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POINTWISE CONTROLLABILITY AS LIMIT OF INTERNAL CONTROLLABILITY FOR THE WAVE EQUATION IN ONE SPACE DIMENSION

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1 – Introduction

In this article, we are interested in the passage from an exact controllability problem for the wave equation in one space dimension when the control acts in an open subset of amplitude ε to an exact controllability problem when the control acts on a point.

We study the case of the wave equation in one space dimension and we start with the solutions of exact controllability problems when the controls have their supports in an interval of the form $]a, a+\varepsilon[$ where a is any fixed point of $]0,\pi[$. The existence of these solutions has been proved by A. Haraux in [3] and we will briefly recall some of his results.

The aim of this paper is to describe what happens when ε tends to zero. We can't hope to get a pointwise control for the limit problem for any point a in $]0,\pi[$ as we know from S. Jaffard (see [5]) and A. Haraux and S. Jaffard (see [4]) that the pointwise exact controllability of the wave equation is possible (in some way) if and only if a is what one calls a strategic point. Results concerning pointwise exact controllability have been obtained by S. Jaffard in [5] proving embeddings between the space of initials datas which are exactly controllable and some Sobolev spaces. We then prove that if a is strategic , we obtain for the limit problem the solution of an exact controllability problem when the control acts on the point a and that for other points, the solutions of internal controllability problems blow up.

In order to solve our problem, we use a method which is quite analogous to the one used in [1] and [2] where we were interested in the limit problem starting with internal controllability problems with controls concentrated in an ε-neighbourhood of a part of the boundary satisfying some geometrical condition.

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We will have to prove some results concerning the behavior of weak solutions of the wave equation in an interval of length ε. As they are independent of the exact controllability problem, we state them in a first section. After recalling the results of A. Haraux mentioned above, we will prove an inverse inequality which will give us the estimates on the internal controls in the case of a strategic point which are necessary to pass to the limit when the amplitude ε tends to zero. We then end up with the limit problem.

2 – Behavior near a point

We consider here the weak solutions of the wave equation in $\Omega\times]0,T[$ where $\Omega=]0,\pi[$. Let h, u^0, u^1 be elements of $L^1(0,T;H^{-1}(\Omega)), L^2(\Omega)$ and $H^{-1}(\Omega)$ and let u be the solution of

$$\begin{cases} u''-\partial^2 u=h \text{ in } \Omega\times[0,T] \\ u=0 \ \text{ in } \partial\Omega\times[0,T] \\ u(0)=u^0, \text{ and } u'(0)=u^1 \ . \end{cases} \tag{2.1}$$

We recall that $u\in C(0,T;L^2(\Omega))\cap C^1(0,T;H^{-1}(\Omega))$ and that such solutions of the wave equation can be defined by transposition (we refer to [6] for more details).

We denote by Ω_ϵ the interval $]a,a+\varepsilon[$, where a is a point of $]0,\pi[$ and we will take $\varepsilon<\pi-a$. For simplicity, we denote by ∂u the derivative with respect to the space variable x of a function u.

The main results of this section concern the regularity of u(a,t) where u is a weak solution of (2.1) and a is any point of $]0,\pi[$ and the behavior of the L^2-norm of u in an ε-neighbourhood of a. These results are gathered in the following

Theorem 2.1. *There exists a constant $c>0$ and a real number $\varepsilon_0>0$ such that for every $\varepsilon\le\varepsilon_0$, and for every (h,u^0,u^1) in $L^1(0,T;H^{-1}(\Omega))\times L^2(\Omega)\times H^{-1}(\Omega)$, the solution u of (2.1) satisfies:*

$$u(a,t)\in L^2(0,T) \ , \tag{2.2}$$

and we also have an estimate such as

$$\frac{1}{\varepsilon}\int_0^T\int_{\Omega_\varepsilon} u^2(x,t)\,dx\,dt \le c(||h||^2_{L^1(0,T,H^{-1}(\Omega))}+||u^0||^2_{L^2(\Omega)}+||u^1||^2_{H^{-1}(\Omega)}) \ . \tag{2.3}$$

Proof: We introduce $v=\Delta^{-1}u$ with Dirichlet conditions on the boundary.

The function v is a solution of

$$
\begin{cases} v'' - \partial^2 v = f \text{ in } \Omega \times [0,T] \\ v = 0 \text{ in } \partial\Omega \times [0,T] \\ v(0) = v^0 \text{ and } v'(0) = v^1 \ , \end{cases} \tag{2.4}
$$

with initial data $v(0) = v^0, v'(0) = v^1$ and with a right hand side f satisfying

$$
\begin{cases} v^0 = \Delta^{-1} u^0 \in H^2(\Omega) \cap H_0^1(\Omega) \\ v^1 = \Delta^{-1} u^1 \in H_0^1(\Omega) \\ f = \Delta^{-1} h \in L^1(0,T; H_0^1(\Omega)) \ . \end{cases}
$$

and it is sufficient to prove the

Theorem 2.2. *There exists a constant $c > 0$ and a real number $\varepsilon_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$, and for every (f, v^0, v^1) in $L^1(0,T; H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, the solution v of (2.4) verifies:*

$$
\partial^2 v(a,.) \in L^2(0,T), \partial v'(a,.) \in L^2(0,T) \tag{2.5}
$$

and the mapping

$$
\begin{aligned} L^1(0,T; H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) &\to L^2(0,T) \times L^2(0,T) \\ (f, v^0, v^1) &\mapsto (\partial^2 v(a,.), \partial v'(a,.)) \end{aligned} \tag{2.6}
$$

is linear continuous.

Furthermore, we have

$$
\begin{aligned} &\frac{1}{\varepsilon} \int_0^T \int_{\Omega_\varepsilon} [(\partial^2 v)^2(x,t) + (\partial v')^2(x,t))] dx dt \le \\ &\qquad c(||f||^2_{L^1(0,T,H_0^1(\Omega))} + ||v^0||^2_{H^2(\Omega)\cap H_0^1(\Omega)} + ||v^1||^2_{H_0^1(\Omega)}) \ . \end{aligned} \tag{2.7}
$$

Proof: Using a density argument, it is sufficient to prove this result for a right hand side f in $\mathcal{D}((0,T) \times \Omega)$. The function $w = v'$ is then a solution with finite energy of

$$
\begin{cases} w'' - \partial^2 w = f' \text{ in } \Omega \times [0,T] \\ w = 0 \text{ in } \partial\Omega \times [0,T] \\ w(0) = v^1, \text{ and } w'(0) = \partial^2 v^0 \ . \end{cases} \tag{2.8}
$$

In order to use the multiplier method, we introduce a function ρ element of $W^{1,\infty}(a,1)$ and we denote by $Q_a =]a,1[\times]0,T[$. Multiplying the equation satisfied by w by $\rho\partial w$, we get

$$\int_{Q_a}(w'' - \partial^2 w)\partial w \rho dxdt = \int_{Q_a} f'\partial w \rho dxdt \ .$$

But, by integrations by parts, we have

$$\begin{aligned}\int_{Q_a} w''\partial w\rho dxdt = &\int_a^1 \partial w(x,T)w'(x,T)\rho(x)dx - \int_a^1 \partial w(x,0)w'(x,0)\rho(x)dx\\ &+\frac{1}{2}\int_{Q_a} w'^2\partial\rho dxdt + \frac{1}{2}\int_0^T w'^2(a,t)\rho(a)dt\\ = &\int_a^1 \partial v'(x,T)\partial^2 v(x,T)\rho(x)dx - \int_a^1 \partial v'(x,0)\partial^2 v(x,0)\rho(x)dx\\ &+\frac{1}{2}\int_{Q_a} v''^2\partial\rho dxdt + \frac{1}{2}\int_0^T v''^2(a,t)\rho(a)dt \ .\end{aligned}$$

As $v'' = f + \partial^2 v$ and $f(0) = f(T) = 0$, we finally obtain

(2.9)

$$\begin{aligned}\int_{Q_a} w''\partial w\rho dxdt = &\int_a^1 \partial v'(x,T)\partial^2 v(x,T)\rho(x)dx - \int_a^1 \partial v'(x,0)\partial^2 v(x,0)\rho(x)dx\\ &+\frac{1}{2}\int_{Q_a} f^2(x,t)\partial\rho(x)dxdt + \frac{1}{2}\int_{Q_a}(\partial^2 v)^2(x,t)\partial\rho(x)dxdt\\ &+\int_{Qa} f(x,t)\partial^2 v(x,t)\partial\rho(x,t)dxdt + \frac{1}{2}\int_0^T f^2(a,t)\rho(a)dt\\ &+\frac{1}{2}\int_0^T(\partial^2 v)^2(a,t)\rho(a)dt + \int_0^T f(a,t)\partial^2 v(a,t)\rho(a)dt \ .\end{aligned}$$

On the other hand, we have

$$\begin{aligned}-\int_{Q_a}\partial^2 w(x,t)\partial w(x,t)\rho(x)dxdt = &\frac{1}{2}\int_{Q_a}(\partial v')^2(x,t)\partial\rho(x)dxdt\\ &+\frac{1}{2}\int_0^T(\partial v')^2(a,t)\rho(a)dt\end{aligned} \tag{2.10}$$

and

(2.11)

$$\begin{aligned}\int_{Q_a} f'(x,t)\partial w(x,t)\rho(x)dxdt &= -\int_{Q_a} f(x,t)\partial w'(x,t)dxdt\rho(x)dxdt\\ &= -\int_{Q_a} f(x,t)\partial(f+\partial^2 v)(x,t)\rho(x,t)dxdt\\ &= \frac{1}{2}\int_{Q_a} f^2(x,t)\partial\rho(x)dxdt + \frac{1}{2}\int_0^T f^2(a,t)\rho(a)dt\\ &\quad+\int_{Q_a} \partial^2 v(x,t)\partial f(x,t)\rho(x,t)dxdt\\ &\quad+\int_{Q_a} f(x,t)\partial^2 v(x,t)\partial\rho(x)dx\\ &\quad+\int_0^T f(a,t)\partial^2 v(a,t)\rho(a)dt\ .\end{aligned}$$

Taking into account (2.9), (2.10) and (2.11), we then obtain

(2.12)

$$\begin{aligned}\frac{1}{2}\int_0^T (\partial v')^2(a,t)\rho(a)dt + \frac{1}{2}\int_0^T (\partial^2 v)^2(a,t)\rho(a)dt &= -\frac{1}{2}\int_{Q_a} (\partial^2 v)^2(x,t)\partial\rho(x)dxdt\\ -\frac{1}{2}\int_{Q_a} (\partial v')^2(x,t)\partial\rho(x)dxdt &+ \int_{Q_a} \partial^2 v(x,t)\partial f(x,t)\rho(x,t)dxdt\\ -\int_a^1 \partial v'(x,T)\partial^2 v(x,T)\rho(x)dx &+ \int_a^1 \partial v'(x,0)\partial^2 v(x,0)\rho(x)dx\ .\end{aligned}$$

Now, choosing $\rho(x)=\pi-x$ in (2.12), one can easily show (2.5) and (2.6). In order to get (2.7), we apply (2.12) to the functions

$$\begin{cases}\rho_\varepsilon(x)=\dfrac{a+\varepsilon-x}{\varepsilon} & \text{in}\quad [a,a+\varepsilon]\\ \rho_\varepsilon(x)=0 \quad \text{in} \quad [a+\varepsilon,1]\ . & \end{cases} \tag{2.13}$$

and we use (2.5) and (2.6). ∎

3 – Internal exact controllability of the wave equation

We recall here some results of A. Haraux that one can find in [3]. Using the "Hilbert Uniqueness Method" introduced by J. L. Lions in [6], A. Haraux proved the exact controllability of the wave equation when the control is distributed and acts on $]a,a+\varepsilon[\times]0,T[$ for initial data in $H_0^1(\Omega)\times L^2(\Omega)$. As usual in exact controllability problems, this result is a consequence of an inverse inequality concerning the solutions of the following homogeneous wave equation:

$$\left\{\begin{array}{l}\varphi'' - \partial^2\varphi = 0 \\ \varphi = 0 \quad \text{in} \quad \partial\Omega\times]0,T[\\ \varphi(0) = \varphi^0, \quad \varphi'(0) = \varphi^1 \ ,\end{array}\right. \tag{3.1}$$

where $\varphi^0 \in L^2(\Omega)$ and $\varphi^1 \in H^{-1}(\Omega)$.

We recall this inequality, whose proof can be found in [3]:

For every $T \geq 2\pi$, there exists $c_\varepsilon > 0$ such that the weak solutions of (3.1) satisfy

$$||\varphi^0||^2_{L^2(\Omega)} + ||\varphi^1||^2_{H^{-1}(\Omega)} \leq c_\varepsilon(\int_{Q_\varepsilon} \varphi^2(x,t)dxdt) \ . \tag{3.2}$$

Furthermore,

$$c_\varepsilon = O(\frac{1}{\varepsilon^3}) \ , \tag{3.3}$$

this estimate being optimal.

Having this inequality, A. Haraux proves that for every $(y^0, y^1) \in H^1_0(\Omega) \times L^2(\Omega)$, the state ψ_ε solution of the exact controllability problem when the control acts on $\Omega_\epsilon\times]0,T[$ with initial data (y^0, y^1) is defined by

$$\left\{\begin{array}{l}\psi_\varepsilon'' - \partial^2\psi_\varepsilon = \tilde{\varphi}_\varepsilon\chi_\varepsilon \\ \psi_\varepsilon = 0 \quad \text{on} \quad \partial\Omega\times]0,T[\\ \psi_\varepsilon(0) = y^0, \quad \psi_\varepsilon'(0) = y^1 \\ \psi_\varepsilon(T) = \psi_\varepsilon'(T) = 0 \ ,\end{array}\right. \tag{3.4}$$

where χ_ε is the characteristic function of $]a, a+\varepsilon[$, $\tilde{\varphi}_\varepsilon\chi_\varepsilon$ is the control and $\tilde{\varphi}_\varepsilon$ is the unique solution of (3.1) whose initial data satisfy

$$-(y^1, \tilde{\varphi}^0_\varepsilon) + \langle y^0, \tilde{\varphi}^1_\varepsilon\rangle = \int_{Q_\varepsilon} \tilde{\varphi}^2_\varepsilon(x,t)dxdt \ . \tag{3.5}$$

As we shall see, (3.3) gives us an estimate on the controls $\tilde{\varphi}_\varepsilon$. However, it is not sufficient to obtain the limit problem and in the case of a strategic point, we will be able to improve it using the next section.

4 – An inverse inequality

We suppose in this section that the point a is strategic which means that the quantity

$$\left(\int_0^T \varphi^2(a,t)\,dt\right)^{(1/2)}$$

defines a norm on the space $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ of the initial data of the solutions of (3.1):

$$\begin{cases} \varphi'' - \partial^2\varphi = 0 \\ \varphi = 0 \quad \text{in} \quad \partial\Omega\times]0,T[\\ \varphi(0) = \varphi^0,\ \varphi'(0) = \varphi^1 \ . \end{cases}$$

J.L. Lions proved in [6] that a point a is strategic if and only if a/π is not an element of Q.

We denote by F the completion of $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ for this norm and we denote by $\|\cdot\|_F$ the following quantity:

$$||\varphi||_F = \left(\int_0^T \varphi^2(a,t)\,dt\right)^{(1/2)} \ .$$

A very simple calculation proves that

$$L^2(\Omega) \times H^{-1}(\Omega) \subset F \ .$$

In fact, S. Jaffard proved (see [5]) that there exists a dense subset of strategic points for which we have the following result:

If $y^0(x) = \sum_{n\ge1} a_n sin(nx)$ and $y^1(x) = \sum_{n\ge1} b_n sin(nx)$, then

$$(y^0, y^1) \in F' \Leftrightarrow \sum_{n\ge1} \frac{n^2 a_n^2 + b_n^2}{sin^2(na)} < \infty \ .$$

The main result of this section is the following:

Theorem 4.1. *For $T \ge 2\pi$, there exists $c > 0$, such that for every $(\varphi^0, \varphi^1) \in F$, the solution φ of (3.1) satisfies*

$$||(\varphi^0, \varphi^1)||_F^2 \le c\left(\frac{1}{\varepsilon}\int_{Q_\varepsilon} \varphi^2(x,t)\,dx\,dt\right) \ . \tag{4.1}$$

Proof: First, we remark that it is sufficient to prove the result for $T = 2\pi$. Indeed, suppose that the theorem is true for $T = 2\pi$ and let us consider a time $T > 2\pi$. Let k be the unique integer such that $2k\pi \le T < 2(k+1)\pi$, we write

$$\varphi(x,t) = \sum_{n\in N} a_n \cos(nt + \alpha_n) \sin(nx) \ ,$$

and we then have

$$\int_0^T \varphi^2(a,t)dt \le \int_0^{2(k+1)\pi} \varphi^2(a,t)dt \ .$$

As the family of functions $(\cos(nt+\alpha_n))_{n\in N}$ is orthogonal in $L^2(0,2(k+1)\pi)$, we obtain

$$\int_0^T \varphi^2(a,t)dt \le (k+1)\pi \sum_{n\in N} a_n^2 \sin^2(na) \ .$$

As

$$(k+1)\pi \sum_{n\in N} a_n^2 \sin^2(na) = (k+1) \int_0^{2\pi} \varphi^2(a,t)dt \ ,$$

we get

$$\begin{aligned}\int_0^T \varphi^2(a,t)dt &\le (k+1)\int_0^{2\pi} \varphi^2(a,t)dt \\ &\le c(k+1)\frac{1}{\varepsilon}\int_0^{2\pi}\int_a^{a+\varepsilon} \varphi^2(x,t)dxdt \\ &\le c\frac{k+1}{k}\frac{1}{\varepsilon}\int_0^{2k\pi}\int_a^{a+\varepsilon} \varphi^2(x,t)dxdt \\ &\le c\frac{k+1}{k}\frac{1}{\varepsilon}\int_0^{T}\int_a^{a+\varepsilon} \varphi^2(x,t)dxdt\end{aligned}$$

and the theorem will be true for every $T \ge 2\pi$.

Now, suppose $T = 2\pi$. We write again

$$\varphi(x,t) = \sum_{n\in N} a_n \cos(nt+\alpha_n)\sin(nx) \ ,$$

and we have

$$\int_{Q_\varepsilon} \varphi^2(x,t)dt = \pi \sum_{n\in N} a_n^2 \int_a^{a+\varepsilon} \sin^2(nx)dx \ .$$

Now, we have to prove that there exists $c > 0$ independent on ε such that for every integer $n \in N$, we have

$$\frac{1}{\varepsilon}\int_a^{a+\varepsilon} \sin^2(nx)dx \ge c\sin^2(na) \ . \tag{4.2}$$

For $b \ge 0, t \ge 0$, we set

$$I(b,t) = \int_0^1 \sin^2(b+tz)dz \ .$$

As

$$\frac{1}{\varepsilon}\int_a^{a+\varepsilon} \sin^2(nx)dx = I(na, n\varepsilon) \ ,$$

it is sufficient to prove that there exists $c > 0$ such that

$$\forall t \geq 0, \quad I(b,t) \geq c \sin^2 b \ . \tag{4.3}$$

We have the formula

$$\forall t \geq 0, \quad I(b,t) = \frac{1}{2}(1 - \frac{\sin t}{t} \cos(2b+t)) \ .$$

If $t \geq \frac{\pi}{2}$, then $I(b,t) \geq \frac{1}{2}(1 - \frac{2}{\pi})$.
If $t < \frac{\pi}{2}$, we distinguish two cases:

Case 1. $]b, b+t[\subset]n\pi, (n+1)\pi[$. It is then enough to consider the case $n = 0$ and as sin is concave on $[0,\pi]$, we obtain

$$\forall z \in [0,1], \quad |\sin(b+tz)| \geq (1-z)|\sin b| \ .$$

Hence

$$\begin{aligned} I(t,b) &\geq \sin^2 b \int_0^1 (1-z)^2 dz \\ &\geq \frac{1}{3} \sin^2 b \ . \end{aligned}$$

Case 2. $(n-1)\pi \leq b \leq n\pi \leq b+t \leq (n+1)\pi$. It is enough here to consider the case $n = 1$ and we write $\pi = b + z_0 t$ with $z_0 \in [0,1]$. We have

$$|\sin(b+tz_0)| \geq (1-z)|\sin b| \ \text{ for } \ z \leq z_0$$

and

$$|\sin(b+tz_0)| \geq z|\sin(b+t)| \ \text{ for } \ z > z_0 \ .$$

Now, if $z_0 \geq \frac{1}{2}$, we find

$$\begin{aligned} I(b,t) &\geq \sin^2 b \int_0^{z_0} (1-z)^2 dz \\ &\geq \frac{7}{24} \sin^2 b \ . \end{aligned}$$

If $z_0 < \frac{1}{2}$, then $b + t - \pi > \pi - b$ and we get the same lower bound.
Combining all the cases, we get assertion (4.3) and the proof of the theorem is complete. ∎

5 – Estimates on the controls

The results of the above section allow us now to get the estimates that we need in order to study the passage to the limit when ε tends to zero in (3.4).

Indeed, using (3.2),(3.3) and (3.5), we obtain the existence of a constant $c > 0$ such that

$$\begin{aligned}||\tilde{\varphi}^0_\varepsilon||^2_{L^2(\Omega)} + ||\tilde{\varphi}^1_\varepsilon||^2_{H^{-1}} &\leq c(\frac{1}{\varepsilon^3}\int_{Q_\varepsilon} \tilde{\varphi}^2_\varepsilon(x,t)dxdt)\\ &\leq \frac{c}{\varepsilon^3}||(y^0,y^1||_{H^1_0(\Omega)\times L^2(\Omega)}||(\tilde{\varphi}^0_\varepsilon,\tilde{\varphi}^1_\varepsilon)||_{L^2(\Omega)\times H^{-1}(\Omega)} \ .\end{aligned}$$

We then deduce

Theorem 5.1. *For every $T \geq 2\pi$, we have the following estimates:*

$$||(\tilde{\varphi}^0_\varepsilon,\tilde{\varphi}^1_\varepsilon)||_{L^2(\Omega)\times H^{-1}(\Omega)} = O(\frac{1}{\varepsilon^3}) \tag{5.1}$$

and

$$\int_{Q_\varepsilon} \tilde{\varphi}^2_\varepsilon(x,t)dxdt) = O(\frac{1}{\varepsilon^3}) \ . \tag{5.2}$$

In fact, when the point a is strategic and if the initial data (y^0, y^1) belong to F', this first estimate can be improved. Indeed, we have from (4.1) and (3.5):

$$\begin{aligned}||(\tilde{\varphi}^0_\varepsilon,\tilde{\varphi}^1_\varepsilon)||^2_F &\leq c(\frac{1}{\varepsilon}\int_{Q_\varepsilon} \tilde{\varphi}^2_\varepsilon(x,t)dxdt)\\ &\leq \frac{c}{\varepsilon}||(y^0,y^1||_{F'}||(\tilde{\varphi}^0_\varepsilon,\tilde{\varphi}^1_\varepsilon)||_F\end{aligned}$$

from which we deduce

Theorem 5.2. *For every $T \geq 2\pi$, if a is strategic and if $(y^0, y^1) \in F'$, the controls $\tilde{\varphi}_\varepsilon$ satisfy the following estimates:*

$$||(\tilde{\varphi}^0_\varepsilon,\tilde{\varphi}^1_\varepsilon)||_F = O(\frac{1}{\varepsilon}) \tag{5.3}$$

and

$$\int_{Q_\varepsilon} \tilde{\varphi}^2_\varepsilon(x,t)dxdt = O(\frac{1}{\varepsilon}) \ . \tag{5.4}$$

The functions $\varphi_\varepsilon = \varepsilon\tilde{\varphi}_\varepsilon$ are bounded in $L^\infty(0,T;F)$ thus, after extraction of a subsequence, they converge for the weak- topology of this space to an element φ which is also solution of (3.1).*

As we shall see, these last estimates are essential to study the limit when ε tends to zero which is done in the next section.

6 – Passage to the limit

We study here the possible convergence of the solutions of the exact controllability problems defined by (3.4). This convergence depends on the nature of the point a and on the space of the initial data y^0 et y^1. When a is strategic and if the initial data (y^0, y^1) belong to F' (which is exactly the space of initial data which are exactly pointwise controllable), the estimates obtained in (5.3) and (5.4) are exactly the one we need to pass to the limit in (3.4). In the others cases, we only have the estimates given by (5.1) and (5.2) and we will see that we can't hope a convergence result concerning the solutions of (3.2) which means that they probably blow up. If the point a is strategic, we have the following

Theorem 6.1. *Suppose that $T \geq 2\pi$. If a is strategic and if the initial data y^0 and y^1 belong to F', the solutions of (3.4) converge for the weak-* topology of $L^\infty(0,T;H_0^1(\Omega))$ to the solution of the following pointwise exact controllability problem:*

$$(6.1)\qquad \begin{cases} \psi'' - \partial^2\psi = v(a,t)\delta_a \\ \psi = 0 \qquad in \qquad \partial\Omega\times]0,T[\\ \psi(0) = y^0, \quad \psi'(0) = y^1 \qquad in \quad \Omega \\ \psi(T) = \psi'(T) = 0 \qquad in \qquad \Omega\,, \end{cases}$$

where v belongs to $L^2(0,T)$ and is the weak limit in $H^{-1}(0,T)$ of $\varphi_\varepsilon(a,.) - \frac{\varepsilon}{2}\partial\varphi_\varepsilon(a,.)$.

Proof: The solutions of (3.4) are defined by a duality process from the weak solution of (2.1) which means that we have:

$\forall(u^0,u^1,h) \in L^2(\Omega)\times H^{-1}(\Omega)\times L^1(0,T;H^{-1}(\Omega))$,

$$(6.2)\qquad \begin{aligned} \langle\langle h,\psi_\varepsilon\rangle\rangle &= \int_{Q_\varepsilon} \tilde\varphi_\varepsilon(x,t)u(x,t)dxdt + (y^1,u^0) - \langle u^1,y^0\rangle \\ &= \frac{1}{\varepsilon}\int_{Q_\varepsilon} \varphi_\varepsilon(x,t)u(x,t)dxdt + (y^1,u^0) - \langle u^1,y^0\rangle \end{aligned}$$

where u is the solution of (2.1) with initial data u^0 and u^1 and right hand side h.

To prove that (ψ_ε) is bounded in $L^\infty(0,T;H^{-1}(\Omega))$, it is sufficient to prove that the linear forms

$$(6.3)\qquad \begin{aligned} L_\varepsilon : L^2(\Omega)\times H^{-1}(\Omega)\times L^1(0,T;H^{-1}(\Omega)) &\to R \\ (u^0,u^1,h) &\mapsto \frac{1}{\varepsilon}\int_{Q_\varepsilon} \varphi_\varepsilon(x,t)u(x,t)dxdt \end{aligned}$$

are bounded in $L^2(\Omega)\times H_0^1(\Omega)\times L^\infty(0,T;H_0^1(\Omega))$. Using Holder inequality, (2.3) and (5.4), we have

$$\begin{aligned}|L_\varepsilon(u^0,u^1,h)|^2 &\leq \Big(\frac{1}{\varepsilon}\int_{Q_\varepsilon}\varphi_\varepsilon^2(x,t)dxdt\Big)\Big(\frac{1}{\varepsilon}\int_{Q_\varepsilon}u^2(x,t)dxdt\Big)\\ &\leq c(||u^0||^2_{L^2(\Omega)}+||u^1||^2_{H^{-1}(\Omega)}+||h||^2_{L^1(0,T;H^{-1}(\Omega))})\ ,\end{aligned}$$

which prove that the linear forms L_ε are bounded in $L^2(\Omega)\times H_0^1(\Omega)\times L^\infty(0,T;H_0^1(\Omega))$.

We then deduce that $(\psi_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$ thus, after extraction of a subsequence, they converge for the weak-* topology of this space to an element ψ of $L^\infty(0,T;H_0^1(\Omega))$. In order to find which equation is satisfied by ψ, we have to pass to the limit in (6.2) that is to determine the limit of the linear forms L_ε. This is given by the

Lemma 6.1. *The linear forms L_ε converge in $L^2(\Omega)\times H_0^1(\Omega)\times L^\infty(0,T;H_0^1(\Omega))$ weak-* to the linear form L defined by*

$$L(u^0,u^1,h)=\int_0^T v(a,t)u(a,t)dt\ , \tag{6.4}$$

where v belongs to $L^2(0,T)$ and is the weak limit in $H^{-1}(0,T)$ of $\varphi_\varepsilon(a,.)-\frac{\varepsilon}{2}\partial\varphi_\varepsilon(a,.)$.

Let us admit this lemma for the moment.

Passing to the limit in (6.2), we get

$\forall(u^0,u^1,h)\in L^2(\Omega)\times H^{-1}(\Omega)\times L^1(0,T;H^{-1}(\Omega))$,

$$\langle\langle h,\psi\rangle\rangle=\int_0^T v(a,t)u(a,t)dxdt+(y^1,u^0)-\langle u^1,y^0\rangle\ , \tag{6.5}$$

where u is the solution of (2.1) with initial data u^0 and u^1 and right hand side h. This proves that ψ is the solution of (6.1).

We now prove Lemma 6.1. In order to do this, we begin with the following result:

Lemma 6.2. *Let φ_ε be a family of solutions of (3.1) with initial data in $L^2(\Omega)\times H^{-1}(\Omega)$ satisfying the following condition:*

$$\int_{Q_\varepsilon}\varphi_\varepsilon^2(x,t)dxdt=O(\varepsilon)\ . \tag{6.6}$$

Then, after extraction of a subsequence, $\varphi_\varepsilon(a,.)-\frac{\varepsilon}{2}\partial\varphi_\varepsilon(a,.)$ converge for the weak topology of $H^{-1}(0,T)$ to an element v of $L^2(0,T)$.

Proof: From (4.1), hypothesis (6.6) implies that the functions φ_ε are bounded in F thus, after extraction of a subsequence, $\varphi_\varepsilon(a,.)$ converge in $L^2(0,T)$ weak. On an other hand, (3.2), (5.3) and (5.4) implies

$$||(\varphi_\varepsilon^0,\varphi_\varepsilon^1)||_{L^2(\Omega)\times H^{-1}(\Omega)} = O(\frac{1}{\varepsilon}) \ .$$

Using (2.5), we can easily prove that the mapping

$$(\varphi^0,\varphi^1)\in L^2(\Omega)\times H^{-1}(\Omega)\to \partial\varphi(a,.)\in H^{-1}(0,T) \ ,$$

where φ is the solution of (3.1), is continuous.

We then have

$$||\partial\varphi_\varepsilon(a,.)||_{H^{-1}(0,T)} = O(\frac{1}{\varepsilon}) \ .$$

In order to finish the proof of Lemma 6.2, we have to prove that $v\in L^2(0,T)$ that is

$$\forall u\in\mathcal{D}(0,T), \qquad |(v,u)_{\mathcal{D}',\mathcal{D}}|\le c||u||_{L^2(0,T)} \ . \tag{6.7}$$

We introduce the following functions:

$$\Phi_\varepsilon(x,t) = \int_0^t \varphi_\varepsilon(x,\tau)d\tau - \Delta^{-1}\varphi_\varepsilon^1 \ ,$$

and

$$s_\varepsilon(x,t) = \int_0^t \Phi_\varepsilon(x,\tau)d\tau - \Delta^{-1}\varphi_\varepsilon^0 \ .$$

The functions Φ_ε and s_ε are solutions of (3.1) with initial data respectively in $H_0^1(\Omega)\times L^2(\Omega)$ and $(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$. Furthermore, we have the following estimates:

$$||(\Phi_\varepsilon^0,\Phi_\varepsilon^1)||_{H_0^1(\Omega)\times L^2(\Omega)} = O(\frac{1}{\varepsilon}) \ ,$$

and

$$||(s_\varepsilon^0,s_\varepsilon^1)||_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)} = O(\frac{1}{\varepsilon}) \ .$$

For $u\in\mathcal{D}(0,T)$, we then write

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^T\int_a^{a+\varepsilon}\varphi_\varepsilon(x,t)u(t)dxdt &= \frac{1}{\varepsilon}\int_0^T\int_a^{a+\varepsilon} s_\varepsilon(x,t)u''(t)dxdt\\ &= \int_0^T (s_\varepsilon(a,t)+\frac{\varepsilon}{2}\partial s_\varepsilon(a,t))u''(t)dt\\ &\quad + \frac{1}{\varepsilon}\int_0^T\int_a^{a+\varepsilon} u''(t)\int_a^x\int_a^y \partial^2 s_\varepsilon(z,t)dzdydxdt \ .\end{aligned}$$

We thus have

$$(s_\varepsilon(a,.)+\frac{\varepsilon}{2}\partial s_\varepsilon(a,.),u'')_{\mathcal{D}',\mathcal{D}}=\frac{1}{\varepsilon}\int_0^T\int_a^{a+\varepsilon}\varphi_\varepsilon(x,t)u(t)dxdt-R_\varepsilon\ ,$$

where

$$R_\varepsilon=\frac{1}{\varepsilon}\int_0^T\int_a^{a+\varepsilon}u''(t)\int_a^x\int_a^y\partial^2 s_\varepsilon(z,t)dzdydxdt\ .$$

Let us prove that

$$\lim_{\varepsilon\to 0}R_\varepsilon=0.$$

Using Holder's inequality, we get,

$$\begin{aligned}|R_\varepsilon|&\le\frac{1}{\varepsilon}||u''||_{L^2(0,T)}(\int_0^T\varepsilon\int_a^{a+\varepsilon}(x-a)\int_a^x(y-a)\int_a^y(\partial^2 s_\varepsilon)^2(z,t)dzdydxdt)^{1/2}\\&\le||u''||_{L^2(0,T)}\frac{\sqrt{\varepsilon}}{\sqrt{8}}||(s_\varepsilon^0,s_\varepsilon^1)||_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}\end{aligned}$$

thus

$$\lim_{\varepsilon\to 0}R_\varepsilon=0\ .$$

Using now (6.6), there exist a constant $c>0$ such that

$$|(s_\varepsilon(a,.)+\frac{\varepsilon}{2}\partial s_\varepsilon(a,.),u'')_{\mathcal{D}',\mathcal{D}}|\le c||u||_{L^2(0,T)}+|R_\varepsilon|\ .$$

Passing to the limit when ε goes to zero, we obtain

$$|(v,u)_{\mathcal{D}',\mathcal{D}}|\le c||u||_{L^2(0,T)}\ ,$$

which prove that v belongs to $L^2(0,T)$. ∎

Proof of Lemma 6.1: As the linear form L defined in (6.4) has a sense on $L^2(\Omega)\times H^{-1}(\Omega)\times L^1(0,T;H^{-1}(\Omega))$), it is sufficient to prove that $(L_\varepsilon)_\varepsilon$ converge to L on a dense subspace of $L^2(\Omega)\times H^{-1}(\Omega)\times L^1(0,T;H^{-1}(\Omega))$ and, for example, on $(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times L^1(0,T;H_0^1(\Omega))$. We then consider $(u^0,u^1,h)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times L^1(0,T;H_0^1(\Omega))$. We know that $u\in C(O,T;H^2(\Omega)\cap H_0^1(\Omega))\cap C^1(0,T;H_0^1(\Omega))$. Using (5.4), one can easily prove that the following linear forms

$$\begin{aligned}&K_\varepsilon:L^2(O,T;H^2(\Omega)\cap H_0^1(\Omega))\to R\\&\qquad u\mapsto\frac{1}{\varepsilon}\int_{Q_\varepsilon}\varphi_\varepsilon(x,t)u(x,t)dxdt\end{aligned}\tag{6.8}$$

are defined and bounded on $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$.

They converge for the weak topology $L^2(0,T,(H^2(\Omega)\cap H_0^1(\Omega)')$ to an element K of $L^2(0,T,(H^2(\Omega)\cap H_0^1(\Omega)')$. In order to determine K, we write for $u \in \mathcal{D}(0,T;C^\infty(\Omega))$:

$$
\begin{aligned}
K_\varepsilon(u) =& \frac{1}{\varepsilon}\int_{Q_\varepsilon} \varphi_\varepsilon(x,t)u(a,t)dxdt \\
&+ \frac{1}{\varepsilon}\int_{Q_\varepsilon} \varphi_\varepsilon(x,t)(\int_a^x \partial u(y,t)dy)dxdt \ .
\end{aligned}
\tag{6.9}
$$

We have already seen that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{Q_\varepsilon} \varphi_\varepsilon(x,t)u(a,t)dxdt = \int_0^T v(t)u(a,t)dt \ .$$

On an other hand, it is easy to prove, using Holder's inequality, that for every $u \in \mathcal{D}(0,T;C^\infty(\Omega))$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{Q_\varepsilon} \varphi_\varepsilon(x,t)(\int_a^x \partial u(y,t)dy)dxdt = 0 \ ,$$

which finishes the proof of the lemma 6.1. ∎

Remark 6.1. In the case of a non strategic point, we still have

$$\langle\langle h,\psi_\varepsilon\rangle\rangle = \int_{Q_\varepsilon} \tilde{\varphi}_\varepsilon(x,t)u(x,t)dxdt + (y^1,u^0) - \langle u^1,y^0\rangle \ .$$

But, we have here,

$$(\int_{Q_\varepsilon} \tilde{\varphi}_\varepsilon(x,t)u(x,t)dxdt)^2 \le \int_{Q_\varepsilon} \tilde{\varphi}_\varepsilon^2(x,t)dxdt \times \int_{Q_\varepsilon} u^2(x,t)dxdt \ .$$

Using (5.2), there exist a constant $c > 0$ such that

$$(\int_{Q_\varepsilon} \tilde{\varphi}_\varepsilon(x,t)u(x,t)dxdt)^2 \le c(\frac{1}{\varepsilon^3}\int_{Q_\varepsilon} u^2(x,t)dxdt) \ .$$

It is clear that we can't hope a result like

$$\frac{1}{\varepsilon^3}\int_{Q_\varepsilon} u^2(x,t)dxdt \le c(||u^0||^2_{L^2(\Omega)} + ||u^1||^2_{H^{-1}(\Omega)} + ||h||^2_{L^1(0,T;H^{-1}(\Omega))}) \ ,$$

with $c > 0$ and independent on ε because this would imply $u(a,.) = 0$ for every u solution of (2.1) with initial data and right side in $L^2(\Omega) \times H^{-1}(\Omega) \times L^1(0,T;H^{-1}(\Omega))$. This explains that the solutions of (3.4) probably blow up in the case of a non strategic point.

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