))^n} = \|\nabla\varphi^0\|_{L^2(\Omega))^n}, \ \forall\, t \in (0,\infty) \ .$$

Roughly, the stabilization problem can be formulated as follows: to introduce a damping term in system (1.1) ensuring the exponential decay of $L^2(\Omega)$ or $H^1(\Omega)$-norms of solutions as t goes to infinity.

Note that solutions of the Schrödinger equation

$$i\,\varphi_t + \Delta\varphi = 0$$

are also solutions of the plate equation

$$\varphi_{tt} + \Delta^2\varphi = 0 \ .$$

Therefore, one can try to obtain stabilization results for Schrödinger equation from the corresponding ones for plate models. The stabilization problem for plate models has been extensively studied during the last years (see, for instance, J. Lagnese [7] and the references therein).

The goal of this paper is to study directly the stabilizability of Schrödinger equation by adapting the multiplier methods developed during the past years in the context of the stabilization of wave and plate equations. We will consider both the boundary and internal stabilization problems. The first one consists in producing the exponential decay by means of suitable dissipative boundary conditions. In the internal stablization problem the damping term is assumed to be supported in a subset of Ω and appears in system (1.1) as a right hand side in the Schrödinger equation. Of course, in borth cases, from a practical view point it is interesting to restrict the support of the damping term to a set as small as possible.

Let x_0 be any point of $\mathbb{R}^n$ and let us define the following partition of the boundary Γ:

$$\text{(1.2)} \qquad \begin{cases} \Gamma_0 = \Big\{x \in \Gamma;\ \ m(x)\cdot\nu(x) > 0\Big\} \\ \Gamma_1 = \Gamma\setminus\Gamma_0 = \Big\{x \in \Gamma;\ \ m(x)\cdot\nu(x) \le 0\Big\}\ , \end{cases}$$

where $m(x) = x - x_0$, $\nu(x)$ denotes the unit outward normal vector to Ω at $x \in \Gamma$ and $\cdot$ denotes the scalar product in $\mathbb{R}^n$.

We will prove the boundary stabilizability of Schrödinger equation with a boundary damping term supported on Γ_0. The internal stabilization will be proved with a damping term supported on a neighborhood of Γ_0 in Ω (by "neighborhood of Γ_0 in Ω" we mean the intersection of Ω with a neighborhood of Γ_0 in $\mathbb{R}^n$). These are the analogues of the stabilization results for wave equations proved in V. Komornik and E. Zuazua [6] and E. Zuazua [12].

Let us state more precisely these two main results.

We introduce the following damped Schrödinger equation with dissipative boundary condition:

$$
\begin{cases} i\, y_t + \Delta y = 0 & \text{in } \Omega \times (0,\infty) \\ \dfrac{\partial y}{\partial \nu} = -(m(x)\cdot \nu(x))\, y_t & \text{on } \Gamma_0 \times (0,\infty) \\ y = 0 & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y^0(x) & \text{in } \Omega\ . \end{cases} \tag{1.3}
$$

The natural space for initial data is

$$
V = \Big\{ \varphi \in H^1(\Omega) \colon\ \varphi = 0 \text{ on } \Gamma_1 \Big\}\ .
$$

When Γ_1 has non-empty interior in Γ, by Poincare's inequality, we have

$$
\|\varphi\|_{L^2(\Omega)} \le \alpha\, \|\nabla\varphi\|_{(L^2(\Omega))^n}\,, \qquad \forall\, \varphi \in V\ . \tag{1.4}
$$

Thus, we may consider V endowed with the norm induced by the scalar product

$$
(\varphi,\psi)_v = \mathrm{Re} \int_\Omega \nabla\varphi \cdot \nabla\overline{\psi}\, dx
$$

which, in V, is equivalent to the norm of $H^1(\Omega)$.

Multiplying in (1.3) by $\overline{y_t}$, integrating by parts and taking the real part we formally obtain that

$$
\frac{dE(t)}{dt} = -\int_{\Gamma_0} (m\cdot\nu)\ |y_t(x,t)|^2\, d\Gamma\,, \qquad \forall\, t > 0\ , \tag{1.5}
$$

where the energy $E(\,\cdot\,)$ is given by

$$
E(t) = \frac{1}{2} \int_\Omega |\nabla y(x,t)|^2\, dx = \frac{1}{2}\, \|y(t)\|_v^2\ . \tag{1.6}
$$

In (1.5), $d\Gamma$ denotes the surface measure on Γ.

Identity (1.5) shows that the boundary conditions of system (1.3) are dissipative in V.

In order to solve system (1.3) we use classical semigroup theory. Let us introduce the linear unbounded operator $A\varphi = i\,\Delta\varphi$ with domain

$$
D(A) = \Big\{ \varphi \in V \colon\ \Delta\varphi \in V,\ \frac{\partial\varphi}{\partial\nu} = -i(m\cdot\nu)\,\Delta\varphi \ \text{ on }\ \Gamma_0 \Big\}\ .
$$

Identity

$$\frac{\partial\varphi}{\partial\nu} = -i(m\cdot\nu)\,\Delta\varphi \quad \text{ on } \Gamma_0$$

has to be understood in the following variational sense:

$$\int_\Omega \nabla\varphi\cdot\nabla\overline{\psi}\,dx + \int_\Omega (\Delta\varphi)\overline{\psi}\,dx + \int_{\Gamma_0} i(m\cdot\nu)\,\Delta\varphi\,\overline{\psi}\,d\Gamma = 0\,, \qquad \forall\,\psi\in V\ .$$

It is easy to see that $(A, D(A))$ is a m-dissipative operator in V. Therefore, by Hille-Yosida's Theorem, for every $y^0\in D(A)$ there exists a unique solution

$$y\in C\Big([0,\infty);\ D(A)\Big)\cap C^1\Big([0,\infty);\ V\Big) \tag{1.7}$$

of

$$\begin{cases} y_t(t) = Ay(t), & \forall\,t\in[0,\infty) \\ y(0) = y^0 \end{cases}$$

or, equivalently, of system (1.3).

Furthermore, $D(A)$ is dense in V and for every $t\in[0,\infty)$, the linear map

$$y^0 \to y(t)$$

extends to a unique contraction $S(t)\colon V\to V$ such that $(S(t))_{t\ge 0}$ is a strongly continuous semigroup of contractions in V.

Therefore, for every $y^0\in V$,

$$y(t) = S(t)y^0\,, \qquad \forall\,t\ge 0\ , \tag{1.8}$$

defines in a unique way a weak solution of (1.3).

Our main boundary stabilization result is as follows.

Theorem 1.1. *Assume that Ω is a bounded domain of class C^3 of $\mathbb{R}^n$ with $n\le 3$. Let be $x_0\in\mathbb{R}^n$ such that Γ_1 has non-empty interior in Γ.*

Then, for every $C>1$ there exists $\gamma>0$ such that for any initial data $y^0\in V$ the energy $E(\,\cdot\,)$ of solution $y(t) = S(t)\,y^0$ of system (1.3) satisfies

$$E(t)\le C\,E(0)\,e^{-\gamma t}\,, \qquad \forall\,t>0\ . \tag{1.9}$$

Remark 1.1. If $\overline{\Gamma}_0\cap\overline{\Gamma}_1=\emptyset$ the restriction on the dimension $n\le 3$ is not needed. Indeed, in Theorem 1.1 we assume that $n\le 3$ in order to use Grisvard's [3] analysis of singularities of solutions of elliptic problems with mixed Dirichlet–Neumann boundary conditions. This analysis of singularities is needed in order to apply multiplier techniques. However, when $\overline{\Gamma}_0\cap\overline{\Gamma}_1=\emptyset$ solutions of (1.3) with smooth initial data satisfying suitable compatibility conditions remain smooth for $t > 0$. Thus, Green's formula suffices to justify the integrations by parts necessary to apply multiplier methods. Notice that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ holds when $\Omega = \Omega_1 \backslash \overline{\Omega}_0$ with Ω_0 and Ω_1 star-shaped domains with respect to $x_0 \in \Omega_0$ if $\overline{\Omega_0} \subset \Omega_1$.

Let us now consider the internal stabilizability problem.

Let $\omega \subset \Omega$ be a neighborhood of $\overline{\Gamma}_0$ in Ω and let be $a = a(x) \in L^\infty(\Omega)$ such that

$$\begin{cases} a \geq 0 \quad \text{a.e. in } \Omega \\ \exists a_0 > 0 \colon\ a \geq a_0 \quad \text{a.e. in } \omega\,. \end{cases} \tag{1.10}$$

Consider the following damped Schrödinger equation:

$$\begin{cases} iy_t + \Delta y + ia(x)y = 0 & \text{in } \Omega \times (0,\infty) \\ y = 0 & \text{on } \Gamma \times (0,\infty) \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{1.11}$$

It is easy to see that for any initial data $y^0 \in L^2(\Omega)$ there exists a unique weak solution of (1.11) in the class

$$y \in C\Big([0,\infty);\ L^2(\Omega)\Big) \cap C^1\Big([0,\infty);\ (H^2(\Omega) \cap H_0^1(\Omega))'\Big)\,. \tag{1.12}$$

Let us define the $L^2(\Omega)$-energy

$$F(t) = \frac{1}{2}\, \|y(t)\|^2_{L^2(\Omega)}\,, \qquad \forall\, t > 0\,. \tag{1.13}$$

We have

$$F(t_2) - F(t_1) = -\int_{t_1}^{t_2} \int_\Omega a(x)\, |y(x,t)|^2\, dx\, dt\,, \qquad \forall\, t_2 > t_1 \geq 0\,. \tag{1.14}$$

Therefore energy $F(\,\cdot\,)$ is non-increasing along trajectories.

We have the following exponential decay result.

Theorem 1.2. *Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with boundary of class C^3. Let be $x_0 \in \mathbb{R}^n$ and $\omega \subset \Omega$ a neighborhood of $\overline{\Gamma}_0$ in Ω. Assume that $a \in L^\infty(\Omega)$ satisfies (1.10).*

Then, for every $C > 1$, there exists $\gamma > 0$ such that

$$F(t) \leq C\, F(0)\, e^{\gamma t}\,, \qquad \forall\, t > 0\,, \tag{1.15}$$

for every solution of (1.11) with initial data $y^0 \in L^2(\Omega)$.

Applying in (1.3) the conjugate Schrödinger operator $-i\,\partial_t + \Delta$, it is easy to see that every solution of (1.3) satisfies also the plate equation

$$y_{tt} + \Delta^2 y = 0 \tag{1.16}$$

with the following boundary conditions:

$$\begin{cases} \dfrac{\partial y}{\partial \nu} = -(m(x)\cdot\nu(x))\, y_t & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial \Delta y}{\partial \nu} = -(m(x)\cdot\nu(x))\, \Delta y_t & \text{on } \Gamma_0 \times (0,\infty) \\ y = \Delta y = 0 & \text{on } \Gamma_1 \times (0,\infty)\ . \end{cases} \tag{1.17}$$

Let us complete system (1.16)–(1.17) with initial conditions

$$y(x,0) = y^0\,, \qquad y_t(x,0) = y^1(x)\ , \tag{1.18}$$

such that $\{y^0, y^1\} \in W$ where

$$W = \left\{(\varphi,\psi) \in V \times V\colon\ \Delta\varphi \in V,\ \frac{\partial\varphi}{\partial\nu} = -(m\cdot\nu)\psi \ \text{ on } \Gamma_0\right\}\ .$$

If Γ_1 has non-empty interior on Γ, the map

$$\{\varphi,\psi\} \in W \to \|\{\varphi,\psi\}\|_W = \left[\|\Delta\varphi\|_v^2 + \|\psi\|_v^2\right]^{1/2}$$

defines a norm in W which is equivalent to the $H^3(\Omega)\times H^1(\Omega)$-norm.

Applying Lebeau's argument [8] (which consists in spliting the solution of the plate equation in two solutions of Schrödinger equations) the semigroup associated to system (1.3) can be extended to a contraction semigroup

$$\tilde S(t)\colon\ W \to W \qquad \text{such that } \ \{y(t), y_t(t)\} = \tilde S(t)\,\{y^0, y^1\}$$

is a weak solution of (1.16)–(1.18) for every $\{y^0, y^1\} \in W$.

The energy of solutions of (1.16)-(1.18) is the following:

$$G(t) = \frac{1}{2}\int_\Omega \left[|\nabla y_t(x,t)|^2 + |\nabla \Delta y(x,t)|^2\right] dx\ .$$

We have the following stabilization result:

Theorem 1.3. *Let* Ω *be a bounded domain of* $\mathbb{R}^n$ *with boundary of class* C^3. *Assume that* $n \le 3$. *Let be* $x_0 \in \mathbb{R}^n$ *such that* Γ_1 *has non-empty interior on* Γ.

Then, for every $C > 1$ there exists $\gamma > 0$ such that

$$G(t) \leq C\, e^{-\gamma t}\, G(0)$$

for every solution of (1.16)–(1.18) with initial data $\{y^0, y^1\} \in W$.

Remark 1.2. If $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ assumption $n \leq 3$ is not necessary.

Remark 1.3. In view of the particular structure of the boundary conditions (1.17) (that allows us to split the solution of (1.16)–(1.18) in two solutions of Schrödinger equations), Grisvard's analysis of singularities of solutions of Laplace's equation with mixed boundary conditions suffices to apply multiplier methods in system (1.16)–(1.18).

A set of the form Γ_0 as in (1.2) is a simple example of subset of the boundary satisfying the "geometric control property" introduced by C. Bardos, G. Lebeau and J. Rauch [1]. This geometric control condition is, essentially, a necessary and sufficient condition for the exact controllability and the stabilizability of wave equations. However, due to infinite speed of propagation, this notion of "geometric control" is not completely natural in the context of the controllability and the stabilizability of Schrödinger equation and plate models. However, G. Lebeau in [8] has proved that this geometric control condition is sufficient to ensure the boundary controllability of Schrödinger equation in $H^{-1}(\Omega)$ with $L^2(\Gamma_0)$ boundary controls. In the special case of Γ_0 satisfying (1.2) this result was proved by E. Machtyngier in [10] and [11]. However, the geometric control property is not necessary to ensure the exact controllability for Schrödinger and plate equations as in shown in A. Haraux [4] and S. Jaffard [5]. Our stabilization results must be understood in this context: sets of the form Γ_0 as in (1.2) are natural candidates to ensure tha stabilizability of Schrödinger equation but they are not optimal from a geometric view-point.

Theorems 1.1 and 1.2 will be proved by using multiplier methods. The method of multipliers has been recently adapted by E. Machtyngier [10], [11] to the study of the exact controllability of Schrödinger equation. Our proofs combine the techniques of [10], [11] with those developed in [6] and [12] for the study of the stabilization problem for wave equations.

Let us finally mention the work by C. Fabre [2] where, in the context of the exact controllability of Schrödinger equation, it is proved that the boundary control can be obtained as limit of internal controls supported on a neighborhood of the boundary as the width of the neighborhood tends to zero. It would be interesting to prove this type of result in the context of the boundary and internal stabilization of Schrödinger equation.

The rest of the paper is organizes as follows. In Section 2, we prove the boundary stabilization result Theorem 1.1. In Section 3 we prove the internal stabilization result Theorem 1.2. In Section 4 we prove the boundary stabilization result Theorem 1.3.

2 – Proof of the boundary stabilization result

Taking into account that $D(A)$ is dense in V and that $S(t): V \to V$ is continuous for every $t \geq 0$, it is sufficient to prove (1.9) for initial data in $D(A)$. Thus, in the sequel, we will assume that solution y of (1.3) belongs to the class (1.7).

For $\varepsilon > 0$ we introduce the functional

$$E_\varepsilon(t) = E(t) + \varepsilon\,\rho(t) \;, \tag{2.1}$$

with

$$\rho(t) = \operatorname{Im} \int_\Omega y(x,t)\, m(x) \cdot \nabla \overline{y}(x,t)\, dx \,, \qquad \forall\, t \geq 0 \;. \tag{2.2}$$

Note that (1.9) follows easily from the existence of positive constants ε_0, C_1 and C_2 such that

$$|\rho(t)| \leq C_1\, E(t)\,, \qquad \forall\, t \geq 0 \;, \tag{2.3}$$

and

$$\frac{dE_\varepsilon(t)}{dt} \leq -C_2\, \varepsilon\, E_\varepsilon(t)\,, \qquad \forall\, t \geq 0,\;\; \forall\, \varepsilon \in (0, \varepsilon_0) \;. \tag{2.4}$$

Using (1.4) we have

$$|\rho(t)| \leq \|y(t)\|_{L^2(\Omega)}\, \|m \cdot \nabla y(t)\|_{L^2(\Omega)} \leq \alpha\, \|m\|_{L^\infty(\Omega)}\, \|y(t)\|_v^2 = 2R\,\alpha\, E(t), \quad \forall\, t \geq 0,$$

with $R = \|m\|_{L^\infty(\Omega)}$. Thus, (2.3) holds with $C_1 = 2R\,\alpha$.

Let us now prove (2.4).

Multiplying equation (1.3) by $\overline{y}_t$ and integrating by parts over Ω we obtain

$$0 = \operatorname{Re} \int_\Omega \Big[i\, |y_t|^2 + \Delta y\, \overline{y}_t \Big]\, dx = -\operatorname{Re} \int_\Omega \nabla y \cdot \nabla \overline{y}_t\, dx + \operatorname{Re} \int_\Gamma \frac{\partial y}{\partial \nu}\, \overline{y}_t\, d\Gamma \;.$$

It follows that

$$\frac{dE(t)}{dt} = E'(t) = \operatorname{Re} \int_\Omega \nabla y \cdot \nabla \overline{y}_t\, dx = -\int_{\Gamma_0} (m \cdot \nu)\, |y_t|^2\, d\Gamma\,, \qquad \forall\, t \geq 0 \;. \tag{2.5}$$

Differentiating in (2.2) we have

$$\rho'(t) = \mathrm{Im} \int_\Omega y_t\, m \cdot \nabla \overline{y}\, dx + \mathrm{Im} \int_\Omega y\, m \cdot \nabla \overline{y}_t\, dx\ . \tag{2.6}$$

By using the divergence theorem we get

$$\begin{aligned}\mathrm{Im} \int_\Omega y\, m \cdot \nabla \overline{y}_t\, dx &= \mathrm{Im} \int_\Gamma (m\cdot\nu)\, y\, \overline{y}_t\, d\Gamma - \mathrm{Im} \int_\Omega m \cdot \nabla y\, \overline{y}_t\, dx - n\ \mathrm{Im} \int_\Omega y\, \overline{y}_t\, dx \\ &= \mathrm{Im} \int_{\Gamma_0} (m\cdot\nu)\, y\, \overline{y}_t\, d\Gamma + \mathrm{Im} \int_\Omega m \cdot \nabla \overline{y}\, y_t\, dx - n\, \mathrm{Im} \int_\Omega y\, \overline{y}_t\, dx\ .\end{aligned}$$

On the other hand, using equation (1.3) we get

$$\begin{aligned}\mathrm{Im} \int_\Omega y\, \overline{y}_t\, dx &= -\,\mathrm{Re} \int_\Omega \Delta y\, \overline{y}\, dx = \mathrm{Re} \int_\Omega \nabla y \cdot \nabla \overline{y}\, dx - \mathrm{Re} \int_\Gamma \frac{\partial y}{\partial \nu}\, \overline{y}\, d\Gamma \\ &= \int_\Omega |\nabla y|^2\, dx + \mathrm{Re} \int_{\Gamma_0} (m\cdot\nu)\, y_t\, \overline{y}\, d\Gamma\end{aligned}$$

and

$$\mathrm{Im} \int_\Omega m \cdot \nabla y\, \overline{y}_t\, dx = \mathrm{Re} \int_\Omega m \cdot \nabla \overline{y} \cdot \Delta y\ dx\ .$$

Thus

$$\rho'(t) = 2Re \int_\Omega \Delta y\, m \cdot \nabla \overline{y}\, dx - n \int_\Omega |\nabla y|^2\, dx - \mathrm{Re} \int_{\Gamma_0} (m\cdot\nu)\, (i+n)\, y\, \overline{y}_t\, d\Gamma\ . \tag{2.7}$$

Now we use the following generalization of Grisvard's inequality [3] proved in [6].

Lemma 2.1. *Assume that $n \le 3$. Let be $\varphi, \psi \in V$ such that $\Delta\varphi \in L^2(\Omega)$ and*

$$\frac{\partial \varphi}{\partial \nu} = -(m\cdot\nu)\, \psi \quad \textit{on}\ \ \Gamma_0\ .$$

Then

$$2 \int_\Omega \Delta\varphi\, m \cdot \nabla \varphi\, dx \le (n-2) \int_\Omega |\nabla \varphi|^2\, dx + 2 \int_\Gamma \frac{\partial \varphi}{\partial \nu}\, m \cdot \nabla \varphi\, d\Gamma - \int_\Gamma (m\cdot\nu)\, |\nabla\varphi|^2\, d\Gamma\ .$$

Note that in Lemma 2.1, φ and ψ are real valued functions. Applying Lemma 2.1 to the solution y of (1.3) we obtain that

$$\begin{aligned}\mathrm{Re} \int_\Omega \Delta y\, m \cdot \nabla \overline{y}\, dx &\le \frac{n-2}{2} \int_\Omega |\nabla y|^2\, dx + \mathrm{Re} \int_\Gamma \frac{\partial y}{\partial \nu}\, m \cdot \nabla \overline{y}\, d\Gamma \\ &\quad - \frac{1}{2} \int_\Gamma (m\cdot\nu)\, |\nabla y|^2\, d\Gamma\ .\end{aligned} \tag{2.8}$$

Combining (2.7)–(2.8) we get that

$$
\begin{aligned}
\rho'(t) &\le -2\int_\Omega |\nabla y|^2\,dx - \mathrm{Re}\int_{\Gamma_0}(m\cdot\nu)\left[2y_t(m\cdot\nabla\overline{y}) + |\nabla y|^2 + (i+n)\,y\,\overline{y}_t\right]d\Gamma \\
&\quad + \mathrm{Re}\int_{\Gamma_1}(m\cdot\nu)\left|\frac{\partial y}{\partial\nu}\right|^2 d\Gamma - \int_{\Gamma_1}(m\cdot\nu)\,|\nabla y|^2\,d\Gamma \le \qquad (2.9)\\
&\le -2\int_\Omega |\nabla y|^2\,dx - \mathrm{Re}\int_{\Gamma_0}(m\cdot\nu)\left[2y_t(m\cdot\nabla\overline{y}) + |\nabla y|^2 + (i+n)\,y\,\overline{y}_t\right]d\Gamma
\end{aligned}
$$

for every $t \ge 0$ since $\nabla y = \frac{\partial y}{\partial\nu}\nu$ on $\Gamma_1\times(0,\infty)$ and $m\cdot\nu < 0$ on Γ_1.

From (2.1), (2.5) and (2.9) we deduce that

$$
E'_\varepsilon(t) \le -4\varepsilon E(t) - \mathrm{Re}\int_{\Gamma_0}(m\cdot\nu)\left[|y_t|^2 + \varepsilon(n+i)\,y\,\overline{y}_t + 2\varepsilon\, y_t\, m\cdot\nabla\overline{y} + \varepsilon|\nabla y|^2\right]d\Gamma. \qquad (2.10)
$$

On the other hand, combining Poincare's inequality (1.4) and the continuity of the trace from $H^1(\Omega)$ into $L^2(\Gamma)$ we deduce the existence of some constant $\beta > 0$ such that

$$
\int_{\Gamma_0}(m\cdot\nu)\,|\varphi|^2\,d\Gamma \le \beta\int_\Omega |\nabla\varphi|^2\,dx\,, \qquad \forall\,\varphi\in V\ .
$$

Hence, it follows that

$$
\begin{aligned}
\left|\int_\Gamma (m\cdot\nu)(n+i)\,y\,\overline{y}_t\,d\Gamma\right| &\le \frac{\beta}{2}(n^2+1)\int_{\Gamma_0}(m\cdot\nu)\,|y_t|^2\,d\Gamma + \frac{1}{2\beta}\int_{\Gamma_0}(m\cdot\nu)\,|y|^2\,d\Gamma \\
&\le \frac{\beta}{2}(n^2+1)\int_{\Gamma_0}(m\cdot\nu)\,|y_t|^2\,d\Gamma + \frac{1}{2}\int_\Omega |\nabla y|^2\,dx\ . \qquad (2.11)
\end{aligned}
$$

On the other hand

$$
|2y_t\, m\cdot\nabla\overline{y}| \le R^2|y_t|^2 + |\nabla y|^2 \quad \text{on}\ \ \Gamma_0\times(0,\infty)\ . \qquad (2.12)
$$

Combining (2.10)–(2.12) we conclude that

$$
E'_\varepsilon(t) \le -3\varepsilon E(t) - \mathrm{Re}\int_{\Gamma_0}(m\cdot\nu)\left[1-\varepsilon\left(R^2 + \frac{(n^2+1)\,\beta}{2}\right)\right]|y_t|^2\,d\Gamma\ .
$$

Choosing $\varepsilon \le \varepsilon_0 = \frac{2}{2R^2+(n^2+1)\beta}$ and taking into account that $m\cdot\nu \ge 0$ on Γ_0 we deduce that

$$
E'_\varepsilon(t) \le -3\varepsilon E(t)\ .
$$

This concludes the proof of (2.4). Theorem 1.1 is proved. ∎

3 – Proof of the internal stabilization result for Schrödinger equation

First we note that, in view of (1.14), it is sufficient to prove the existence of a time $T>0$ and a constant $C_0>0$ such that

$$F(T) \le C_0 \int_0^T \int_\Omega a(x)\,|y(x,t)|^2\;dx\,dt \tag{3.1}$$

for every solution of (1.11) with initial data $y^0 \in L^2(\Omega)$. (In fact we will prove that (3.1) holds for any $T>0$ and some constant $C=C(T)>0$.)

Indeed, combining (1.14) and (3.1) it follows that

$$F(T) \le \frac{C_0}{1+C_0}\,F(0) \tag{3.2}$$

which, combined with the semigroup property, yields (1.15) with

$$C = 1+\frac{1}{C_0}\,; \qquad \gamma = \frac{1}{T}\log\left(1+\frac{1}{C_0}\right). \tag{3.3}$$

In order to prove (3.1) we write the solution y of (1.11) as $y=\varphi+z$ where $\varphi=\varphi(x,t)$ solves

$$\begin{cases} i\,\varphi_t + \Delta\varphi = 0 & \text{in } \Omega\times(0,\infty) \\ \varphi = 0 & \text{on } \partial\Omega\times(0,\infty) \\ \varphi(0) = y^0 & \text{in } \Omega \end{cases} \tag{3.4}$$

and $z=z(x,t)$ satisfies

$$\begin{cases} i\,z_t + \Delta z = -i\,a(x)\,y & \text{in } \Omega\times(0,\infty) \\ z = 0 & \text{on } \partial\Omega\times(0,\infty) \\ z(0) = 0 & \text{in } \Omega\,. \end{cases} \tag{3.5}$$

Using the non-increasing character of the energy $F(\,\cdot\,)$ we get

$$F(T) \le F(0) = \frac{1}{2}\,\|\varphi(0)\|^2_{L^2(\Omega)}\,. \tag{3.6}$$

Now we use the following observability estimate wich is due to E. Machtyngier, [10], [11].

Proposition 3.1 ([10], [11]). *Let Ω and ω be as in the statement of Theorem 1.2. Then, for every $T > 0$ there exists a positive constant $C_1 = C_1(T)$ such that*

$$\|\varphi(0)\|^2_{L^2(\Omega)} \le C_1 \int_0^T \int_\omega |\varphi(x,t)|^2 \, dx\, dt \tag{3.7}$$

for every solution of (3.4) with initial data $y^0 \in L^2(\Omega)$.

Combining (3.6)-(3.7) and using (1.10) we get

$$\begin{aligned} F(T) &\le \frac{C_1}{2} \int_0^T \int_\omega |\varphi(x,t)|^2 \, dx\, dt \\ &\le \frac{C_1}{2a_0} \int_0^T \int_\Omega a(x)\, |\varphi(x,t)|^2 \, dx\, dt \\ &\le \frac{C_1}{a_0} \int_0^T \int_\Omega a(x) \Big[|y|^2 + |z|^2\Big] \, dx\, dt \\ &\le \frac{C_1}{a_0} \int_0^T \int_\Omega a(x)\, |y|^2 \, dx\, dt \,+\, \frac{C_1 \|a\|_\infty}{a_0} \int_0^T \|z(t)\|^2_{L^2(\Omega)} \, dt \;. \end{aligned} \tag{3.8}$$

By classical estimates on Schrödinger equation we have

$$\begin{aligned} \|z\|^2_{L^\infty(0,T;L^2(\Omega))} &\le C\, \|i\, a(x)\, y\|^2_{L^2(\Omega\times(0,T))} \\ &\le C\, \|a\|_\infty \int_0^T \int_\Omega a(x)\, |y(x,t)|^2 \, dx\, dt \;. \end{aligned} \tag{3.9}$$

Combining (3.8)–(3.9), (3.1) follows. This completes the proof of Theorem 1.2. ∎

4 – Boundary stabilization of the plate model

Given $\{y^0, y^1\} \in D(A)$ satisfying

$$\begin{cases} -\Delta v^0 = \dfrac{i}{2}\, y^1 - \dfrac{1}{2} \Delta y^0 & \text{in } \Omega \\ v^0 = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial v^0}{\partial \nu} = -\frac{(m\cdot\nu)}{2}\, (y^1 + i\, \Delta y^0) & \text{on } \Gamma_0 \;. \end{cases} \tag{4.1}$$

Define $v=v(x,t)$ as the solution of

$$\tag{4.2} \begin{cases} -i\,v_t+\Delta v=0 & \text{in } \Omega\times(0,\infty) \\ v=0 & \text{on } \Gamma_1\times(0,\infty) \\ \dfrac{\partial v}{\partial\nu}=-(m\cdot\nu)\,v_t & \text{on } \Gamma_0\times(0,\infty) \\ v(0)=v^0\ . \end{cases}$$

Solution v of (4.2) is given by $v(t)=S(t)\,v^0$ and belongs to $C([0,\infty);\,D(A))\cap C^1([0,\infty);\,V)$.

It is easy to check that $y(x,t)=v(x,t)+\overline{v}(x,t)$ satisfies (1.16)–(1.18). Thus, (1.16)–(1.18) we have a contraction semigroup $\{\tilde{S}(t)\}_{t\geq 0}$ in W associated to system (1.16)–(1.18) such that

$$\{y(t),y_t(t)\}=\tilde{S}(t)\,\{y^0,y^1\}=\Big\{S(t)\,v^0+\overline{S(t)\,v^0},\ i(\Delta S(t))\,v^0-\Delta\overline{S(t)\,v^0})\Big\}$$

is the unique solution of (1.16)–(1.18) in $C([0,\infty);W)$ for every $\{y^0,y^1\}\in W$.

In order to prove Theorem 1.3 it is sufficient to prove the stabilization of Schrödinger equation (4.2) in $D(A)$, i.e.

$$\tag{4.3} \|\nabla\Delta v(t)\|^2_{L^2(\Omega)}\leq C\,e^{-\gamma t}\|\nabla\Delta v^0\|^2_{L^2(\Omega)}\ .$$

In Section 2 we have proved the exponential decay in V. In order to prove it in $D(A)$ it is sufficient to observe that, if $v\in C([0,\infty);\,D(A))\cap C^1([0,\infty);\,V)$ is solution of (4.2) then

$$v_t(t)=i\,\Delta v(t)=S(t)\,[i\,\Delta v^0]\in C([0,\infty);V)$$

is weak solution of (4.2) with initial data $i\,\Delta v^0\in V$. In view of the exponential decay of the semigroup $\{S(t)\}_{t\geq 0}$ in V we have

$$\|\Delta v(t)\|^2_V\leq C\,e^{-\gamma t}\,\|\Delta v^0\|^2_V$$

which is equivalent to (4.3).

This conclude the proof of Theorem 1.3. ∎

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