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PERIODIC SOLUTIONS TO RETARDED AND PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: The existence of mild and strong periodic solutions to a retarded functional differential equation in a Banach space is established upon the condition that the non-linear term is periodic. These results are then applied to a class of parabolic partial functional differential equations.

0 – Introduction

In this note we are interested in studying periodicity questions concerning "different kinds" (mild, strong) of solutions to the retarded functional differential equation

$$\begin{aligned}&\frac{du(t)}{dt} + Au(t) = F(t, u_t)\,, \qquad t > 0\,,\\ &u_0 = \varphi\,,\end{aligned} \tag{0.1}$$

where A is the infinitesimal generator of a semigroup of linear operators $T(t)$, $t \geq 0$ and F is nonlinear, satisfying assumptions to be specified in the subsequent section, and periodic.

Moreover, the results related to the above RFDE will serve as a basis in establishing existence of periodic solutions for the partial functional differential equation

$$v_t(x,t) = v_{xx}(x,t) + f\Big(t, v(x, t-r),\ v_x(x, t-r)\Big), \quad (x,t) \in [0,\pi] \times \mathbb{R}^+\,, \tag{0.2}$$

satisfying conditions of the form

$$v(0,t) = v(\pi,t) = 0 \ \text{ and } \ v(x,t) = \varphi(x,t)\,, \qquad t \in [-r, 0]\,, \tag{0.3}$$

for suitable f.

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Periodicity problems for functional differential equations have been studied by many authors, see for example, [1], [5], [7], and the references therein. Travis and Webb [12], [13] have studied the problems of existence (and stability) of solutions of (0.1), using methods derived from the fundamental results of Segal [11].

Section 1 contains definitions and preliminaries to be used in the subsequent development. In Section 2 we establish the main results, assuming that the forcing term F is periodic. In Section 3, we use the results of Section 2 to study the equation $((0.2),(0.3))$.

1 – Notation and preliminaries

Throughout this paper E will denote a Banach space over a real or complex field with norm $\|\cdot\|$. $C := C([-r,0];E)$ will denote the Banach space of continuous E-valued functions on $[-r,0]$, with the supremum norm, r being a positive real number. If u is a function with domain $[\sigma-r,\sigma+b)$, then for any $t\in[\sigma,\sigma+b)$, u_t will denote the element of C, defined by $u_t(\theta)=u(t+\theta)$, $-r\le\theta\le 0$. $B(E,E)$ will denote the space of bounded, linear, everywhere defined operators from E to E. A strongly continuous semigroup on E is a family $T(t)$, $t\ge 0$, of everywhere defined (possibly nonlinear) operators from E to E, satisfying $T(t+s)=T(t)\,T(s)$, $s,t\ge 0$, and $T(t)\,x$ is continuous as a function from $[0,\infty)$ to E for each fixed $x\in E$. The infinitesimal generator A of $T(t)$, $t\ge 0$, is the function from E to E defined by $A\,x=\lim_{t\to 0_+}\frac{T(t)x-x}{t}$, with domain $D(A)$ the set of all x for which this limit exists.

We will be dealing with the abstract ordinary functional differential equation in E, of the form

$$\begin{aligned} &\frac{du(t)}{dt}+Au(t)=F(t,u_t)\,, \qquad t>0\;,\\ &u_0=\varphi\;. \end{aligned}\tag{1.1}$$

We will make the following assumptions on the operator A:

(A1) A is closed, densely defined linear operator in E, and $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$, $t\ge 0$, satisfying

$$\|T(t)\,x\|<\mu\,e^{\gamma t}\,\|x\| \quad \text{ for } \; t>0, \;\; x\in E\;,$$

where μ and γ are real constants.

Moreover we shall need assumptions on the fractional power A^a of A:

(A2) For $a \in [0,1)$, $\|A^a\, T(t)\, x\| \le \mu_a\, t^{-a}\, e^{\gamma t}\|x\|$, for $t > 0$, $x \in E$, where μ_a is a real positive constant;

(A3) $A^{-a} \in B(E,E)$; so $E_a := D(A^a)$ is a Banach space when endowed with the norm $\|x\|_a = \|A^a\, x\|$ for $x \in E_a$;

(A4) A^{-a} satisfies $\|[T(t)-I]A^{-a}\| \le v_a\, t^a$ for $t > 0$, where v_a is a real positive constant;

(A5) $T(t)$ is compact for each $t > 0$.

C_a will denote the Banach space of continuous functions $C([-r,0];E_a)$ with the norm $\|\varphi\|_{c_a} = \sup\{\|A^a\, \varphi(\theta)\| :\ \theta \in [-r,0]\}$. F is supposed to satisfy the assumption:

(F1) $F\colon\ D \to E$ is continuous, where D is an open set in $\mathbb{R} \times C_a$.

To conclude this section we give the definitions of the terms "mild" and "strong" solution of (1.1): as is well known, to (1.1) corresponds the following integral equation,

$$(1.2)\qquad \begin{aligned} & u(t) = T(t-\sigma)\,\varphi(0) + \int_\sigma^t T(t-s)\, F(s,u_s)\, ds, && t \in [\sigma, \sigma + n_\varphi)\ , \\ & u_\sigma = \varphi, && t \in [-r,0]\ . \end{aligned}$$

Then

i) u is a mild solution of (1.1), if it satisfies (1.2) and $u \in C([\sigma-r, \sigma+n_\varphi); E_a)$;

ii) u is a strong solution of (1.1) if it satisfies (1.2) and $u \in C([\sigma - r, \sigma + n_\varphi); E_a) \cap C^1((\sigma, \sigma + n_\varphi); E)$.

2 – Main results

We shall establish the proofs of our main results in a series of lemmata:

Lemma 2.1 (Local existence). *Suppose that (A1), (A2), (A3), (A4), (A5) and (F1) hold. For each $(\sigma,\varphi) \in D$ there exists $n_\varphi > 0$, such that the problem*

$$(2.1)\qquad \begin{aligned} & \frac{du(t)}{dt} + Au(t) = F(t,u_t)\,, \qquad t \in [\sigma, \sigma + n_\varphi)\ , \\ & u_\sigma = \varphi\ , \end{aligned}$$

has a mild solution.

The proof can be found in [13].

Now let F be ω-periodic in t, i.e.

(F2) There exists $\omega > 0$: $F(t+\omega,\psi) = F(t,\psi)$, $t \in \mathbb{R}^+$, $\psi \in C_a$.

In order to be able to discuss periodicity of solutions, we need a continuation result:

Lemma 2.2 (continuation of solutions). *Suppose that (A1) to (A5) hold and that (F1) is substitued by the stronger hypothesis*

(F3) *$F:\ D \to E$ is continuous, and maps closed, bounded sets of D into bounded sets in E.*

Let u defined on $[\sigma - r, T)$ be a non-continuable beyond T solution of (2.1). Then either $T = +\infty$, or

for any closed, bounded set W in D, there is a t_w such that

$$(t,u_t) \notin W \quad \text{for } t_w \le t < T\ .$$

Proof: The idea of the proof is that of Theorem 3.2 of [5], and is performed along the lines of the proof of Proposition 3.1 of [13]. The details follow: assume $T < +\infty$. Let us suppose, for contradiction, that the conclusion of the Lemma is not correct. Then we can find a closed, bounded set W in D such that for $\sigma \le t < T$, $(t,u_t) \in W$.

Denote by ξ the $\sup\{F(t,\psi)\colon (t,\psi) \in W\}$. Then

$$\begin{aligned}
\Big\|u(t+h) - u(t)\Big\|_a &\le \Big\|(T(h)-I)\,A^{-(\beta-a)}\,A^\beta\,u(t)\Big\| \\
&\quad + \Big\|\int_t^{t+h} A^a\,T(t+h-s)\,F(s,u_s)\,ds\Big\| \\
&\le v_{\beta-a}\,h^{\beta-a}\|u(t)\|_\beta + \frac{\xi}{1-a}\,\mu_a \max(1,e^{\gamma T})\,h^{1-a}
\end{aligned}$$

for $a < \beta < 1$ and $t, t+h \in (\sigma,T)$, $h>0$.

On the other hand,

$$\begin{aligned}
\|u(t)\|_\beta &\le \Big\|A^{\beta-a}\,T(t-\sigma)\,A^a\,\varphi(0)\Big\| + \Big\|\int_\sigma^t A^\beta\,T(t-s)\,F(s,u_s)\,ds\Big\| \\
&\le \mu_{\beta-a}\,e^{\gamma(t-\sigma)}(t-\sigma)^{a-\beta}\|\varphi\|_a + \mu_\beta\,\xi \int_0^{t-a} e^{\gamma s}\,s^{-\beta}\,ds\ ,
\end{aligned}$$

whereby $\|u(t)\|_\beta$ is bounded on compact subsets of (σ,T). Therefore

$$\Big\|u(t+h) - u(t)\Big\|_a \le k\,h^\theta\ ,$$

for suitable $\theta = \theta(a, \beta)$, and hence u is uniformly continuous on $[\sigma - r, T)$, thus establishing the existence of $\lim_{t \to T} u(t)$. In this way, u can be continuously extended to $[\sigma - r, T]$. But $(T, u_T) \in D$, and so there exists a solution through (T, u_T) beyond T, contradicting the hypothesis of the Lemma. ∎

If we consider F defined on $[\sigma, \infty) \times C_a$, then we can directly prove the following:

Corollary 2.1. *Suppose that all hypothesis of Lemma 2.2 are valid. Then, either $T = +\infty$, or if $T < +\infty$ then*

$$\limsup_{t \to T^-} \|u_t\|_{C_a} = +\infty \ . \tag{2.2}$$

Under the assumptions of Corollary 2.1, it follows that there exists a global solution, provided

$$\limsup_{t \to T^-} \|u_t\|_{C_a} < +\infty \ .$$

We shall also need the following

Lemma 2.3. *Let A satisfy (A1) and consider the initial value problem*

$$\begin{aligned} &\frac{du(t)}{dt} + Au(t) = g(t), \qquad && t > 0 \ , \\ &u(0) = x, && x \in E \ , \end{aligned} \tag{2.3}$$

where g is an E-valued continuous function. Then (2.3) has a unique mild solution.

For a proof see [9], p. 106.

Now we consider the problem

$$\begin{aligned} &\frac{du(t)}{dt} + Au(t) = F(t, u_t) \ , \\ &u(0) = \varphi \ , \end{aligned} \tag{2.4}$$

and we suppose that it has a global solution $u(t)$.

We also consider the initial value problem for the inhomogeneous ordinary differential equation

$$\begin{aligned} &\frac{dz(t)}{dt} + Az(t) = F(t, u_t) \ , \\ &z(0) = u(0) \ . \end{aligned} \tag{2.5}$$

By Lemma 2.3, the problem (2.5) has a unique mild solution $z(t)$. Let $P\colon C\left([-r,n_\varphi];E_a\right)\to E$ be the Poincar mapping, defined by

$$P\,u = z(\omega)\ . \tag{2.6}$$

Finally, consider the initial value problem,

$$\begin{aligned}&\frac{du(t)}{dt}+Au(t)=F(t,u_t)\ ,\\ &u(0)=P\,u\ ,\end{aligned} \tag{2.7}$$

which – by Lemma 2.3 – has, also, a unique mild solution $u(t)$.

Let $S\colon C\left([-r,n_\varphi];E_a\right)\to C\left([-r,n_\varphi];E\right)$ be the mapping defined by

$$S\,u = u\ .$$

We are now in a position to state and prove the basic tool for the proof of our main results.

Lemma 2.4. *The problem (2.4) has a periodic solution if and only if the mapping S has a fixed point.*

Proof: Let u be an ω-periodic solution of (2.4). Then u is, clearly, an ω-periodic solution of (2.5) and hence $Pu=u(\omega)$. Since u is ω-periodic $u(0)=u(\omega)$, and therefore $u(0)=Pu$, whereby u satisfies (2.7) and so $Su=u$. Conversely, let u be a fixed point of S. By definition, u satisfies (2.5) and since $u(0)=z(0)$, Lemma 2.3 shows that $u(t)\equiv z(t)$ and hence $u(\omega)=z(\omega)$. Since $Su=u$, (2.7) gives $u(0)=Pu=z(\omega)$. We have thus concluded that $u(0)=u(\omega)$. But F is ω-periodic, and therefore $u(t)=u(t+\omega)$, $t\in\mathbb{R}^+$, i.e. (2.4) has a periodic solution. ∎

To finish up, we have:

Lemma 2.5. *Let Q be the set defined by*

$$Q=\Big\{\xi\in C([-r,n_\varphi];E_a)\colon\ \xi_0=0,\ \|\xi_t\|_{C_a}\le\gamma\ \ \textit{for} t\in[0,n_\varphi]\Big\}\ .$$

Then the mapping $S\colon Q\to C\left([-r,n_\varphi];E_a\right)$ has at least one fixed point.

In order to prove Lemma 2.5 one can use Schauder's fixed point theorem. It therefore suffices to show that S maps Q into itself, S is continuous, and $S(Q)$ is relatively compact. These follow as in the proof of Lemma 2.1, of which Lemma 2.5 consists a restatement.

The proof of the following theorem is a direct combination of the previous lemmata.

Theorem 2.1. *Consider the problem (2.4):*

$$\frac{du(t)}{dt} + Au(t) = F(t, u_t) \ ,$$
$$u_0 = \varphi \ ,$$

and let A satisfy (A1) to (A5) and F satisfy (F2) and (F3), with $D = [\sigma, \infty) \times C_a$. Let, moreover, u defined on $[\sigma - r, T)$ satisfy

$$\limsup_{t \to T^-} \|u_t\|_{C_a} < +\infty \ . \tag{2.8}$$

Then (2.4) has an ω-periodic mild solution.

We proceed to investigating under what conditions the mild solutions of (2.4) are actually strong solutions. Arguing as in [13], it is easy to prove the following:

Theorem 2.2. *Consider (2.4), and suppose that A satisfies (A1) to (A5), while F satisfies (F2), (F3) with $D = [\sigma, \infty) \times C_a$, and*

(F4) *There exist constants $\lambda > 0$ and $\theta \in (0, 1]$ such that*

$$\Big\| F(t_1, \psi_1) - F(t_2, \psi_2) \Big\| \le \lambda \Big\{ |t_1 - t_2|^\theta + \|\psi_1 - \psi_2\|_{C_a}^\theta \Big\}$$

holds in a neighbourhood Λ of any point of D for which $(t_1, \psi_1), (t_2, \psi_2) \in \Lambda$.

Let u satisfy (2.8). Then every ω-periodic mild solution of (2.4) is an ω-periodic strong solution of (2.4).

We are concluding this section with a result concerning the positivity of solutions of (2.4). Its proof follows from standard arguments and is omitted for the sake of brevity.

Theorem 2.3. *Suppose, additionally, that E is a partially ordered Banach space with a closed cone E^+. If F is positive, the semigroup $T(t)$ is positive, and $\varphi \in C^+$, where $C^+ = \{h \in C \colon\ h(t) \in E^+ \ \forall t \in [-r, 0]\}$ then the ω-periodic mild (cf. Theorem 2.1) or strong (cf. Theorem 2.2) solutions of (2.4) are positive.*

3 – Application to parabolic partial functional differential equations

In this last section, we shall apply the results of Section 2 to the following problem, whose autonomous analogue is studied in [13]. Consider the problem

$$u_t(x,t) = u_{xx}(x,t) + f\Big(t, u(x,t-r), u_x(x,t-r)\Big) \;, \tag{3.1}$$

where $(x,t) \in [0,\pi] \times \mathbb{R}^+$, $r \in \mathbb{R}^+$, and

$$\begin{aligned} &u(0,t) = u(\pi,t) = 0, && t \geq 0 \;, \\ &u(x,t) = g(x,t), && (x,t) \in [0,\pi] \times [-r,0] \;. \end{aligned} \tag{3.2}$$

Let $f\colon\, \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous in its first and second variables and Lipschitz continuous in its third variable, satisfying $f(0,0,0) = 0$.

Assume that f satisfies, moreover,

$$|f(t,\xi,\eta)| \leq k(t)\,(1 + |\xi| + |\eta|) \;, \tag{3.3}$$

where $k(\cdot)$ is continuous on (σ,∞).

Such a condition has been considered by other authors as well, (e.g. see [6]).

We suppose that f is ω-periodic in t:

$$f(t+\omega,\xi,\eta) = f(t,\xi,\eta) \;, \qquad t \geq 0 \;. \tag{3.4}$$

Let $E = L^2([0,\pi])$ and let $A\colon\, E \to E$ be defined by

$$Az = -z'' \;,$$

$$D(A) = \Big\{ z \in E\colon\; z,\, z' \text{ are absolutely continuous, } z'' \in E \text{ and } z(0) = z(\pi) = 0 \Big\} .$$

Let $a_m(T) = (\frac{2}{\pi})^{1/2} \sin mT$, $m = 1,2,\ldots$, be the orthonormal set of eigenvectors of A, and so

$$Az = \sum_{m=1}^{\infty} m^2 (z, a_m)\, a_m \;, \qquad z \in D(A) \;.$$

It is well known that $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$, in E, given by

$$T(t)\, z = \sum_{m=1}^{\infty} e^{-m^2 t} (z, a_m)\, a_m \;, \qquad z \in E \;.$$

This $T(t)$ satisfies the inequality in (A1) with $\mu \geq 1$, $\gamma \geq -1$. Since

$$A^{1/2}\, T(t)\, z = \sum_{m=1}^{\infty} m\, e^{-m^2 t} (z, a_m)\, a_m \;, \qquad z \in E \;,$$

and

$$\|A^{1/2}\,T(t)\,z\|^2 \le \sup_{m\ge 1}\{m^2\,e^{-2m^2 t}\}\,\|z\|^2\,, \qquad z\in E\ ,$$

and, moreover,

$$t\,m^2\,e^{-2t(m^2+\gamma)} \le \frac{1}{2e(1+\gamma)}\ .$$

A satisfies (A2) with $a=\frac{1}{2}$, $-1<\gamma<0$, $\mu_{1/2}=\frac{1}{(2e(1+\gamma))^{1/2}}$. On the other hand, since

$$A^{-1/2}z=\sum_{m=1}^{\infty}\frac{1}{m}(z,a_m)\,a_m\,, \qquad z\in E\ ,$$

and

$$A^{-1/2}\,T(t)\,z=\sum_{m=1}^{\infty}\frac{1}{m}\,e^{-m^2 t}(z,a_m)\,a_m\,, \qquad z\in E\ ,$$

we have

$$\Big\|(T(t)-I)\,A^{-1/2}z\Big\|^2 \le \sup_{m\ge 1}\Big\{\frac{1}{m^2}\,(e^{-m^2 t}-1)^2\Big\}\,\|z\|^2$$

and since

$$\frac{1}{m^2}\,(e^{-m^2 t}-1)^2 \le \frac{t}{2}$$

(A3) is satisfied, as well as (A4) with $v_{1/2}=\frac{\sqrt{2}}{2}$.

Noting that the eigenvalues of $A^{1/2}$ are $\lambda_m=\frac{1}{m}$, we have that $A^{-1/2}$ is compact. But this is a necessary and sufficient condition for an analytic semigroup $T(t)$, $t>0$ to be compact, and hence (A5) is also satisfied.

Now, let $F\colon \mathbb{R}^+\times C_{1/2}\to E$ be defined by

$$F(t,\varphi)(x)=f\Big(t,\varphi(-r)(x),\varphi(-r)'(x)\Big)\,, \qquad \varphi\in C_{1/2},\ \ x\in[0,\pi]\ .$$

Since f is continuous in its first and second variables and Lipschitz continuous in the third, and since $z\in D(A^{1/2})$ implies that z is absolutely continuous and $z'\in E$, F is well defined.

To prove the continuity of F, it suffices to show that the mapping $\phi\colon \mathbb{R}^+\times C_{1/2}\to\mathbb{R}^+\times E\times E$ defined by

$$\phi(t,\varphi)=(t,\varphi(-r),\varphi(-r)')$$

is continuous.

Indeed, we observe that

$$\begin{aligned}
\Big\|\varphi_1(-r)-\varphi_2(-r)\Big\|_E^2 &= \int_0^\pi \Big|\varphi_1(-r)(T)-\varphi_2(-r)(T)\Big|^2\,dT \\
&= \sum_{m=1}^\infty \Big(\varphi_1(-r)-\varphi_2(-r),a_m\Big)^2 \\
&\le \sum_{m=1}^\infty m^2\Big(\varphi_1(-r)-\varphi_2(-r),a_m\Big)^2 \\
&\le \Big\|A^{1/2}(\varphi_1(-r)-\varphi_2(-r))\Big\|^2 \\
&\le \|\varphi_1-\varphi_2\|_{C_{1/2}}^2\ ,
\end{aligned}$$

while, on the other hand

$$\begin{aligned}
\Big\|\varphi_1(-r)'-\varphi_2(-r)'\Big\|_E^2 &= \int_0^\pi \Big|\varphi_1(-r)'(T)-\varphi_2(-r)'(T)\Big|\,dt = \\
&= \sum_{m=1}^\infty\sum_{k=1}^\infty \Big(\varphi_1(-r)-\varphi_2(-r),a_m\Big)\Big(\varphi_1(-r)-\varphi_2(-r),a_k\Big)(a_m',a_k') \\
&= \sum_{m=1}^\infty\sum_{k=1}^\infty \Big(\varphi_1(-r)-\varphi_2(-r),a_m\Big)\Big(\varphi_1(-r)-\varphi_2(-r),a_k\Big)(-a_m'',a_k) \\
&= \sum_{m=1}^\infty \Big(\varphi_1(-r)-\varphi_2(-r),a_m\Big)^2 m^2 \\
&= \Big\|A^{1/2}(\varphi_1(-r)-\varphi_2(-r))\Big\|^2 \\
&\le \|\varphi_1-\varphi_2\|_{C_{1/2}}^2\ ,
\end{aligned}$$

thus proving the referred to continuity.

By (3.3) it follows that F satisfies

$$\|F(t,\varphi)\| \le k(t)\Big(1+\|\varphi\|_{C_{1/2}}\Big)\ ,$$

whereby we have that F maps closed, bounded subsets of $\mathbb{R}^+\times C_{1/2}$ into bounded sets of $L^2([0,\pi])$.

It remains to show that the solutions of ((3.1), (3.2)) are defined globally. This can fail only if there exist $t_n\to T<\infty$, such that $\|u_{t_n}\|_{C_{1/2}}\to\infty$. However, since

$$\|u_t\|_{C_{1/2}} = \sup\Big\{\|A^{1/2}u(\theta)\|\colon\ \theta\in[-r,0]\Big\}$$

and u satisfies (1.1), we have

$$\|u_t\|_{C_{1/2}} \le \left\| A^{1/2-a}\, T(t-\sigma)\, A^a g(0) \right\| + \left\| \int_\sigma^t A^{1/2}\, T(t-s)\, F(s,u_s)\, ds \right\| \le$$
$$\le \mu_{1/2-a} (t-\sigma)^{a-1/2}\, e^{\gamma(t-a)} \|g\|_{C_a} + \int_\sigma^t \|A^{1/2}\, T(t-s)\|\, k(s) \Big(1 + \|u_s\|_{C_{1/2}}\Big)\, ds$$

for $\gamma \in (-1,0)$, $\mu_{1/2-a} \in \mathbb{R}^+$, $a \in [0,1/2)$.

By Gronwall's inequality, it follows that $\|u_t\|_{C_{1/2}}$ remains bounded as $t \to T$.

We have, therefore, passed in the setting of Section 2, and Theorem 2.1 ensures that the equation (3.1) with data (3.2), and f satisfying the aforementioned assumptions, prossesses an ω-periodic mild solution.

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