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ON CERTAIN DIAMETERS OF BOUNDED SETS

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Abstract: In this paper we prove that if the balanced convex closed subset A of the normed linear space X, has a certain property called the property P_0, then the Gelfand n-width $d^n(A,X)$ is attained. If A is a balanced and compact subset of X then the Bernstein n-width $b^n(A,X)$ is attained, and if A is a subset of the dual space X^*, and A contains a ball $B(0,r)$ of positive radius, then the linear n-width $\delta_n(A,X^*)$ is attained. It is also shown that if X has a certain property called the property P_1 then the compact width $a(A,X)$ is attained.

1 – Introduction

If A is a subset of the normed linear space X, then for each $x \in X$, the distance of x from A, $d(x,A)$ is defined to be

$$d(x,A) = \inf\Big\{\|x-y\|;\ \ y \in A\Big\}\ .$$

If B is another subset of X, then the deviation of A from B, $\delta(A,B)$ is defined to be

$$\delta(A,B) = \sup\Big\{d(x,B);\ \ x \in A\Big\}\ .$$

If $n \geq 0$ is a non-negative integer, then the Kolmogorov n-width of the set A in X, $d_n(A,X)$ id defined to be

$$d_n(A,X) = \inf\Big\{\delta(A,N);\ \ N \text{ is an } n\text{-dimensional subspace of } X\Big\}\ .$$

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The Kolmogorov n-width is an important diameter for sets, it has many applications in approxiamtion theory (see for example Brown [3], Deutsch, Mach and Saatkamp [4], Kamal [8] and [9]). Beside the Kolmogorov n-width there are other diameters of sets, which have their applications in approximation theory (see for example Brown [3], Micchelli and Pinkus [14] and [15]). The Gelfand n-width is defined to be

$$d^n(A,X) = \inf\Big\{\sup\{\|x\|; x\in A\cap L_n\};\ L_n \text{ is an } n\text{-codimensional subspace of } X\Big\}.$$

If $S(X)$ is the unit sphere of the space X, then the Bernstein n-width is defined to be

$$\begin{aligned} b_n(A,X) = \sup\Big\{&\sup\{\lambda;\ \lambda S(X_{n+1}) \subseteq A\};\\ &X_{n+1} \text{ is an } (n+1) \text{ dimensional subspace of } X\Big\}\ . \end{aligned}$$

If $F_n(X,X)$ is the set of all bounded linear operators on X of rank $\leq n$, then the linear n-width is defined to be

$$\delta_n(A,X) = \inf\Big\{\sup\{\|x - T(x)\|;\ x \in A\};\ T \in F_n(X,X)\Big\}\ .$$

The properties of the n-widths, and the relations among them were studied by several authors, for example, Singer [17], Pinkus [16] and Garkavi [6]. The n-width $d_n(A,X)$ (resp. $d^n(A,X)$) is said to be attained, if there is an n-dimensional (resp. n-codimensional) subspace Y of X satisfying that, $d_n(A,X) = \delta(A,Y)$ (resp. $d^n(A,X) = \sup\{\|x\|;\ x \in A\cap Y\}$). In this case the subspace Y is said to be an optimal subspace for $d_n(A,X)$ (resp. $d^n(A,X)$). The n-width $b_n(A,X)$ is attained if there is an $(n+1)$ dimensional subspace Y of X, satisfying $b_n(A,X) = \sup\{\lambda; \lambda S(X) \subseteq A\}$. Also in this case the subspace Y is an optimal subspace for $b_n(A,X)$. The width $\delta_n(A,X)$ is attained if there is a bounded linear operator $T\colon\ X \to X$ of rank $\leq n$, such that $\delta_n(A,X) = \sup\{\|x - T(x)\|;\ x \in A\}$.

Another important diameter for sets is the compact width. If A is a bounded subset of the normed linear space X, then the compact width of A, $a(A,X)$ is defined to be

$$a(A,X) = \inf\Big\{\delta(A,K);\ K \text{ is a compact subset of } X\Big\}\ .$$

This width is attained if there is a compact set K in X, such that $a(A,X) = \delta(A,K)$. For some applications of the compact width one may refer to Feder [5].

In most of the applications, the authors were concerned with certain known sets. They study the relation among the diameters, and the optimization of each of the diameters. (See for example Brown [3], Pinkus [16], Micchelli and Pinkus [14] and [15].) The existence of optimal subspaces of sets depends on the sets itself and the space X. In some spaces X, a certain width is attained for all bounded sets, and other specific widths are attained for sets with certain properties regardless of the space X. Garkavi [6] showed that if X^* is the dual space of X, then the Kolmogorov width $d_n(A, X^*)$ is attained for any bounded set A in X^*. Pinkus [16] showed that if A is a bounded convex closed subset of X, and A contains a ball of positive radius centered at zero, then the Gelfand width $d^n(A, X)$ is attained. The question that one may ask is: are there other results of this type?

In this paper the author studies this problem. In section two it is shown that if A is a balanced convex closed subset of the normed linear space X, and A has a certain property called P_0, then $d^n(A, X)$ is attained. It is shown that if the balanced convex closed set A contains a ball of positive radius centered at zero, then it has the property P_0, and thus the result of this section includes the result of Pinkus [16]. Other examples of sets having the property P_0, and examples of sets that do not have the property P_0 are given. In section three it is shown that if A is a balanced compact subset of the normed linear space X, then $b_n(A, X)$ is attained. Also in section three it is shown that if the non-empty subset A, of the dual space X^* contains a ball of positive radius centered at zero, then $\delta_n(A, X^*)$ is attained. In section four it is shown that if the Banach space X has a certain property called P_1, then for each bounded set A in X, the compact width, $a(A, X)$ is attained. The property P_1 has many applications in approximation theory. It is known that if X is uniformly convex or if $X = C(Q)$, then X has this property. In section four, examples of spaces having the property P_1, and other examples of spaces that do not have the property P_1 are given. It is also shown that there are spaces X for which $a(A, X)$ is attained for each bounded set A, but X does not have the property P_1.

The rest of this introduction will cover some definitions and known results. If Q is a compact Hausdorff space, and X is a normed linear space, then $B(Q, X)$ is the space of all bounded functions from Q to X, and $C(Q, X)$ is the space of all continuous functions from Q to X. The norm defined on both $B(Q, X)$ and $C(Q, X)$ is the uniform norm; that is

$$\|f\| = \sup\Big\{\|f(x)\|;\ x \in Q\Big\}\ .$$

If $X = \mathbb{R}$ (the space of real numbers), then $B(Q, \mathbb{R})$ and $C(Q, \mathbb{R})$ are denoted by $B(Q)$ and $C(Q)$. The set A is said to be balanced if it is centrally symmetric. If A is balanced, convex and closed, then the boundary of A, $bd(A) = \{x \in A;\ \lambda x \notin A$ whenever $|\lambda| > 1\}$. If A is a subset of X, then $[A]$ denotes the subspace of X generated by A. If A is balanced and convex then $x \in [A]$ iff for each $\epsilon > 0$ there is $z \in A$ and a real number $\lambda > 0$, such that $\|x - \lambda z\| < \epsilon$. If Y is a closed subspace of the normed linear space X, then $Y^\perp$ denotes the supspace of X^*

consisting of all $f \in X^*$ satisfying $f(x) = 0$ for each $x \in Y$. As in most texts, $B(x, r)$ is the ball of radius r centered at x, and $S(x, r)$ is the sphere of radius r centered at x.

The proof of the following proposition can be found in Singer [17, page 22].

Proposition 1.1. *Let X be a normed linear space, Y be a closed subspace of X, and assume that x_0 is an element in X. Then*

$$d(x_0, Y) = \sup\Big\{|f(x_0)|;\ \|f\| \le 1 \ \text{and}\ f \in Y^\perp\Big\}\ .$$

2 – Optimal subspaces for the Gelfand width

Definition 2.1. Let X be a normed linear space, A a closed balanced convex subset of X. The set A is said to have the property P_0 if for each $x_0 \in bd(A)$, there is $\epsilon > 0$ and $0 < \delta < 1$, such that for each $x \in B(x_0, \epsilon) \cap S(0, \|x_0\|)$, if $\beta x \in bd(A)$ then $|\beta| \ge \delta$.

The property P_0 is related to the smoothness of the boundary of A. If $x_0 \in bd(A)$, then for all the points x within a certain distance from x_0 on the sphere $S(0, \|x_0\|)$, if the line $[x]$ intersects the boundary of A, then one of the points of intersection should be within a given distance from x_0. If $\dim[A] < \infty$ then A has the property P_0, but if $[A]$ is of infinite dimension, then the boundary of A should satisfy a certain weak form of smoothness in order to have the property P_0. For example if A contains a ball $B(0, r)$ for some $r > 0$ in $[A]$, then A has the property P_0. Indeed in this case one may choose $\epsilon > 0$ to be any positive number, and $\delta < 1$ satisfying that $\delta > 1 - \frac{r}{\|x_0\|}$. Not all balanced convex closed sets have the property P_0, the following example illustrates this.

Example 2.2. In the classical Banach spaces of sequences l_1, let

$$x_0 = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right)$$

and for each positive integer $n \ge 1$, let

$$x_n = \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}, \ldots, \frac{1}{2^{2n}}, 0, 0, \ldots\right).$$

Let A be the closed balanced convex hull of $\{x_0, x_1, x_2, \ldots\}$, then for each $n = 0, 1, 2, \ldots$, $x_n \in bd(A)$. Also $\lim 2^n x_n = x_0$. For each $n = 1, 2, \ldots$, let $\delta_n = \frac{2}{\|2^n x_n\|}$ then $\lim \delta_n = 1$ and $\|2^n \lambda_n x_n\| = \|x_0\|$. Let $z_n = 2^n \lambda_n x_n$, then $\lim z_n = x_0$.

Therefore for each $\epsilon > 0$ and $0 < \delta < 1$, there is a positive integer $M \geq 1$ for which $z_n \in B(x_0, \epsilon) \cap S(0, \|x_0\|)$ for each $n \geq M$. But since $x_n = \frac{1}{2^n \lambda_n} z_n$ is a point of intersection of the line $[z_n]$ with $bd(A)$, it follows that $\beta_n = \frac{1}{2^n \lambda_n} < \delta$ for all n that are large enough.

In Theorem 2.6 it will be shown that if A is a closed balanced convex subset of the normed linear space X, satisfying the property P_0, then for each non-negative integer $n \geq 0$, the Gelfand width $d^n(A, X)$ is attained. This result includes the result of Pinkus [16].

The following proposition is due to Helfrich [7]. Its proof can be found also in Pinkus [16].

Proposition 2.3. *Let X be a normed linear space, and let Y be a subspace of X. If A is a subset of Y, then for any non-negative integer $n \geq 0$, $d^n(A, X) = d^n(A, Y)$.*

Let X be any normed linear space, and let A be a closed balanced convex subset of X. By Proposition 2.3 one may assume that $X = [A]$. Let $n \geq 0$ be any non-negative integer, if $n = 0$ then $Y_0 = X$ is the zero-codimensional subspace of X optimal for $d^n(A, X)$. Thus one may assume that $n \geq 1$. Let $a = d^n(A, X)$. If $a = \infty$ then any n-codimensional subspace Y_0 of X is optimal subspace for $d^n(A, X)$, and if $a = 0$ then $\dim[A] = n$; that is, $\dim X = n$, so $Y_0 = \{0\}$ is an optimal subspace for $d^n(A, X)$. Therefore one may assume that $0 < a < \infty$. Define

$$S = \left\{ x \in A;\ \|x\| \leq a, \text{ and if } \alpha x \in A \text{ then } \|\alpha x\| \leq a \right\},$$

and for each $i = 1, 2, \ldots$ define

$$S_i = \left\{ x \in A;\ \|x\| \leq a + \frac{1}{i}, \text{ and if } \alpha x \in A \text{ then } \|\alpha x\| \leq a + \frac{1}{i} \right\}.$$

The following lemma includes some of the properties of S and S_i, its proof follows from their definitions.

Lemma 2.4. *Let X be a normed linear space, $n \geq 1$ be a positive integer, and let A be a balanced convex closed subset of X, satisfying $0 \leq d^n(A, X) < \infty$. If a, S, and S_i are as in the preceding argument, then the following statements hold.*

a) *For each $i = 1, 2, \ldots$, $S \subseteq S_{i+1} \subseteq S_i$.*

b) *If $x_i \in S_i$ for each $i = 1, 2, \ldots$, and $\lim x_i = x_0$ then $x_0 \in S$.*

c) *For each $x \in bd(A)$, if $\|x\| \leq a$ then $x \in S$, and if $\|x\| \leq a + \frac{1}{i}$ then $x \in S_i$.*

Lemma 2.5. *In Lemma 2.4, assume that $X = [A]$, and for each $i = 1, 2, \ldots,$ let Y_i be an n-codimensional subspace of X satisfying* $\sup\{\|x\|;\, x \in Y_i \cap A\} \leq a + \frac{l}{i}$, *then*

a) $Y_i \cap A \subseteq S_i$.

b) *For each $x \in Y_m$, there is a sequence $\{x_i\}$ in S_m, and a sequence of real numbers $\{\alpha_i\}$, such that $\lim \alpha_i x_i = x$.*

Proof: **a)** Let $x \in Y_i \cap A$ and assume that $x \notin S_i$, then either $\|x\| > a + \frac{l}{i}$, or there is a real number α, such that $\alpha x \in A$ and $\|\alpha x\| > a + \frac{1}{i}$. In both cases one has $\sup\{\|x\|;\, x \in Y_i \cap A\} > a + \frac{1}{i}$.

b) Let $x_0 \in Y_m$. If $x_0 \in [Y_m \cap A]$ then there is nothing to prove, so one may assume that $x_0 \notin [Y_0 \cap A]$. Since $X = [A]$, and A is balanced and convex, it follows that there is a sequence $\{x_i\}$ in A, and a sequence of non-negative numbers $\{\alpha_i\}$, such that $\lim \alpha_i x_i = x_0$. Also since A is balanced and closed, one can choose each $x_i \in bd(A)$. If $\{\alpha_i\}$ has a subsequence $\{\alpha_j\}$ for each $\lim \alpha_j = \alpha_0 < \infty$, then if $\alpha_0 = 0$ it follows that $0 = x_0 \in [Y_m \cap A]$, and if $\alpha_0 \neq 0$ then $\lim x_j = \frac{x_0}{\alpha_0} \in A$. Therefore $x_0 \in [Y_m \cap A]$, so one may assume that $\lim \alpha_i = \infty$. But in this case $\lim \|x_i\| = 0$. Thus there is a positive integer $N \geq 1$, such that for each $i > N$, $\lim \|x_i\| \leq a + \frac{1}{m}$. Applying Lemma 2.4.c, one can show that the sequence $\{x_{i+N}\}$ is in S_m. ∎

2.6 Theorem. *Let A be a closed balanced convex subset of the normed linear space X, and let $n \geq 0$ be a non-negative integer. If A has the property P_0 then $d^n(A, X)$ is attained.*

Proof: Let $a = d^n(A, X)$. By the argument preceding Lemma 2.4, one may assume that $0 < \alpha < \infty$, $n \geq 1$, and $X = [A]$. Let S, S_i, and Y_i be as in Lemma 2.5. Since for each $i = 1, 2, \ldots, \operatorname{cod} Y_i = n$, there are $\{f_1^i, f_2^i, \ldots, f_n^i\}$ in X^*, satisfying $\|f_k^i\| = 1$ for each $k = 1, 2, \ldots, n$ and such that

$$Y_i = \left\{x \in X;\ f_1^i(x) = f_2^i(x) = \ldots = f_n^i(x) = 0\right\}.$$

The sequence $\{(f_1^i, f_2^i, \ldots, f_n^i)\}_{i=1}^{\infty}$ is bounded in $\prod_{i=1}^{n} X^*$, so there is an element $(f_1^0, f_2^0, \ldots, f_n^0) \in \prod_{i=1}^{n} X^*$, and a subsequence $\{(f_1^j, f_2^j, \ldots, f_n^j)\}_{j=1}^{\infty}$ of $\{(f_1^i, f_2^i, \ldots, f_n^i)\}$, such that for each $k = 1, 2, \ldots, n$, f_k^0 is the w^*-limit of the sequence $\{f_k^j\}_{j=1}^{\infty}$ in X^*. Without loss of generality, one may assume that for each $x \in X$, the sequence $\{(f_1^i(x), f_2^i(x), \ldots, f_n^i(x))\}_{i=1}^{\infty}$ converges to $(f_1^0(x), f_2^0(x), \ldots, f_n^0(x))$. Let $Y_0 = \{x \in X;\ f_1^0(x) = f_2^0(x) = \ldots = f_n^0(x) = 0\}$, then $\operatorname{cod} Y_0 \leq n$. It will be shown that $Y_0 \cap A \subseteq S$. If this is true then $\sup\{\|x\|;\, x \in Y_0 \cap A\} \leq a$, therefore any subspace Y of Y_0 which is of codimension n in X, is an optimal subspace for $d^n(A, X)$. Let

$x_0 \in Y_0 \cap A$, and assume that $x_0 \in bd(A)$. Since $\{(f_1^i(x_0), f_2^i(x_0), ..., f_n^i(x_0))\}_{i=1}^{\infty}$ converges to $(0, 0, ..., 0)$, it follows by Proposition 1.1 that $\lim[d(x_0, Y_I)] = 0$. Thus, for each $i = 1, 2, ...,$ there is $y_i \in Y_i$ such that the sequence $\{y_i\}$ converges to x_0. Therefore by Lemma 2.5, one can easily show that for each $i = 1, 2, ...,$ there is $x_i \in bd(A) \cap S_i$, and a non-negative number α_i such that $\lim \alpha_i x_i = x_0$. If $x_0 = 0$ then $x_0 \in S$, so one may assume that $x_0 \neq 0$. Let $\epsilon > 0$ and $\delta > 0$ be the two positive numbers corresponding to x_0 in the definition of Property P_0 of A, and for each $i = 1, 2, ...,$ let

$$z_i = \frac{\alpha_i \|x_0\|}{\|\alpha_i x_i\|} \cdot x_i \ ,$$

then $\lim z_i = x_0$ and for each $i = 1, 2, ..., z_i \in B(0, \|x_0\|)$. Thus there is a positive integer $M \geq 1$, such that for each $i \geq M$ one has $z_i \in B(x_0, \epsilon) \cap S(0, \|x_0\|)$. On the other hand,

$$x_i = \frac{\|\alpha_i x_i\|}{\alpha_i \|x_0\|} z_i \ \in \ bd(A) \ ,$$

hence by the definition of Property P_0,

$$\frac{\|\alpha_i x_i\|}{\alpha_i \|x_0\|} \geq 1 - \delta \qquad \text{for each} \quad i \geq M \ .$$

Thus

$$\alpha_I \leq \frac{\|\alpha_i x_i\|}{(1-\delta) \|x_0\|} \qquad \text{for each} \quad i \geq M \ .$$

But $\lim \frac{\|\alpha_i x_i\|}{\alpha_i \|x_0\|} = 1$, so the sequence $\{x_i\}$ is bounded in $\mathbb{R}$ (the set of real numbers). Let $\{\alpha_j\}$ be a convergent subsequence of $\{\alpha_i\}$, and assume that $\lim \alpha_j = \alpha_0$. Then $0 < \alpha_0 < \infty$, and therefore $\lim x_j = \frac{x_0}{\alpha_0}$. Thus by Lemma 2.4.b, $\frac{x_0}{\alpha_0} \in S$, and since $x_0 = \alpha_0(\frac{x_0}{\alpha_0}) \in A$, it follows that $x_0 \in S$. ∎

3 – Optimal subspaces for other n-widths

In this section it will be shown that if A is a balanced compact subset of the normed linear space X, then for each non-negative integer $n \geq 0$, the Bernstein n-width $b_n(A, X)$ is attained. It will be shown also that if A is a non-empty subset of the dual space X^*, and A contains a ball $B(0, r)$ for some $r > 0$, then for each non-negative integer $n \geq 0$, the linear n-width $\delta_n(A, X^*)$ is attained.

The linearly independent set $\{x_1, x_2, ..., x_n\}$ of the normed linear space X is said to be an Auerbach set, if $\|x_k\| = 1$ for each $k = 1, 2, ..., n$, and there are linear functionals $\{f_1, f_2, ..., f_n\}$ in the dual space X^* of X such that for each

$k = 1, 2, ..., n$, $\|f_k\| = 1$, and $f_k(x_i) = \delta_{ki}$; that is, $f_k(x_k) = 1$ and $f_k(x_i) = 0$ for $i \neq k$. By Proposition 1.1, if $\{x_1, x_2, ..., x_n\}$ is an Auerbach set in X, and for any $k = 1, 2, ..., n$, Y_k is the subspace of X generated by $\{x_1, x_2, ..., x_n\} \backslash \{x_k\}$, then $d(x_k, Y_k) = 1$. Therefore if $\{(x_1^i, x_2^i, ..., x_n^i)\}_{i=1}^{\infty}$ is a sequence in $\prod_{i=1}^{n} X$ such that for each $i = 1, 2, ...,$ the set $\{x_1^i, x_2^i, ..., x_n^i\}$ is an Auerbach set, and for each $i = 1, 2, ..., n$, the sequence $\{x_k^i\}$ converges to x_k^0 in X, then the set $\{x_1^0, x_2^0, ..., x_n^0\}$ is a linearly independent set in X. If the Auerbach set $\{x_1, x_2, ..., x_n\}$ is a basis for the n-dimensional subspace Y of X, then it is called an Auerbach basis for Y. In this case if $x = a_1x_1 + a_2x_2 + ... + a_nx_n$, then for each $k = 1, 2, ..., n$, $|a_k| \leq \|x\|$. Indeed if $f_k \in X^*$ satisfies $\|f_k\| = 1$ and $f_k(x_i) = \delta_{ki}$, then $\|x\| \geq \|f_k(x)\| = |a_k|$.

Proposition 3.1 shows that each n-dimensional normed linear space X has an Auerbach basis. The proof of this proposition can be found in Lindenstrauss and Tzafriri [11, page 16].

3.1 Proposition. *If X is an n-dimensional normed linear space, then X has an Auerbach basis.*

3.2 Theorem. *Let A be a non-empty balanced subset of the normed linear space X, and let $n \geq 0$ be a non-negative integer. If A is compact then the Bernstein n-width, $b_n(A, X)$ is attained.*

Proof: If $n = 0$ or $b_n(A, X) = 0$ then the proof is obvious, so one may assume that $n \geq 0$ and $b_n(A, X) > 0$. Let $\alpha = b_n(A, X)$, and let $\{\alpha_i\}$ be a strictly increasing sequence of positive numbers converging to α. For each $i = 1, 2, ...,$ let Y_i be an $(n + 1)$ dimensional subspace of X satisfying $\alpha_i\, S(Y_i) \subseteq A$, where $S(Y_i)$ is the unit sphere of Y_i, and let $\{y_1^i, y_2^i, ..., y_{n+1}^i\}$ be an Auerbach basis for Y_i. Furthermore for each $k = 1, 2, ..., n + 1$, let $x_k^i = \alpha_i\, y_k^i$, then $\{x_1^i, x_2^i, x_{n+1}^i\} \subseteq A$. The set A is compact, so there is a point

$$(x_1^0, x_2^0, ..., x_{n+1}^0) \in \prod_{i=1}^{n+1} A ,$$

and a subsequence $\{(x_1^j, x_2^j, ..., x_{n+1}^j)\}_{j=1}^{\infty}$ of $\{(x_1^i, x_2^i, ..., x_{n+1}^i)\}_{i=1}^{\infty}$, such that for each $k = 1, 2, ..., n$, $\lim x_k^j = x_k^0$. Let $y_k^0 = \frac{1}{\alpha}\, x_k^0$, then $\|y_k^0\| = 1$, and $\lim_{j \to \infty} y_k^j = y_k^0$. By the argument preceding Proposition 3.1, the set $\{y_1^0, y_2^0, ..., y_{n+1}^0\}$ is linearly independent. Let Y_0 be the $(n + 1)$ dimensional subspace of X generated by $\{y_1^0, y_2^0, ..., y_{n+1}^0\}$. It will be shown that $\alpha S(Y_0) \subseteq A$. That is if $x \in S(Y_0)$ then $\alpha x \in A$. If this is true then Y_0 is an optimal subspace for $b_n(A, X)$. Let $x = a_1y_1^0, a_2y_2^0, ..., a_{n+1}, y_{n+1}^0$ be in $S(Y_0)$, and let $\epsilon > 0$ be given. Since A is compact, it is enough to show that $d(\alpha x, A) < \epsilon$. For each $k = 1, 2, ..., n + 1$, the sequence $\{y_k^j\}$ converges to y_k^0. Therefore if $x_j = \alpha_1 y_1^j + \alpha_2 y_2^j + ... + \alpha_{n+1} y_{n+1}^j$, then the sequence $\{x_j\}$ converges to x. Letting $\epsilon' = \frac{\epsilon}{2+\alpha}$, there is a positive integer $m \geq 1$ such that for each $j \geq M$ one has $\|x - x_j\| < \epsilon'$. That is, for each $j \geq M$,

$$\|x_j\| \leq \|x\| + \|x - x_j\| < 1 + \epsilon' \ .$$

Using the fact that A is balanced and that $\alpha S(Y_j) \subseteq A$, it follows that for each $j \geq M$ one has $\frac{\alpha_j}{1+\epsilon'} x_j \in A$. Therefore for each $j \geq M$

$$d(\alpha x, A) \leq \left\| \alpha x - \frac{\alpha_j}{1+\epsilon'}\, x_j \right\| \leq \frac{1}{1+\epsilon'}\, \|\alpha x - \alpha_j x_j\| + \frac{\alpha\epsilon'}{1+\epsilon'}\, \|x\| \ .$$

But $\lim \alpha_j = \alpha$, so one can choose $j_0 \geq M$ such that $\|\alpha x - \alpha_{j_0} x_{j_0}\| < 2\epsilon'$. Therefore

$$\begin{aligned} d(\alpha x, A) &\leq \frac{1}{1+\epsilon'}\, \|\alpha x - \alpha_{j_0} x_{j_0}\| + \frac{\alpha\epsilon'}{1+\epsilon'}\, \|x\| \\ &< \frac{2\epsilon'}{1+\epsilon'} + \frac{\alpha\epsilon'}{1+\epsilon'} < 2\epsilon' + \alpha\epsilon' < \epsilon \ . \blacksquare \end{aligned}$$

3.3 Theorem. *Let A be a non-empty subset of the dual space X^* of the normed linear space X, and let $n \geq 0$ be a non-negative integer. If there is $\lambda > 0$ such that $B(0, \lambda) \subseteq A$, then the linear n-width $\delta_n(A, X)$ is attained.*

Proof: If $n = 0$ then the proof is obvious, thus one may assume that $n \geq 1$. Let $a = \delta_n(A, X)$, and for each $i = 1, 2, ...$, let $F_i \colon X^* \to X^*$ be a bounded linear operator of rank $= n$ satisfying $\sup\{\|x - F_i(x)\|;\ x \in A\} \leq a + \frac{1}{i}$. For each $x \in B(0,1)$, $\lambda x \in B(0, \lambda) \subseteq A$. Therefore,

$$\begin{aligned} \|F_i\| &\leq \frac{1}{\lambda} \sup\Big\{\|F_i(x)\|;\ \|x\| \leq \lambda\Big\} \\ &\leq \frac{1}{\lambda} \sup\Big\{\|x - F_i(x)\|;\ \|x\| \leq \lambda\Big\} + \frac{1}{\lambda} \sup\Big\{\|x\|;\ \|x\| \leq \lambda\Big\} \\ &\leq \frac{1}{\lambda} \sup\Big\{\|x - F_i(x)\|;\ x \in A\Big\} + 1 \\ &\leq \frac{1}{\lambda}\left(a + \frac{1}{i}\right) + 1 \leq \frac{1}{\lambda}\,(a+1) + 1 \ . \end{aligned}$$

So the sequence $\{F_i\}$ is bounded in $F_n(X^*, X^*)$. For each $i = 1, 2, ...$, let $\{x_1^i, x_2^i, ..., x_n^i\}$ be an Auerbach basis for the n-dimensional subspace $Y_i = F_i(X^*)$ of X^*, and let $\{f_1^i, f_2^i, ..., f_n^i\}$ be the linear functionals on X^*, such that for each $k = 1, 2, ..., n$, $\|f_k^i\| = 1$ and $f_k^i(x_j^i) = \delta_{kj}$. Then for each $x \in Y_i$,

$$x = \sum_{k=1}^{n} f_k^i(x)\, x_k^i \ .$$

Let $g_k^i = f_k^i \odot F_i$ then $g_k^i \in X^{**}$, $\|g_k^i\| \le \|F_i\|$, and for each $x \in X^*$,

$$F_i(x) = \sum_{k=1}^{n} g_k^i(x)\, x_k^i \ .$$

The sequence $\{(g_1^i, g_2^i, ..., g_n^i, x_1^i, x_2^i, ..., x_n^i)\}_{i=1}^{\infty}$ is bounded in $(\prod_{i=1}^{n} X^{**}) \times (\prod_{i=1}^{n} X^*)$, therefore there is an element $(g_1^0, g_2^0, ..., g_n^0, x_1^0, x_2^0, ..., x_n^0)$ in $(\prod_{i=1}^{n} X^{**}) \times (\prod_{i=1}^{n} X^*)$, and a subsequence $\{(g_1^j, g_2^j, ..., g_n^j, x_1^j, x_2^j, ..., x_n^j)\}_{j=1}^{\infty}$ of $\{(g_1^i, g_2^i, ..., g_n^i, x_1^i, x_2^i, ..., x_n^i)\}_{i=1}^{\infty}$ satisfying for each $k = 1, 2, ..., n$, g_k^0 is the w^*-limit of $\{g_k^j\}$ in X^{**}, and x_k^0 is the w^*-limit of $\{x_k^j\}$ in X^*. Define $F_0 \colon X^* \to X^*$ by $F_0(x) = \sum_{k=1}^{n} g_k^0(x)\, x_k^0$, then F_0 is a bounded linear operator on X^* of rank $\le n$, and for each $x \in X^*$, $F_0(x)$ is the w^*-limit of the sequence $\{F_j(x)\}$ in X^*. Thus for each x in A

$$\|x - F_0(x)\| \le \underline{\lim}\, \|x - F_j(x)\| \le a \ .$$

So $\sup\{\|x - F_0(x)\|;\ x \in A\} = \delta_n(A, X^*)$. ∎

4 – Optimal sets for the compact width

4.1 Definition. The normed linear space X is said to have the property P_1 if for each $\epsilon > 0$ and $r > 0$ there is $\delta > 0$ such that for each x and y in X, there is $z \in B(x, \epsilon)$ satisfying the property that for each $0 < \theta < \delta$

$$B(x, r+\delta) \cap B(y, r+\theta) \subseteq B(z, r+\theta) \ .$$

The property P_1 was studied by several authors, for example Mach [12], Mach [13], Lau [10] and Amir, Mach and Saatkamp [1]. The following proposition summarizes some of their results.

4.2 Proposition.

a) *Lau [10]: If X is uniformly convex, then X has the property P_1.*

b) *Mach [13]: If Q is a compact Hausdorff space, then both $B(Q)$ and $C(Q)$ have the property P_1.*

c) *Mach [12]: If Q is a compact Hausdorff space, and X is a uniformly convex space, then both $B(Q, X)$ and $C(Q, X)$ have the property P_1.*

d) *Amir, Mach, Saatkamp [1]: If $X = L_1(\mu)$ where μ is a finite positive measure, and $\dim X = \infty$, then X does not have the property P_1. Also there is a finite dimensional space X, that does not have the property P_1.*

In Theorem 4.3 it will be shown that if the Banach space X has the property P_1, then for each bounded subset A of X, the compact width $a(A, X)$ is attained. The inverse of this statement need not be true. In Example 4.4, it will be shown that there are spaces that do not have the property P_1, but the compact width of any of its bounded subsets is attained.

4.3 Theorem. *Let X be a Banach space. If X has the property P_1, then for each bounded subset A of X, $a(A, X)$ is attained.*

Proof: Let $r = a(A, X)$, if $r = 0$ then the proof is obvious, so one may assume that $r > 0$. For each $i = 1, 2, ...$, let $\epsilon_i = \frac{1}{2^i}$, and let $\delta_i > 0$ corresponds to ϵ_i, and r in the definition of the property P_1. Without loss of generality one may assume that $\delta_{i+1} < \delta_i$ and that $\lim \delta_i = 0$. For each $i = 1, 2, ...$, there is a finite set $D_i = \{x_1^i, x_2^i, ..., x_m^i\}$ in X satisfying $\delta(A, D_i) \leq r + \delta_i$. In what follows an infinite sequence $\{C_i\}$ of bounded subsets of X will be constructed with the following properties

1) For each $i = 1, 2, ...,$ C_i consists of a finite number of elements.

2) For each $i = 1, 2, ...,$ $\delta(A, C_i) \leq r + \delta_i$.

3) For each $i = 1, 2, ...,$ $\delta(C_{i+1}, C_i) \leq \epsilon_i$.

Let $C_1 = D_1$ and assume that for some positive $n \geq 1$ the sets $C_1, C_2, ..., C_n$ have been chosen with the required properties. By the property P_1 and the fact that $\delta_{n+1} < \delta_n$, it follows that for each $y \in D_{n+1}$ and each $x \in C_n$, there is z_{xy} in $B(x, \epsilon)$ satisfying

$$B(x, r + \delta_n) \cap B(y, r + \delta_{n+1}) \subseteq B(z_{xy}, r + \delta_{n+1}) \ .$$

Let $C_{n+1} = C_n \cup \{z_{xy};\ x \in C_n \text{ and } y \in D_{n+1}\}$. Then $\delta(A, C_{n+1}) \leq r + \delta_{n+1}$ and $\delta(C_{n+1}, C_n) \leq \epsilon_n$. Let $E = \bigcup_{n=1}^{\infty} C_n$ and let $C = \overline{E}$, the closure of E, then $\delta(A, C) = r$, so to complete the proof it is enough to show that E is relatively compact. Let $\epsilon > 0$ be given, and assume that $\epsilon \geq \epsilon_m$ for some positive integer m. If $x \in E$ then $x \in C_{m+k}$ for some $k = 1, 2, ...$ Thus

$$d(x, C_m) \leq \sum_{i=m}^{m+k} \delta(C_{i+1}, C_i) \leq \sum_{i=m}^{m+k} \epsilon_i < \sum_{i=m}^{\infty} \epsilon_i = \sum_{i=m}^{\infty} \frac{1}{2^i} = \frac{1}{2^{m-1}} \leq \epsilon \ .$$

Therefore since C_m is a finite subset of E, it follows that E is relatively compact. ∎

4.4 Example:

a) Let X be the classical Banach space of sequence l_1. By Feder [5, Proposition 2.b and Proposition 6], $a(A, l_1)$ is attained for each bounded separable subset A of l_1. But the space l_1 is separable, so for each bounded subset

A of l_1, the compact width $a(A, l_1)$ is attained. On the other hand, by Proposition 4.2d, the space l_1 does not have the property P_1.

b) If A is a bounded subset of the finite dimensional space X, then $\overline{A}$, the closure of A, is compact, so $a(A, X) = 0$, and $\overline{A}$ is an optimal compact set for $a(A, x)$. Therefore if X is a finite dimensional normed linear space, then for each bounded set A in X, the compact width $a(A, X)$ is attained. On the other hand, by Proposition 4.2.d there is a finite dimensional normed linear space X that does not have the property P_1.

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