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SOME DISTRIBUTIONAL PRODUCTS WITH RELATIVISTIC INVARIANCE

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Abstract: In [1, 2] we introduced a distribution product, invariant under the action of compact Lie groups of linear transformations. This product has application to non-relativistic physics. Here we present a general version of the product, invariant under various groups, including the Lorentz group.

0 – Introduction

A Lorentz invariant product of distributions based in [1, 2] will imply the existence of a Lorents invariant function $\alpha \in \mathcal{D}(\mathbb{R}^4)$ such that $\int_{\mathbb{R}^4} \alpha = 1$. If such a function α exists, it will be invariant with the linear transformation defined by the matrix.

$$\begin{bmatrix} \operatorname{ch}\theta & 0 & 0 & \operatorname{sh}\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \operatorname{sh}\theta & 0 & 0 & \operatorname{ch}\theta \end{bmatrix} \quad \text{with} \quad \theta \in \mathbb{R} \;,$$

which is the same to say that

$$\alpha(x \operatorname{ch}\theta + t \operatorname{sh}\theta,\, y,\, z,\, x \operatorname{sh}\theta + t \operatorname{ch}\theta) = \alpha(x,y,z,t) \quad \text{for all} \;\; x,y,z,t \in \mathbb{R} \;.$$

If we fix a point (x_0, y_0, z_0, t_0) in the "space-time" we see that α is a constant function on the line

$$(x,y,z,t) = (x_0 \operatorname{ch}\theta + t_0 \operatorname{sh}\theta,\, y_0,\, z_0,\, x_0 \operatorname{sh}\theta + t_0 \operatorname{ch}\theta) \quad \text{with} \quad \theta \in \mathbb{R} \;.$$

This line is not bounded in $\mathbb{R}^4$ because his projection on the xot-plane is the hiperbole $x^2 - t^2 = x_0^2 - t_0^2$. Thus, the only function $\alpha \in \mathcal{D}(\mathbb{R}^4)$ which is Lorentz invariant is the zero-function and we cannot have $\int_{\mathbb{R}^4} \alpha = 1$.

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Hence, it is not possible to have a Lorentz invariant product of distributions applying the theory of [1, 2]. In the following we define a family of distributional products with relativistic invariance based in a generalization of the methods of [1, 2] applying the concepts of order of growth and integral of a distribution introduced by Sebastião e Silva [4, 5].

1 – Preliminaries on limits, integrals and orders of growth of distributions

Let $N \in \mathbb{N} = \{1, 2, 3, \ldots\}$. By C_∞ we mean the set of complex valued distributions of finite order defined on $\mathbb{R}^N$. Recall the following concepts introduced by J. Sebastião e Silva [4, 5].

1.1. Let $f \in C_\infty$ and $\lambda \in \mathbb{C}$. We say that $f(t_1, \ldots, t_N)$ converge to λ as $(t_1, \ldots, t_N) \to (+\infty, \cdots, +\infty)$ and we write $f(+\infty, \ldots, +\infty) = \lambda$ iff there exists a sistem of integers $r_1, \ldots, r_N \geq 0$ and a continous complex valued function $F(t_1, \ldots, t_N)$ defined on $\mathbb{R}^N$ such that

a) $D_1^{r_1} \ldots D_N^{r_n}\, F(t_1, \ldots, t_N) = f(t_1, \ldots, t_N)$ on $\mathbb{R}^N$;

b) $\forall\, \delta > 0\; \exists\, L > 0$: $t_1, \ldots, t_N > L \;\Rightarrow\; \left| \dfrac{F(t_1, \ldots, t_N)}{t_1^{r_1} \cdots t_N^{r_N}} - \dfrac{\lambda}{r_1! \cdots r_N!} \right| < \delta$.

D_k means the usual derivation operator relative to the variable t_k. Note that the concept of convergence when some variables converge to $+\infty$ and others to $-\infty$ is analogous.

1.2. We can define the generalized De Barrow symbol $[T(t_1, \ldots, t_N)]_{(u_1,\ldots,u_N)}^{(v_1,\ldots,v_N)}$ by setting

$$\Big[T(t_1, \ldots, t_N)\Big]_{(u_1)}^{(v_1)} = T(v_1, t_2, \ldots, t_N) - T(u_1, t_2, \ldots, t_N)$$

and

$$\Big[T(t_1, \ldots, t_N)\Big]_{(u_1,\ldots,u_N)}^{(v_1,\ldots,v_N)} = \Big[T(t_1, \ldots, t_{N-1}, v_N) - T(t_1, \ldots, t_{N-1}, u_N)\Big]_{(u_1,\ldots,u_{N-1})}^{(v_1,\ldots,v_{N-1})}.$$

Now, we will say that $f \in C_\infty$ is Silva-integrable on $\mathbb{R}^N$ iff there exists a complex valued distribution $T(t_1, \ldots, t_N)$ such that

a) $D_1 \ldots D_N T(t_1, \ldots, t_N) = f(t_1, \ldots, t_N)$ on $\mathbb{R}^N$;

b) All parcels of the distribution defined on $\mathbb{R}^{2N}$ by $[T(t_1, \ldots, t_N)]_{(u_1,\ldots,u_N)}^{(v_1,\ldots,v_N)}$ converge as $(v_1, ..., v_N) \to (+\infty, ..., +\infty)$ and $(u_1, ..., u_N) \to (-\infty, ..., -\infty)$.

In this case, we will write

$$\begin{aligned}\int_{\mathbb{R}^N} f &= \lim_{\substack{(v_1,\dots,v_N)\to(+\infty,\dots,+\infty)\\(u_1,\dots,u_N)\to(-\infty,\dots,-\infty)}} \Big[T(t_1,\dots,t_N)\Big]_{(u_1,\dots,u_N)}^{(v_1,\dots,v_N)}\\ &= \Big[T(t_1,\dots,t_N)\Big]_{(-\infty,\dots,-\infty)}^{(+\infty,\dots,+\infty)}\,.\end{aligned}$$

For instance, with $N=1$ $\int_{\mathbb{R}} \cos t = 0$ in the sense of Silva, because $\cos t = D\sin t$ and $[\sin t]_{-\infty}^{+\infty} = \sin(+\infty) - \sin(-\infty)$. Meanwhile, $\sin(+\infty) = 0$ on account of $\sin t = D[-\cos t]$ and $\frac{\cos t}{t} \to \frac{0}{1!}$ as $t \to +\infty$. Also $\sin(-\infty) = 0$.

1.3. $f \in C_\infty$ is bounded on $\mathbb{R}^N$ in the sense of Silva iff there exists a sistem of integers $r_1,\dots,r_N \geq 0$ and a complex valued continuous function F defined on $\mathbb{R}^N$ such that

a) $D_1^{r_1}\dots D_N^{r_n} F(t_1,\dots,t_N) = f(t_1,\dots,t_N)$ on $\mathbb{R}^N$;

b) For any invertible linear transformation $A : \mathbb{R}^N \to \mathbb{R}^N$ the function $\frac{F\circ A(t_1,\dots,t_N)}{t_1^{r_1}\dots t_N^{r_n}}$ is bounded in the usual sense on the region $|t_1| > k$, ..., $|t_N| > k$ where k is a positive number.

Let $\alpha \in \mathbb{R}$ and $f \in C_\infty$. We write $f \in \mathrm{O}(\|t\|^\alpha)$ as $\|t\| \to \infty$ in the sense of Silva if there are $f_0 \in C_\infty$ bounded in the sense of Silva and a number $\varepsilon > 0$ such that $f(t) = \|t\|^\alpha f_0(t)$ when $\|t\| > \varepsilon$. Recall that the concept of bounded distribution in the sense of Silva is more general that the same concept in the sense of Schwartz [3].

2 – The family of products

Let G be a group of unimodular transformations $L\colon \mathbb{R}^N \to \mathbb{R}^N$ ($|\det L| = 1$) such that there exists a C^∞-function $\alpha\colon \mathbb{R}^N \to \mathbb{C}$ obeying the conditions:

a) α is G-invariant;

b) $\alpha \in \mathrm{O}(\|t\|^p)$ as $\|t\| \to \infty$ with $p < -N$, in the sense of Silva;

c) $\int \alpha = 1$.

(Unless otherwise specified, all integrals are over $\mathbb{R}^N$ and in the sense of Silva.)

2.1 Definition. Let $T \in \mathcal{D}'$ and $U \in \mathcal{D}'_n$ ($\mathcal{D}'_n$ denotes the space of distributions with nowhere dense support). We say that there exists the product of T by U relative to the pair (G,α) iff

d) For each $x \in \mathcal{D}$ there exists an integer $p < -N$ such that

$$T[\alpha * (Ux)] \in \mathrm{O}(\|t\|^p) \quad \text{as } \ \|t\| \to \infty \ \text{ in the sense of Silva} ,$$

e) The functional $x \to \int T[\alpha * (Ux)]$ is continuous on $\mathcal{D}$ (endowed with the usual topology),

and we call product of T by U relative to the pair (G, α), the distribution $T \underset{\alpha}{\cdot} U$ defined by

$$\left\langle T \underset{\alpha}{\cdot} U, x \right\rangle = \int T[\alpha * (Ux)] \quad \text{for all } \ x \in \mathcal{D} .$$

Note that if $f \in C_\infty \cap \mathrm{O}(\|t\|^p)$ with $p < -N$ then f is Silva integrable (see [4]).

This product verifies the distributive properties and is consistent with the product of $T \in \mathcal{D}'$ by $U \in \mathcal{D}'_n$ defined in [1, 2]. Indeed, if $\alpha \in \mathcal{D}$ then $\langle T \underset{\alpha}{\cdot} U, x\rangle = \langle T, \alpha * (Ux)\rangle = \langle T\zeta^{-1}(U), x\rangle$ which is the (G, α)-result we have obtained in [1, 2].

Observe that the family of products defined here is really more general than that considered in [1, 2] in the sense that it can be applied to a wider class of groups. For instance:

2.2 Proposition. *Let G be the Lorentz group in the 4-dimensional space. Then, there exists a C^∞-function $\alpha\colon \mathbb{R}^4 \to \mathbb{C}$ such that conditions a), b), c) are verified.*

Proof: Let $\alpha(x,y,z,t) = \frac{1}{\pi^2 i}\, e^{i(x^2+y^2+z^2-t^2)}$. It is obvious that α is Lorentz invariant. To see that α verifies conditions b), let us note that $x\alpha = \frac{1}{2i}\, D_x\alpha$ and for $p = 2,3,4,\ldots$, $x^p\,\alpha = \frac{1}{2i}[D_x(x^{p-1}\alpha) - (p-1)x^{p-2}\alpha]$. Thus, for $p = 0,1,2,\ldots$, $x^p\,\alpha = \sum_{j\in\mathbb{N}_0} a_j\, D^j\alpha$ where only a finite number of $a_j \in \mathbb{C}$ are different from zero ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). By a similar process we conclude that for $q = 0,1,2,\ldots$, $y^q\,\alpha = \sum_{\ell\in\mathbb{N}_0} b_\ell\, D_y^\ell\alpha$ and so

$$x^p\, y^q\, z^r\, t^s\, \alpha = \sum_{(j,\ell,m,n)\in\mathbb{N}_0^4} d_{j\ell mn}\, D_x^j D_y^\ell D_z^m D_t^n \alpha\ ,$$

where only a finite number of $d_{j\ell mn} \in \mathbb{C}$ are different from zero.

But, for every invertible linear transformation $A : \mathbb{R}^4 \to \mathbb{R}^4$ the function $\frac{\alpha\circ A(x,y,z,t)}{x^j y^\ell z^m t^n}$ is bounded in the usual sense in some region $|x| > x_0$, $|y| > y_0$, $|z| > z_0$, $|t| > t_0$ with $x_0, y_0, z_0, t_0 > 0$ and we conclude that $x^p\, y^q\, z^r\, t^s\, \alpha$ is bounded on $\mathbb{R}^4$ in the sense of Silva. Then, $(x^2+y^2+z^2+t^2)^3\,\alpha$ is bounded on $\mathbb{R}^4$ in the same sense, which proves that

$$\alpha \in \mathrm{O}\Big(\|(x,y,z,t)\|^{-6}\Big) \quad \text{as } \ \|(x,y,z,t)\| \to \infty \ \text{ in the sense of Silva} .$$

Applying Theorem 14.2 of [4] we have

$$\int_{\mathbb{R}^4} \alpha = \frac{1}{\pi^2 i} \Big(\int_{\mathbb{R}} e^{ix^2}\, dx\Big) \Big(\int_{\mathbb{R}} e^{iy^2}\, dy\Big) \Big(\int_{\mathbb{R}} e^{iz^2}\, dz\Big) \Big(\int_{\mathbb{R}} e^{-it^2}\, dt\Big) = 1$$

because $\int_{\mathbb{R}} \sin(x^2)\, dx = \int_{\mathbb{R}} \cos(x^2)\, dx = 2\sqrt{\frac{\pi}{8}}$ are simply convergent integrals (Fresnel's integrals) and so the corresponding Silva-integrals exist and have the same value. ∎

Meanwhile, the product of this approach is certainly more restricted in what concerns the left-hand side factor.

Applying 1.2 and 1.1 we can prove a proposition we need, which is an example of the power of the concepts introduced by S. Silva in [4, 5].

2.3 Proposition. *If $f \in C_\infty$ is Silva integrable on $\mathbb{R}^N$, then all partial derivatives of f are Silva integrable and the integral of anyone of them is zero.*

Now we can prove the usual derivation rule.

2.4 Proposition. *If $T \in \mathcal{D}'$, $U \in \mathcal{D}'_n$, there exists $T \underset{\alpha}{\cdot} U$ and one of the products $(D_k T) \underset{\alpha}{\cdot} U$ or $T \underset{\alpha}{\cdot} (D_k U)$ then there exists the other product and we have*

$$D_k(T \underset{\alpha}{\cdot} U) = (D_k T) \underset{\alpha}{\cdot} U + T \underset{\alpha}{\cdot} (D_k U)\,, \qquad k = 1, \ldots, N\ .$$

Proof: Let $x \in \mathcal{D}$. It is easy to prove that

$$D_k[T(\alpha * Ux)] - (D_k T)(\alpha * Ux) - T[\alpha * (D_k U)x] = T[\alpha * U(D_k x)]\ .$$

By assumption we have

$$\int D_k[T(\alpha * Ux)] - \int (D_k T)\,(\alpha * Ux) - \int T[\alpha * (D_k U)x] = \int T[\alpha * U(D_k x)]\ ,$$

the first term is zero by Proposition 2.3 and we can write

$$-\Big\langle (D_k T) \underset{\alpha}{\cdot} U, x\Big\rangle - \Big\langle T \underset{\alpha}{\cdot} (D_k U), x\Big\rangle = \Big\langle T \underset{\alpha}{\cdot} U, D_k x\Big\rangle = -\Big\langle D_k(T \underset{\alpha}{\cdot} U), x\Big\rangle$$

which proves the proposition. ∎

This product is also invariant with translations and all transformations of G.

2.5 Proposition. *If $T \in \mathcal{D}'$, $U \in \mathcal{D}'_n$, $L \in G$, $a \in \mathbb{R}^N$ and the product $T \underset{\alpha}{\cdot} U$ exists, then:*

a) $\tau_a(T \underset{\alpha}{\cdot} U) = (\tau_a T) \underset{\alpha}{\cdot} (\tau_a U)$;

b) $(T \underset{\alpha}{\cdot} U) \circ L = (T \circ L) \underset{\alpha}{\cdot} (U \circ L)$.

Proof: Let $x \in \mathcal{D}$.

a) $$\Big\langle \tau_a(T \underset{\alpha}{\cdot} U), x \Big\rangle = \Big\langle T \underset{\alpha}{\cdot} U, \tau_{-a}\, x \Big\rangle = \int T[\alpha * U(\tau_{-a}\, x)] = \int T\, \tau_{-a}[\alpha * (\tau_a\, U) x]$$

and applying theorem 13.8 of [4] we can write

$$\Big\langle \tau_a(T \underset{\alpha}{\cdot} U), x \Big\rangle = \int (\tau_a\, T)\, [\alpha * (\tau_a U) x] = \Big\langle (\tau_a T) \underset{\alpha}{\cdot} (\tau_a U), x \Big\rangle$$

and the invariance of the product with translations is proved.

b) $$\Big\langle T \underset{\alpha}{\cdot} U \circ L, x \Big\rangle = \Big\langle T \underset{\alpha}{\cdot} U, x \circ L^{-1} \Big\rangle = \int T[\alpha * U(x \circ L^{-1})]$$
$$= \int T\Big[\alpha * (((U \circ L)x) \circ L^{-1})\Big]$$

and again by theorem 13.8 of [4] we can write

$$\Big\langle T \underset{\alpha}{\cdot} U \circ L, x \Big\rangle = \int (T \circ L) \Big[(\alpha * (((U \circ L)x) \circ L^{-1})) \circ L\Big] \ .$$

Noting that, if V is a distribution with compact support, $\alpha \in C^\infty$ and $L\colon \mathbb{R}^N \to \mathbb{R}^N$ is unimodular, then $(\alpha * V) \circ L = (\alpha \circ L) * (V \circ L)$, we have

$$\Big\langle T \underset{\alpha}{\cdot} U \circ L, x \Big\rangle = \int (T \circ L) \Big[(\alpha \circ L) * ((U \circ L)x)\Big] = \int (T \circ L) \Big[\alpha * ((U \circ L)x)\Big]$$
$$= \Big\langle (T \circ L) \underset{\alpha}{\cdot} (U \circ L), x \Big\rangle \ ,$$

which proves that the product is G-invariant. ∎

3 – Examples and comments

3.1. Notice that if T is a distribution with compact support and $U \in \mathcal{D}'_n$ then there exists always the product $T \underset{\alpha}{\cdot} U$ and we dont need the Silva integral to compute it because, in this case we have

$$\Big\langle T \underset{\alpha}{\cdot} U, x \Big\rangle = \int T(\alpha * Ux) = \Big\langle T(\alpha * Ux), 1 \Big\rangle = \Big\langle T, \alpha * Ux \Big\rangle \ .$$

For instance, let G be the Lorentz group in $\mathbb{R}^4$ and let α a C^∞-function obeying the conditions 2a), b), c). The product of two Dirac delta distributions is easily obtained

$$\Big\langle \delta \underset{\alpha}{\cdot} \delta, x \Big\rangle = \langle \delta, \alpha * \delta x \rangle = \langle \delta, x(0) \cdot \alpha \rangle = x(0)\, \alpha(0) = \langle \alpha(0) \delta, x \rangle \ .$$

Thus, we can write $\delta \underset{\alpha}{\cdot} \delta = \alpha(0)\,\delta$. Formally this is the same result we have obtained in [1] 1.5.6 relative to a group of a more restricted class which does not include the Lorentz group.

3.2. Let G be the Lorentz group in $\mathbb{R}^4$, α a C^∞-function obeying the conditions 2a), b), c) and H the Heaviside function defined on $\mathbb{R}^4$. We can compute $H \underset{\alpha}{\cdot} \delta$ where δ is the Dirac distribution defined on $\mathbb{R}^4$ noting that

a) $H(\alpha * \delta\phi) = \phi(0)\,H\alpha$ for all $\phi \in \mathcal{D}(\mathbb{R}^4)$ (now we reserve the x letter to note the first coordinate of the generic point (x,y,z,t) of the "space-time") and $H\alpha \in \mathrm{O}(\|(x,y,z,t)\|^p)$ in the sense of Silva when $\|(x,y,z,t)\| \to \infty$ with $p < -4$ on account of condition 2b) for α.

b) The functional $\phi \to \int_{\mathbb{R}^4} \phi(0)\,H\alpha$ is continuous on $\mathcal{D}(\mathbb{R}^4)$ because $H\alpha$ is Silva-integrable on $\mathbb{R}^4$ and so this functional is equal to the distribution $(\int_{\mathbb{R}^4} H\alpha)\,\delta$.

Now, we will prove that $\int_{\mathbb{R}^4} H\alpha = \frac{1}{16}$ in the sense of Silva. Let $R\colon \mathbb{R}^4 \to \mathbb{R}^4$ be any one of the 16 transformations $(x,y,z,t) \to (\pm x, \pm y \pm z, \pm t)$ and $P(x,y,z,t) = x^2+y^2+z^2-t^2$. We have $P \circ L = P$ for all Lorentz transformation L and so $P \circ L \circ R = P \circ R = P$ which proves that $L \circ R$ is also a Lorentz transformation and condition 2a) yields $\alpha \circ L \circ R = \alpha$. Then $\alpha \circ R = \alpha$ which proves that α is R-invariant. Putting $A = \{-1, 1\}$ we can write α as sum of 16 terms

$$\alpha(x,y,z,t) = \sum_{(i,j,k,\ell)\in A^4} \alpha(x,y,z,t)\,H(ix,jy,kz,\ell t)\ ,$$

thus

$$1 = \int_{\mathbb{R}^4} \alpha = \sum_{(i,j,k,\ell)\in A^4} \int_{\mathbb{R}^4} \alpha(x,y,z,t)\,H(ix,jy,kz,\ell t)\,dx\,dy\,dz\,dt$$

and it is easy to see that all terms of this sum are identical: for instance, applying the change of variable $x=-s$, $y=u$, $z=v$, $t=w$ in the Silva integral we have

$$\int_{\mathbb{R}^4} \alpha(x,y,z,t)\,H(-x,y,z,t)\,dx\,dy\,dz\,dt = \\ = \int_{\mathbb{R}^4} \alpha(s,u,v,w)\,H(s,u,v,w)\,ds\,du\,dv\,dw\ .$$

Then $\int_{\mathbb{R}^4} H\alpha = \frac{1}{16}$ and it is proved that $H \underset{\alpha}{\cdot} \delta = \frac{1}{16}\,\delta$ is independent of the α function.

3.3. It is possible to extend this product (and also the product defined in [1, 2]) to a larger class of situations in a simple way. Let p be an integer ≥ 0 or $p = \infty$ and $\mathcal{D}'^p$ the space of distributions of order $\leq p$ in the sense of Schwartz

[3]. If $T \in \mathcal{D}'^p$, $S = \beta + U \in C^p \oplus \mathcal{D}'_n$ and there exists $T \underset{\alpha}{\cdot} U$ in the sense of 2.1 we can always define $T \underset{\alpha}{\cdot} S$ putting $T \underset{\alpha}{\cdot} S = T\beta + T \underset{\alpha}{\cdot} U$ where $T\beta$ is the product in the sense of Schwartz [3].

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REFERENCES

[1] Sarrico, C.O.R. – About a family of distributional products important in the applications, *Portugaliae Mathematica*, 45(3) (1988), 295–316.
[2] Sarrico, C.O.R. – A note on a family of distributional products important in the applications, *Note di Matematica*, VII (1987), 151–158.
[3] Schwartz, L. – *Théorie des distributions*, Hermann, Paris, 1966.
[4] Silva, J. Sebastião – *Integrals and orders of growth of distributions*, reprinted from "Theory of Distribution", Proceedings of an International Summer Institute held in Lisbon, September 1964.
[5] Silva, J. Sebastião – Novos elementos para a teoria do integral no campo das distribuições, *Boletim da Academia das Ciências de Lisboa*, XXXV (1963), 175–184.

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