PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 3 – 1994

ON THE CENTER OF THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF A HYPERPLANE ARRANGEMENT

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Abstract: Let $\mathcal{A}$ be a simplicial hyperplane arrangement in $\mathbf{R}^d$. We prove that the center of the fundamental group of the manifold $\mathbf{C}^d \setminus \bigcup\{H \otimes \mathbf{C} : H \in \mathcal{A}\}$ is a direct product of infinite cyclic subgroups.

1 – Introduction

Let $\mathcal{A}$ be a *(central) hyperplane arrangement* in $\mathbf{R}^d$, i.e. a finite set $H_1, H_2, \ldots, H_m$ of subspaces of codimension 1 in $\mathbf{R}^d$. Consider now the manifold $M = M(\mathcal{A}) = \mathbf{C}^d \setminus \bigcup\{H \otimes \mathbf{C} : H \in \mathcal{A}\}$. M is an open, smooth, parallelizable manifold of real dimension $2d$ (see [OT, Proposition 5.1.3]). The homotopy type of M is non trivial and has been recently an active area of research. An important topological invariant of the manifold $M(\mathcal{A})$ is certainly its fundamental group $\pi_1(M)$. A reduced presentation of this group was obtained by Randell [R] and Salvetti [Sa] (see also [CG]). In this paper we determine the center of $\pi_1(M)$ for *simplicial (hyperplane) arrangements* (i.e., such that every component of $\mathbf{R}^d \setminus \cup\{H \colon H \in \mathcal{A}\}$ is an open polyhedral simplicial cone).

We use as general reference on arrangements of hyperplanes, the recent book with the same title by Orlik and Terao [OT]. We recall that a hyperplane arrangement $\mathcal{A}$ in $\mathbf{R}^d$ is *reducible* [OT] if there are two arrangements $\mathcal{A}_1$ in $\mathbf{R}^{d_1}$ and $\mathcal{A}_2$ in $\mathbf{R}^{d_2}$ such that, after a change of coordinates:

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \stackrel{\text{def}}{=} \Big\{H \oplus \mathbf{R}^{d_2} \colon H \in \mathcal{A}_1\Big\} \cup \Big\{\mathbf{R}^{d_1} \oplus H \colon H \in \mathcal{A}_2\Big\} .$$

Otherwise $\mathcal{A}$ is *irreducible*.

Received: July 3, 1992.
1991 Mathematics Subject Classification: Primary 55P20; Secondary 05B35.
(*) Partially supported by the project PMCT/C/CEN/37/90 of J.N.I.C.T.

The results presented here depend of the following preliminary observations:

- Suppose that $\mathcal{A}$ is not *essential* (i.e., $\bigcap_{H\in\mathcal{A}} H = X \neq 0$). Let Y be the orthogonal complement of X in $\mathbf{R}^d$. Then $\mathcal{A}' = \{H \cap Y : H \in \mathcal{A}\}$ is an essential hyperplane arrangement and the manifolds $M(\mathcal{A})$ and $M(\mathcal{A}')$ have the same homotopy type [OT, Sa].
- Suppose that $\mathcal{A}$ is essential.Then $\mathcal{A}$ is irreducible if and only if the matroid $\mathcal{M}(\mathcal{A})$ determined by the nonempty intersections of the hyperplanes of $\mathcal{A}$ is a connected one.
- In order to study decompositions of the group $\pi_1(M(\mathcal{A}))$, reducibility is not the "right concept" [CG]. However for simplicial arrangements the two concepts coincide (see Theorem 2.1 and Proposition 2.2 below).
- If $\mathcal{A}$ is a simplicial arrangement and $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ then $\mathcal{A}_1$ and $\mathcal{A}_2$ are also simplicial arrangements.
- If $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_n$ then $\pi_1(M(\mathcal{A})) \simeq \pi_1(M(\mathcal{A}_1)) \times ... \times \pi_1(M(\mathcal{A}_n))$.

Our main result is:

1.1 Theorem. *Suppose $\mathcal{A}$ is an irreducible simplicial hyperplane arrangement in $\mathbf{R}^d$.Then the center of the fundamental group of the manifold $M(\mathcal{A}) = \mathbf{C}^d \setminus \bigcup\{H \otimes \mathbf{C} \colon H \in \mathcal{A}\}$ is an infinite cyclic subgroup.* ∎

We prove in Section 3 the oriented matroid generalization of this theorem (see Theorem 3.4 below).When $\mathcal{A}$ is an irreducible *Coxeter arrangement* [Bo,H], Theorem 1.1 can be deduced from well known results. Indeed let W be the reflection group determined by $\mathcal{A}$. Let $\mathcal{H}_{\mathbf{C}} = \{H \otimes \mathbf{C} \colon H \in \mathcal{A}\}$. Let $\mathcal{H}'_{\mathbf{C}}$ be the image of $\mathcal{H}_{\mathbf{C}}$ in the quocient map $q\colon \mathbf{C}^d \to \mathbf{C}^d/W$. Consider now the manifolds: $M(\mathcal{A}) = \mathbf{C}^d \backslash \mathcal{H}_{\mathbf{C}}$, $N(\mathcal{A}) = (\mathbf{C}^d/W) \backslash \mathcal{H}'_{\mathbf{C}}$. The fundamental groups $\pi_1(M(\mathcal{A}))$ and $\pi_1(N(\mathcal{A}))$ are, respectively, *the generalized pure (or coloured) braid group* and *generalized braid group (or Artin group)* determined by the Coxeter arrangement $\mathcal{A}$ [BS]. From the covering $M(\mathcal{A}) \to N(\mathcal{A})$ we deduce the following short exact sequence [Br]:

$$\{1\} \to \pi_1(M(\mathcal{A})) \to \pi_1(N(\mathcal{A})) \to W \to \{1\} \ . \tag{1.1}$$

The center of $\pi_1(N(\mathcal{A}))$ was already calculated: it is an infinite cyclic subgroup [BS,D]. Besides if W is irreducible it is known that its center $Z(W)$ is $\{1\}$ or $\{1, -1\}$ (see [Bo, Ch.V,Sect. 4, exerc.3] or [H, Sect. 6.3, exerc.1]). From these results it is easy to deduce that the center $Z(\pi_1(M(\mathcal{A})))$ is also an infinite cyclic subgroup. Note also that a direct computation of the center of the pure braid groups was done by Chow [Ch] (see also [Bi, Corollary 1.8.4]).

The reader is assumed to have some familiarity with the oriented matroid theory namely the knowledge of the main definitions and results.The best general reference is [BLSWZ].

We assume also some familiarity with the *Salvetti complexes* determined by a real hyperplane arrangement [Sa] and its construction arising from a given oriented matroid [BLSWZ].

We use the notations introduced in [C,CG], a survey of which is given in the next section.

2 – Notations and definitions

Assume $\mathcal{A}$ is an essencial hyperplane arrangement in $\mathbf{R}^d$. Consider the intersection of the hyperplanes of $\mathcal{A}$ with the unit sphere $S^{d-1} \subset \mathbf{R}^d$. This intersection determines a regular cell decomposition Σ of the sphere S^{d-1}. Let $P = P(\mathcal{A})$ be the poset of the closed cells of Σ ordered by inclusion. The poset P determines the regular CW complex Σ up to homeomorphism (see [BLSWZ,Proposition 4.7.8]). Let $H_1, H_2, ..., H_n$ be an ordering of the hyperplanes of $\mathcal{A}$. For every hyperplane H_i we choose positive and negative sides ${H_i}^+$ and ${H_i}^-$, respectively. To every open cell $\sigma \in \Sigma$ we associate a "signed vector" $\omega(\sigma) \in \{+,-,0\}^{\{1,\dots,n\}}$, called *covector* and defined in the following way. Pick up an element x of σ, then: $\omega(\sigma)_i = +,-,0$ if $x \in H^+$, $x \in H^-$, $x \in H$, respectively. The set of the covectors constructed in this way, ordered componentwise according to the relations $0 < +, 0 < -$, is a poset isomorphic to $P(\mathcal{A})$ (and therefore determines the regular CW complex Σ up to homeomorphism). This poset $\mathcal{L} = \mathcal{L}(\mathcal{A})$ is called *the oriented matroid determined by the real hyperplane arrangement* $\mathcal{A}$.

The theory of oriented matroids can be seen as the "right axiomatization" of the posets of covectors of the type $\mathcal{L}(\mathcal{A})$. (Note that in the standard notation a bottom element is adjoined to $\mathcal{L}$.) There are many oriented matroids not corresponding to real hyperplane arrangements.However $\mathcal{L}$ is always the poset of closed cells,ordered by inclusion, of a regular cell decomposition of a sphere (see [BLSWZ],Theorem 5.2.1]).

Using the standard notations we suppose $\mathcal{L}$ adjoined with a bottom element $\widehat{0} = (0,\dots,0)$. $\mathcal{L}$ is a graded poset; its elements of maximal rank are called *topes*. In this paper we consider only loopless oriented matroids without parallel elements. We say that i, $1 \le i \le n$ is a *wall of the tope* T if there is another tope $\widetilde{T}$ such that $T_j = \widetilde{T}_j$ for every j, $j \ne i$, $1 \le j \le n$, and $T_i = -\widetilde{T}_i$. (We are using the notation $-(-) = +$ and $-(+) = -$.) Note that the "signed vector" w such that $w_j = T_j$, $j \ne i$, $1 \le j \le n$, and $w_i = 0$ is a covector of $\mathcal{L}$ of corank 1 covered by T and $\widetilde{T}$. We denote by wall(T) the set of the walls of the tope T. It is known that $|\,\mathrm{wall}(T)| \geq \mathrm{rank}(T)$ [BLSWZ]; in the equality case, T is called a *simplicial tope*. $\mathcal{L}$ is a *simplicial oriented matroid* if all its topes are simplicial ones. If $\mathcal{A}$ is a simplicial hyperplane arrangement then $\mathcal{L}(\mathcal{A})$ is a simplicial oriented matroid.

Now let $\mathcal{M}$ be a (nonoriented) matroid. By definition, *the graph $G_c(\mathcal{M})$ determined by $\mathcal{M}$* is the graph whose vertex set is $E(\mathcal{M})$ and where $\{a,b\}$ is an edge if the line $\overline{\{a,b\}}$ contains at least a third element of E. If $\mathcal{L}$ is an oriented matroid and $\mathcal{M}(\mathcal{L})$ is its underlying nonoriented matroid we set $G_c(\mathcal{L}) = G_c(\mathcal{M}(\mathcal{L}))$. If $\mathcal{A}$ is a hyperplane arrangement, we set by definition $G_c(\mathcal{A}) = G_c(\mathcal{L}(\mathcal{A}))$. A *connected component* of $\mathcal{A}$ is naturally a subarrangement $\mathcal{A}'$ such that $G_c(\mathcal{A}')$ is a connected component of $G_c(\mathcal{A})$.The following result has justified the introduction of the graph G_c [CG]:

2.1 Theorem. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be the connected components of a hyperplane arrangement $\mathcal{A}$. Then*

$$\pi_1(M(\mathcal{A})) \simeq \pi_1(M(\mathcal{A}_1)) \times \ldots \times \pi_1(M(\mathcal{A}_n))\ . \blacksquare$$

We remark that if $\mathcal{A}$ is a simplicial arrangement then $G_c(\mathcal{A})$ is connected if and only if $\mathcal{A}$ is irreducible. This is a consequence of the following useful fact:

2.2 Proposition. *Let $\mathcal{L}$ be a simplicial oriented matroid.Then the following two conditions are equivalent:*

2.2.1. *The graph $G_c(\mathcal{L})$ is connected;*

2.2.2. *The underlying matroid $\mathcal{M}(\mathcal{L})$ is connected.*

Proof: We will prove the non trivial implication $\sim(2.2.1) \Rightarrow \sim(2.2.2)$, by induction on $\mathrm{rank}(\mathcal{M})$.

If $\mathrm{rank}(\mathcal{M}) = 1$ or 2 there is nothing to prove. Suppose the implication true for matroids of rank $< r$ and set $\mathrm{rank}(\mathcal{M}) = r$. Let $X_1 \uplus \ldots \uplus X_n = E(\mathcal{L})$ be the partition of the vertices of $G_c(\mathcal{L})$ corresponding to the connected components. Set $A = X_1$ and $B = X_2 \uplus \ldots \uplus X_n$.

Let H be a hyperplane of $\mathcal{M}(E)$ such that $H \cap A \neq \emptyset$ and $H \cap B \neq \emptyset$. The restriction of $\mathcal{L}$ to the flat H is also a simplicial matroid and then by induction hypothesis we know that $\mathcal{M}(H) = \mathcal{M}(H \cap A) \oplus \mathcal{M}(H \cap B)$. Let C be the covector of $\mathcal{L}$ such that $C_i = +$ if $i \in E \setminus H$ and $C_i = 0$ otherwise.Let T be a tope of $\mathcal{L}$ such that $C \leq T$. Set $W = \mathrm{wall}(T)$. Then clearly $W_1 = W \cap A \supset W \cap (H \cap A) \neq \emptyset$ and $W_2 = W \cap B \supset W \cap (H \cap B) \neq \emptyset$. Suppose $\widetilde{T}$ a tope such that $\widetilde{T}_j = -T_j$ for some $j \in W_1$, and $\widetilde{T}_i = T_i$ if $i \neq j$. Set $\widetilde{W} = \mathrm{wall}(\widetilde{T})$. We claim that $\widetilde{W} \cap B = W_2$. Indeed pick $x \in W_2$ and let X be a covector such that $X \leq T$ and $X_i = X_j = 0$. Set $F = \{i' : X_{i'} = 0\}$. F is a flat of $\mathcal{M}$ and by induction hypothesis we know that $\mathcal{M}(F) = \mathcal{M}(F \cap A) \oplus \cap M(F \cap B)$; as $X \leq \widetilde{T}$ we conclude that $x \in \widetilde{W} \cap B$. As $\mathcal{L}$ is simplicial we also deduce that $|(\widetilde{W} \cap A)| = |W_1|$.

Now let $\widetilde{T}$ be an arbitrary tope of $\mathcal{L}$. It is well known that there is a path of adjacent topes $X_0 = T, \ldots, X_m = \widetilde{T}$ connecting T and $\widetilde{T}$. From the above reasoning we conclude that if $\widetilde{W} = \mathrm{wall}(\widetilde{T})$ then $|(\widetilde{W} \cap A)| = |W_1|$ and $|(\widetilde{W} \cap B)| = |W_2|$. These equalities are only possible if $\mathcal{M}(E) = \mathcal{M}(A) \oplus \mathcal{M}(B)$. ∎

Suppose now that $\mathcal{T}$ is the set topes of an oriented matroid $\mathcal{L}$. *The Salvetti complex* $\Delta_{Sal}(\mathcal{L})$, *determined by* $\mathcal{L}$, is the finite regular CW complex (determined up to homeomorphism) whose poset of closed cells is the set $\{[w, T] \in \mathcal{L} \times \mathcal{T} : w \leq T\}$ with the parcial order $[\widetilde{w}, \widetilde{T}] \preceq [w, T]$ if $w \leq \widetilde{w}$ and $\widetilde{T} = \widetilde{w} \circ T$. By abuse of language, and if no confusion is possible, we denote by the same symbol a geometric realization of the Salvetti complex $\Delta_{Sal}(\mathcal{L})$ and its poset of closed cells.

We have the following nice theorem [Sa]:

2.3 Theorem. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbf{R}^d$. Then the finite regular CW complex $\Delta_{Sal}(\mathcal{L}(\mathcal{A}))$ has the homotopy type of the open manifold*

$$M(\mathcal{A}) = \mathbf{C}^d \setminus \bigcup \{H \otimes \mathbf{C} \colon H \in \mathcal{A}\} \ . ∎$$

We will consider the 1-skeleton $\Delta^{(1)}_{Sal}(\mathcal{L})$ as an oriented graph. Its vertex set might be seen as the set of topes $\mathcal{T}$ of $\mathcal{L}$, and the 1-cells $[w, T]$ as the directed edge with T as its initial vertex. Unless indicated otherwise we will suppose that the initial vertex of a given edge path α of $\Delta^{(1)}_{Sal}(\mathcal{L})$ is a fixed vertex O.Note that if σ is a 2-cell of Δ_{Sal} then there are two minimal length positive edge paths γ_1 and γ_2 in $\Delta^{(1)}$ such that $\gamma_1 \cdot \gamma_2^{-1}$ is the oriented boundary path of σ.

Edge paths are denoted by Greek letters α, β and γ. For any edge path α we denote $\alpha(T)$ the subpath of α ending at the vertex T. We denote by the same letter an edge path and the homotopic equivalence class it determined.By convention the empty path $\emptyset$ is considered a positive closed edge path.

The homotopic equivalence of the edge paths, which we denote $\simeq$ is generated by the following two *"elementary discrete moves"*:

(m_1) *Insert or remove an edge that runs back and forth*;

Let γ_1 and γ_2 be two minimal length positive edge paths such that $\gamma_1 \cdot \gamma_2^{-1}$ is the oriented boundary path of a 2-cell of Δ_{Sal}. Then:

(m_2) *Substitute γ_1 by γ_2 or γ_1^{-1} by γ_2^{-1}.*

Move (m_2) generates an equivalence relation on the set of positive edge paths of $\Delta^{(1)}_{Sal}(\mathcal{L})$; it is denoted by $\overset{+}{\simeq}$ (*positive equivalence*).

The following theorem is the first of the crucial results concerning positive edge paths in the Salvetti complex (see[CM]; for the simplicial and realizable cases see [D] and [Sa], respectively).

2.4 Theorem. *Let α and β be two positive edge paths in $\Delta_{Sal}^{(1)}$ with the same end points, which have minimal length (among all paths with these end points). Then $\alpha \stackrel{+}{\simeq} \beta$.* ■

Using Deligne's notations we denote $u(T,\widetilde{T})$ the positive equivalence class of minimal positive edge paths from T to $\widetilde{T}$. Suppose now that T and $\widetilde{T}$ are two adjacent topes. Set $u(T,\widetilde{T})^t = u(-T,-\widetilde{T})$. Let $u(T,\widetilde{T})^{-1}$ denote the edge path from $\widetilde{T}$ to T such that the edge $u(T,\widetilde{T})$ is traversed along the opposite direction. These notions are eassily extended to edge paths noting that for every edge path $\gamma = \gamma_1 \cdot \gamma_2$, $\gamma^{-1} = \gamma_2^{-1} \cdot \gamma_1^{-1}$ and $\gamma^t = \gamma_1^t \cdot \gamma_2^t$.

By convenience of the calculations we denote by the same symbol ∇ every minimal positive edge path joining two arbitrary opposite topes (i.e. $\nabla = u(T,-T)$, for some tope T). Note that if $u(T,\widetilde{T})$ is a directed edge then

$$u(T,\widetilde{T})^{-1} \simeq \nabla^{-1} \cdot \nabla \cdot u(T,\widetilde{T})^{-1} \simeq \nabla^{-1} \cdot u(-\widetilde{T}) \cdot u(T,\widetilde{T}^{-1}) \simeq \nabla^{-1} \cdot u(-\widetilde{T},T)\ .$$

Note also that

$$u(T,\widetilde{T}) \cdot \nabla^{-1} \simeq u(T,\widetilde{T}) \cdot u(T,\widetilde{T})^{-1} \cdot u(-\widetilde{T},T)^{-1}\ ,$$

$$\nabla^{-1} \cdot u(T,\widetilde{T})^{t} \simeq u(-\widetilde{T},T)^{-1} \cdot u(-T,-\widetilde{T})^{-1} \cdot u(-T,-\widetilde{T})\ ,$$

and then

$$u(T,\widetilde{T}) \cdot \nabla^{-1} \simeq \nabla^{-1} \cdot u(T,\widetilde{T})^{t}\ .$$

Therefore for every edge path α there is a smallest positive integer n such that $\alpha \simeq \nabla^{-n} \cdot \widetilde{\alpha}$, for some positive edge path $\widetilde{\alpha}$.

The following two theorems are the key to problems concerning positive edge paths in the Salvetti complexes determined by simplicial oriented matroids.

The proof of Theorem 2.5 is non trivial but similar to [D, Propositions 1.19 and 1.27] and omitted here (see [Sa2] for a recent and detailed proof).

2.5 Theorem. *Let α, β and γ be positive edge paths of Salvetti complex $\Delta_{Sal}(\mathcal{L})$ determined by a simplicial oriented matroid $\mathcal{L}$.Then:*

2.5.1. $\alpha \cdot \gamma \stackrel{+}{\simeq} \alpha \cdot \beta \Rightarrow \gamma \stackrel{+}{\simeq} \beta$ *(left cancellation);*

2.5.2. $\beta \cdot \alpha \stackrel{+}{\simeq} \beta \cdot \alpha \Rightarrow \gamma \stackrel{+}{\simeq} \beta$ *(right cancellation);*

2.5.3. $\gamma \simeq \beta \Rightarrow \gamma \stackrel{+}{\simeq} \beta$. ■

We also need some information concerning the "ends" of the homotopic equivalence class determined by a positive edge path $\alpha(T)$. We denote by $L = L(\alpha)$ the associate poset on the set $\{\widetilde{T}\colon \widetilde{T}$ tope of $\mathcal{L}$ and $\alpha \simeq \widetilde{\alpha} \cdot u(\widetilde{T}, T)$ for some positive edge path $\widetilde{\alpha}\}$ with the partial order, $T_1 \le T_2$ if $u(T_2, T) \simeq u(T_2, T_1) \cdot u(T_1, T)$. Then [C] (compare [D,Proposition 1.19 iii]):

2.6 Theorem. *$L(\alpha)$ is a lattice.* ∎

3 – Theorem

The following algorithm describes a construction method for the lattice $L(\alpha)$.

Suppose that the bottom and top element of $L(\alpha)$ are respectively T and $\widehat{T}$. Let $\widetilde{T}$ be an atom of $L(\alpha)$ obtained from T crossing the wall i.

Set $D_i(\widetilde{T}) = \{X\colon X$ is a tope of $\mathcal{L}$ and $X_i = \widetilde{T}_i\}$. Then:

3.1 Algorithm. $L(\alpha(\widetilde{T})) \cap D_i(\widetilde{T}) = L(u(\widehat{T}, \widetilde{T}))$.

Algorithm 3.1 can be proved similarly to [D, Algorithme 1.22]. We give here a proof by completeness.

Proof: We will prove

3.1.1. $L(u(\widehat{T}, \widetilde{T})) \subset L(\alpha(\widetilde{T})) \cap D_i(\widetilde{T})$.

Suppose that X belongs to the first member of the inclusion 3.1.1. Note that as $\widehat{T} \in D_i(\widetilde{T})$ we have also $X \in D_i(\widetilde{T})$. We claim that $X \in L(\alpha(\widetilde{T}))$. Indeed we know from the definitions that $\alpha \overset{+}{\simeq} \alpha(X) \cdot u(X, \widetilde{T}) \cdot u(\widetilde{T}, T) \overset{+}{\simeq} \alpha(\widetilde{T}) \cdot u(\widetilde{T}, T)$. Using the right cancellation 2.5.2 we conclude that $\alpha(X) \cdot u(X, \widetilde{T}) \overset{+}{\simeq} \alpha(\widetilde{T})$. Therefore $X \in L(\alpha(\widetilde{T}))$ and the inclusion 3.1.1 follows.

Now we will prove

3.1.2. $L(\alpha(\widetilde{T})) \cap D_i(\widetilde{T}) \subset L(u(\widehat{T}, \widetilde{T}))$.

Suppose that X belongs to the first member of the inclusion 3.1.2.Then

$$\alpha(\widetilde{T}) \simeq \alpha(X) \cdot u(X, \widetilde{T})$$

and

$$\alpha \simeq \alpha(\widetilde{T}) \cdot u(\widetilde{T}, T) \simeq \alpha(X) \cdot u(X, \widetilde{T}) \cdot u(\widetilde{T}, T) \simeq \alpha(X) \cdot u(X, T) \ .$$

Therefore $X \in L(\alpha)$. The implication 2.5.3 entails

$$\alpha(\widehat{T}) \cdot u(\widehat{T}, \widetilde{T}) \cdot u(\widetilde{T}, T) \overset{+}{\simeq} \alpha(\widehat{T}) \cdot u(\widehat{T}, X) \cdot u(X, \widetilde{T}) \cdot u(\widetilde{T}, T) \ .$$

Using the left and right cancellations 2.5.1 and 2.5.2 we conclude that

$$u(\widehat{T},\widetilde{T}) \stackrel{+}{\simeq} u(\widehat{T},X)\cdot u(X,\widetilde{T})$$

and therefore $X \in L(u(\widehat{T},\widetilde{T}))$. ∎

The gist of the proof of the main theorem is the following proposition.

3.2 Proposition. *Let $\Delta_{Sal}(\mathcal{L})$ be the Salvetti complex determined by an irreducible simplicial oriented matroid $\mathcal{L}$.*

Suppose that one of the following conditions holds:

3.2.1. *α is a positive closed edge path with base point O, and for any other positive closed edge path β, with base point O, we have $\alpha\cdot\beta \simeq \beta\cdot\alpha$;*

3.2.2. *α is a positive edge path from $-O$ to O, and for any other positive closed edge path β,with base point O, we have $\alpha\cdot\beta\simeq\beta^t\cdot\alpha$.*

Then $L(\alpha) = L(u(-O,O))$.

To prove the proposition we need the following lemma.

3.3 Lemma. *Suppose the graph $G_c(\mathcal{L})$ connected.*
Let $T,\widetilde{T}$ be two different topes such that $\mathrm{wall}(T) = \mathrm{wall}(\widetilde{T})$.
Then $T = -\widetilde{T}$.

Proof: Suppose by absurd that $T \neq -\widetilde{T}$. Set $A = \{i \in E(\mathcal{L})\colon T_i = -\widetilde{T}_i\} \neq \emptyset$. Note that $(T \neq -\widetilde{T}) \Leftrightarrow (A \neq E(\mathcal{L}))$.

Let F be the closure of $\{\mathrm{wall}(T)\cap A\}$ in the underlying matroid $\mathcal{M}$. From the theory of convexity in oriented matroids [BLSWZ] we know that $F \subset A$ and there is a tope X of $\mathcal{L}$ such that $F = \{i \in E(\mathcal{L})\colon T_i = -X_i\}$.

We claim that $F = A(\Leftrightarrow(X = -\widetilde{T}))$.
Indeed let $Y_0 = T,\dots,Y_j = X,\dots,Y_n = \widetilde{T}$ be a sequence of minimal length of adjacent topes from T to $\widetilde{T}$ using X. Suppose $X \neq \widetilde{T}$ and let i be the wall crossed by the edge $u(Y_{n-1},\widetilde{T})$. Then $i \in \mathrm{wall}(\widetilde{T})$ and the sequence is not minimal, because by hypothesis $\mathrm{wall}(T) = \mathrm{wall}(\widetilde{T})$. Hence A is a closed set of $\mathcal{M}(E)$.

As $\mathrm{wall}(-\widetilde{T}) = \mathrm{wall}(\widetilde{T})$ we have also $\mathrm{wall}(T) = \mathrm{wall}(-\widetilde{T})$. Using the above argument we conclude that $E\setminus A$ is also a closed set of $\mathcal{M}(E)$.

But then no element of A can be connected to an element of $E\setminus A$ in the graph $G_c(\mathcal{L})$, a contradiction. ∎

Proof of Proposition 3.2: Let $\widehat{T}$ be the top element of the lattice $L(\alpha)$. From Lemma 3.3 it is enough to prove $\mathrm{wall}(O) = \mathrm{wall}(\widehat{T})$.

Let X be an arbitrary tope adjacent to O and i be the common wall of X and O such that $O_i = -X_i$. Set $\beta = u(O,X) \cdot u(X,O)$ and let $\widehat{X}$ be the top element of the lattice $L(\alpha \cdot u(O,X))$.

Using Algorithm 3.1 we know that $L(u(\widehat{X},O)) = L(\alpha) \cap D_i(O)$ and then

$$u(\widehat{T},O) \overset{+}{\simeq} u(\widehat{T},\widehat{X}) \cdot u(\widehat{X},O), \;\; u(\widehat{X},X) \overset{+}{\simeq} u(\widehat{X},O) \cdot u(O,X) \; .$$

We conclude that

$$u(\widehat{T},O) \cdot u(O,X) \overset{+}{\simeq} u(\widehat{T},\widehat{X}) \cdot u(\widehat{X},X)$$

and if $\widehat{T} \neq \widehat{X}$ then $u(\widehat{T},\widehat{X})$ is an edge crossing the wall i.

Now, denote by $\widehat{Y}$ the top element of the lattice $L(\alpha \cdot \beta)$. From Algorithm 3.1 we know that $L(u(\widehat{Y},X)) = L(\alpha \cdot u(O,X)) \cap D_i(X)$ and then $\widehat{Y}_i = -\widehat{X}_i$,

$$u(\widehat{Y},O) \overset{+}{\simeq} u(\widehat{Y},X) \cdot u(X,O) \, , \quad u(\widehat{X},X) \overset{+}{\simeq} u(\widehat{X},\widehat{Y}) \cdot u(\widehat{Y},X) \; .$$

We remark that if $\widehat{T} = \widehat{X}$ then $u(\widehat{T},\widehat{Y})$ is an edge crossing the wall i (see the above argument).

From our hypothesis we have $\alpha \cdot \beta \simeq \beta \cdot \alpha$ or $\alpha \cdot \beta \simeq \beta^t \cdot \alpha$. We conclude that $\widehat{T} \in L(\alpha \cdot \beta)$. From the above equivalences we deduce

$$u(\widehat{Y},O) \overset{+}{\simeq} u(\widehat{Y},\widehat{T}) \cdot u(\widehat{T},\widehat{X}) \cdot u(\widehat{X},O) \; .$$

If $\widehat{T} \neq \widehat{X}$ and $\widehat{T} \neq \widehat{Y}$ both the edge pahts $u(\widehat{Y},\widehat{T})$ and $u(\widehat{T},\widehat{X})$ cross the wall i, an impossibity. Then ($\widehat{T} \neq \widehat{X}$ and $\widehat{T} = \widehat{Y}$) or ($\widehat{T} \neq \widehat{Y}$ and $\widehat{T} = \widehat{X}$), and $i \in \mathrm{wall}(\widehat{T})$. As i is an arbitrary wall of the tope O, $\mathrm{wall}(O) \subset \mathrm{wall}(\widehat{T})$. As $\mathcal{L}$ is supposed simplicial, we conclude that $\mathrm{wall}(O) = \mathrm{wall}(\widehat{T})$. ∎

The following result is the "oriented matroid generalization" of Theorem 1.1.

We remember that the elements of the fundamental group $\pi_1(\Delta_{Sal})$ are the homotopic equivalence classes of closed edge paths starting from the base O. We denote by the same letter a closed path and the equivalence class it determines.

3.4 Theorem. *Let $\mathcal{L}$ be an irreducible simplicial oriented matroid and $\Delta_{Sal}(\mathcal{L})$ be the Salvetti complex determined by $\mathcal{L}$.*

Then the center of the fundamental group of $\Delta_{Sal}(\mathcal{L})$ is the infinite cyclic sugroup generated by the equivalence class determined the positive closed edge path $\nabla^2 = u(O,-O) \cdot u(-O,O)$.

Proof: For every closed edge path β we have $\nabla^2 \cdot \beta \simeq \nabla \cdot \beta^t \cdot \nabla \simeq \beta \cdot \nabla^2$ and ∇^2 is an element of the center of $\pi_1(\Delta_{Sal})$. Note that ∇^2 is an element of infinite order [CG, Theorem 4.2].

Suppose now that α is an arbitrary element of the center of $\pi_1(\Delta_{Sal})$. Let n be the smallest positive integer such that $\alpha \simeq \nabla^{-n} \cdot \widetilde{\alpha}$, where $\widetilde{\alpha}$ denotes a positive edge path. Then $\nabla \cdot \widetilde{\alpha}$ or $\widetilde{\alpha}$ is also an element of the center. Suppose that $\beta = \nabla \cdot \widetilde{\alpha}$ [resp. $\beta = \widetilde{\alpha}$] is an element of the center $\neq \emptyset$. Note that in both cases $\widetilde{\alpha} \neq \emptyset$. Then $L(\widetilde{\alpha}) = L(\nabla)$ from Proposition 3.2, and there is a positive edge path γ such that $\widetilde{\alpha} \simeq \gamma \cdot \nabla \simeq \nabla \cdot \gamma^t$. Therefore $\alpha \simeq \nabla^{-n} \cdot \nabla \cdot \gamma^t$ a contradiction with the definition of n. So $\beta = \emptyset$, and $\alpha \simeq \nabla^{2m}$ for some $m \in \mathbf{Z}$. ∎

ACKNOWLEDGEMENT – This work was completed at Institut Mittag-Leffler during the spring term of the program "Combinatorics 1991/92". We gratefully acknowledge their hospitality and support.

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