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SYMMETRIC EQUILIBRIA FOR A BEAM WITH A NONLINEAR ELASTIC FOUNDATION

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Abstract: We study the existence of symmetric solutions of a nonlinear fourth order O.D.E. with nonlinear boundary conditions arising in the theory of elastic beams. Variational methods are used, namely, duality, minimization and mountain pass.

1 – Introduction

This paper is concerned with the study of symmetric solutions of the boundary value problem

$$u^{(iv)} + g(x,u) = 0 \;, \tag{1.1}$$

$$u''(0) = u''(1) = 0 \;, \tag{1.2}$$

$$u'''(0) = -f(u(0)) \quad \text{and} \quad u'''(1) = f(u(1)) \;, \tag{1.3}$$

where $g\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ are real continuous functions with

$$f(s) = 0 \quad \text{iff} \;\; s = 0 \;. \tag{1.4}$$

In order to look for symmetric solutions, i.e. solutions such that

$$u(x) = u(1-x) \;,$$

the function g will be supposed to satisfy

$$g(x,u) = g(1-x,u) \tag{1.5}$$

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for all $u \in \mathbb{R}$ and $x \in [0,1]$. Variational methods will be used throughout. More precisely we shall work in the Hilbert subspace of the Sobolev space $H^2(0,1)$ that consists of all symmetric functions, that is,

$$H_s^2(0,1) = \Big\{u \in H^2(0,1)\colon\ u(x) = u(1-x)\Big\} .$$

In an analogous way we denote by $W_s^{m,p}(0,1)$ and $L_s^p(0,1)$ the subspaces of symmetric functions that belong to $W^{m,p}(0,1)$ and $L^p(0,1)$, respectively.

Thus problem (1.1)–(1.3) becomes

$$(P) \qquad \begin{cases} u^{(iv)} + g(x,u(x)) = 0 \\ u''(0) = 0 \\ u'''(0) = -f(u(0)) \\ u(x) = u(1-x)\,, \qquad x \in [0,1]\ . \end{cases}$$

(Equivalently, one might replace the symmetry condition by the addition of the boundary conditions $u'(\frac{1}{2}) = 0 = u'''(\frac{1}{2})$ relative to the interval $(0,\frac{1}{2})$).

Here we look for solutions of (P) when g and f are either monotone or enjoy some generalized form of monotonicity (see conditions (2.7)–(2.8)). It should be noted that, if $g(x,0) \equiv 0$, then $u = 0$ is a (trivial) solution of (P).

The study of existence and multiplicity of solutions of the original problem (1.1)–(1.3) will appear in a forthcoming paper. There we study the critical points in $H^2(0,1)$ of the functional

$$(1.6) \qquad \varphi(u) = \int_0^1 \left[\frac{u''^2}{2} + G(x,u)\right] dx + F(u(0)) + F(u(1))\ .$$

Our problem is related to the classical bending theory of elastic beams resting on a nonlinear elastic foundation. Let us consider an elastic beam of length 1. Here $g(x,u)$ represents the force exerted on the beam by the nonlinear elastic foundation when indented by the displacement field u. The nonlinear boundary condition (1.3) states that the beam rests on two bearings with an elastic response characterized by the function f. A solution of (1.1)–(1.3) describes then the bending equilibrium state of the beam when it is subjected to a force developed by the foundation and given by the function $g(x,u)$. In fact, in view of (1.1) and (1.3) we have

$$(1.7) \qquad f(u(0)) + f(u(1)) + \int_0^1 g(x,u(x))\,dx = 0\ .$$

We remark that condition (1.2) means that the bending moment at the ends is zero and that the condition (1.4) means that the only position where there is no elastic response of the two bearings is achieved at $u \equiv 0$. We refer the reader to the references [14] and [6] for a physical justification of this model.

Studies on fourth order O.D.E. have been made by several authors and a large literature is available today. We refer for example [1], [7], [13], [15] and their references. In all the above referred papers, only linear boundary conditions are considered. We point out that in E. Feireisl [6] an analogous linear fourth order time periodic equation with nonlinear boundary conditions was studied. He has used the Rayleigh-Ritz approximation method to analyse a problem that concerns the slow oscillations of beams on elastic bearings.

The paper is organized as follows: we end this section with the variational formulation of problem (P) and some remarks on the symmetric properties of the associated linear eigenvalue problem. In section 2, we state a necessary and sufficient condition for the existence of solutions of (P) when f and $g(x,\cdot)$ are both strictly monotone functions, and a result based on a variational reduction method. In section 3, we consider the special case when the function f of the nonlinear boundary condition (1.3) is defined only in an open interval $(a,b) \subset \mathbb{R}$ with $f(s) \to \pm\infty$ as $s \to a, b$. Existence results for this singular case are obtained by a dual variational procedure.

Let $E = H^2_s(0,1)$ be the subspace of $H^2(0,1)$ defined before with norm

$$\|u\|_E^2 = \|u''\|_2^2 + \|u\|_2^2$$

where $\|\cdot\|_p$ denotes $L^p(0,1)$ norm. We consider the functional $J\colon\ E \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_0^1 |u''(x)|^2\,dx + \int_0^1 G(x,u(x))\,dx + 2\,F(u(0)) \tag{1.8}$$

where

$$G(x,u) = \int_0^u g(x,t)\,dt \quad \text{and} \quad F(s) = \int_0^s f(t)\,dt$$

are primitives of g and f, respectively. Then, by the continuity of the functions g and f, J is of class C^1 and weakly lower semicontinuous in E. In addition,

$$J'(u)\phi = \int_0^1 u''(x)\,\phi''(x)\,dx + \int_0^1 g(x,u(x))\,\phi(x)\,dx + 2f(u(0))\,\phi(0) \tag{1.9}$$

for all $\phi \in E$. Of course, by a standard argument of "symmetric criticality", $u \in E$ is a critical point of J if and only if it is a classical solution of the problem (P).

Let us recall some spectral properties of the linear eigenvalue problem

$$(L) \qquad \begin{cases} u^{(iv)} = \lambda u \\ u''(0) = u''(1) = 0 \\ u'''(0) = u'''(1) = 0 \ . \end{cases}$$

It can be found in [6] or [16] that the eigenvalue problem (L) possesses a sequence of eigenvalues (λ_k), $k \geq -1$, such that

$$\lambda_{-1} = \lambda_0 = 0 \quad \text{ and } \quad 0 < \lambda_1 < \lambda_2 < \lambda_3 < ...$$

More precisely for $k > 0$, $\lambda_k = \mu_k^4$ where $(\cosh \mu_k) \cos \mu_k = 1$. In particular, $\lambda_1 \approx 500.6$. The associated eigenfunctions are then

$$\begin{cases} \phi_{-1} = 1, \quad \phi_0 = x - \frac{1}{2} \ , \\ \phi_k(x) = \cosh \mu_k x + \cos \mu_k x - K_k(\sinh \mu_k x + \sin \mu_k x), \qquad \text{for} \quad k > 0 \ , \end{cases}$$

with $K_k = (\cosh \mu_k - \cos \mu_k)/(\sinh \mu_k - \sin \mu_k)$. It is easy to see that ϕ_k is symmetric in $(0,1)$ for $k = -1, 1, 3, \cdots$ and antisymmetric for $k = 0, 2, 4, \cdots$. The eigenfunctions $\{1, \phi_1, \phi_3, \phi_5, \cdots\}$ provide an orthogonal basis for both $L^2_s(0,1)$ and $H^2_s(0,1)$.

Let us consider the orthogonal decomposition $E = Y \oplus W$, where Y is the subspace of the constant functions (and therefore isomorphic to the set of real numbers) and W is the subspace spanned by $\{\phi_1, \phi_3, \phi_5, ...\}$. As notation we put for $u \in E$, $u = t + w$ with $t \in Y$ and $w \in W$. In fact we have

$$t = \int_0^1 u(x)\, dx \quad \text{ and } \quad w = u - t \ .$$

By the variational characterization of the eigenvalue λ_1 , it is clear that for all $w \in W$

$$\int w''^2 \, \geq \lambda_1 \int w^2 \ .$$

This shows that $\|w''\|_2$ is a norm equivalent to $\| \cdot \|_E$ in W.

2 – Symmetric solutions

Let us begin by remarking that, in this section, we always assume that the function g is continuous and satisfies condition (1.5) and f is a continuous function satisfying (1.4), as stated in section 1.

Lemma 1. *Given $v \in L^1_s(0,1)$ and $\alpha \in \mathbb{R}$, there exists $w \in W$ solution of*

$$(W) \qquad \begin{cases} w^{(iv)} = v \\ w''(0) = 0 \\ w'''(0) = \alpha \end{cases}$$

if and only if $2\alpha = -\int v(t)\,dt$.

Proof: Suppose that $2\alpha = -\int_0^1 v(t)\,dt$. Then the function

$$z(x) = \frac{1}{6}\int_0^x (x-t)^3\, v(t)\,dt + \frac{\alpha}{6}\,x^3$$

satisfies $z^{(iv)}(x) = v(x)$, $z''(0) = z''(1) = 0$, $z'''(0) = -z'''(1) = \alpha$. So, putting $\overline{z}(x) = \frac{1}{2}(z(x) + z(1-x))$ then $\overline{z} \in H^2_s(0,1)$ and is a solution of (W). Then we take $w(x) = \overline{z}(x) - \int \overline{z}\,dx$ that belongs to W and satisfies (W). The necessary condition is trivial. ∎

Our first result gives a necessary and suficient condition for the existence of solution of (P) when the functions $g(x,\cdot)$ and f are increasing. This result is based on the convexity of the functional J and properties of the kernel of the associated linear problem. The proof is very close to one given by Mawhin [10]. Note that (1.7) may be seen as a motivation for this theorem.

Theorem 2. *Suppose that $g(x,\cdot)$ and f are increasing functions, $x \in [0,1]$. Then problem (P) has a solution if and only if there exists $a \in Y$ such that*

$$\int_0^1 g(x,a)\,dx + 2f(a) = 0 \;. \tag{2.1}$$

Proof: If $u \in H^2_s(0,1)$ is a solution of (P) then

$$\int_0^1 g(x,u(x))\,dx + 2f(u(0)) = 0 \;. \tag{2.2}$$

Now, as $|u(x)| \le \|u\|_\infty$, by the monotonicity of $g(x,\cdot)$ and f we have

$$\int_0^1 g(x,-\|u\|_\infty)\,dx + 2f(-\|u\|_\infty) \le 0 \le \int_0^1 g(x,\|u\|_\infty)\,dx + 2f(\|u\|_\infty) \;.$$

Therefore by continuity there exists $a \in \mathbb{R}$, $-\|u\|_\infty \le a \le \|u\|_\infty$, that satisfies (2.1). Conversely, suppose that (2.1) holds for some $a \in Y$. We have then three possibilities. Suppose first that for all $t > a$

$$\int_0^1 g(x,t)\,dx + 2f(t) = \int_0^1 g(x,a)\,dx + 2f(a) \tag{2.3}$$

then by monotonicity arguments we have that $g(x,t) = g(x,a)$ and $f(t) = f(a)$ for all $t > a$, $x \in (0,1)$. As $g(\cdot,a)$ is a symmetric function, from the above lemma there exists $w \in W$ satisfying

$$\begin{cases} w^{(iv)} + g(x,a) = 0 \\ w''(0) = 0 \\ w'''(0) = -f(a) \ . \end{cases}$$

Selecting $b > a + \|w\|_\infty$ we have $u(x) = b + w(x) > a$ for all $x \in [0,1]$ and therefore $g(x,u(x)) = g(x,a)$ and $f(u(0)) = f(a)$ for all $x \in [0,1]$. This shows that $u = b+w$ is a solution of (P). Similarily, if (2.3) holds for all $s < a$, problem (P) has a solution as well. The third possibility occurs when there exist constants c_1 and c_2, $c_1 < a < c_2$ such that

$$\int_0^1 g(x,c_1)\,dx + 2f(c_1) < 0 < \int_0^1 g(x,c_2)\,dx + 2f(c_2) \ .$$

In this case, the monotonicity of g and f shows that

$$\int_0^1 G(x,s)\,dx + 2F(s) \to +\infty \quad \text{as} \quad |s| \to +\infty \ . \tag{2.4}$$

Next, we show that J has a minimum. For $u = t + w \in E$,

$$\begin{aligned} J(u) &= \frac{1}{2}\int_0^1 |w''(x)|^2\,dx + \int_0^1 G(x,t+w(x))\,dx + 2F(t+w(0)) \\ &\geq \frac{1}{2}\,\|w''\|_2^2 + \int_0^1 g(x,a)\,(t-a+w)\,dx \\ &\quad + \int_0^1 G(x,a)\,dx + 2f(a)\,(t-a+w(0)) + 2F(a) \ , \end{aligned}$$

where we have used the convexity of $G(x,\cdot)$ and F. Using (2.1) it follows that for some constant $C > 0$

$$J(u) \geq \frac{1}{2}\,\|w''\|_2^2 - \|g(\cdot,a)\|_1\,\|w\|_\infty - 2\,|f(a)|\,\|w\|_\infty - C > -\infty \ . \tag{2.5}$$

Therefore J is bounded below. Let $u_n = t_n + w_n$ be a minimizing sequence of J. We show that u_n has a bounded subsequence, which will guarantee the existence of a minimum of J since it is weakly lower semicontinuous. Suppose by contradiction that $\|u_n\| \to \infty$. Since by (2.5) w_n is bounded then $|t_n| \to \infty$. By convexity of $G(x,\cdot)$ and F we have

$$G(x,u_n) \geq 2G(x,\tfrac{1}{2}t_n) - G(x,-w_n) \ ,$$

$$F(u_n(0)) \geq 2F(\tfrac{1}{2}t_n) - F(-w_n(0)) \ ,$$

$$J(u_n) \geq 2\int_0^1 G(x,\tfrac{1}{2}t_n)\,dx - \int_0^1 G(x,-w_n)\,dx + 4F(\tfrac{1}{2}t_n) - 2F(-w_n(0)) \ .$$

Since w_n is bounded, we see that for some constant $C > 0$,

$$\frac{1}{2} J(u_n) \geq \int_0^1 G(x, \tfrac{1}{2}t_n)\, dx + 2F(\tfrac{1}{2}t_n) - C \;\longrightarrow\; +\infty$$

by (2.4), what yields a contradition. This ends the proof. ∎

Remark 1. If $g(x,\cdot)$ and f are increasing functions satisfying (2.4), Theorem 2 is applicable.

Remark 2. The above result has not an analogous one for the problem (1.1)–(1.3). In fact, the condition corresponding to (2.1) would be: There exists $v \in Y = \mathrm{Span}\{1, x\}$ such that

$$\int_0^1 g(x, v(x))\, v(x)\, dx + f(v(0))\, v(0) + f(v(1))\, v(1) = 0\ . \tag{2.6}$$

Suppose $g \equiv 1$ and $f \equiv 0$. So, g and f are both increasing functions and (2.6) holds for $v(x) = -1 + 2x$. But clearly, there is no solution in $H^2(0,1)$ for the problem $u^{(iv)} = -1$, $u''(0) = u''(1) = u'''(0) = u'''(1) = 0$.

The next result is based on a variational reduction method introduced by Castro [4], Lazer-Landesman-Meyers [8] and used for example by Lazer-McKenna [9]. The main idea is to transform the problem of finding critical points of J in $H_s^2(0,1) = Y \oplus W$ into a problem of finding critical points of a suitable function in Y.

Proposition 3. *Suppose that there exist nonnegative constants β and γ such that for all $u, v \in Y$*

$$(g(x,u) - g(x,v))\,(u - v) \geq -\beta(u - v)^2\ , \tag{2.7}$$

$$(f(u) - f(v))\,(u - v) \geq -\gamma(u - v)^2\ , \tag{2.8}$$

and take $a \in Y$ arbitrarily fixed. Then if $\beta < \lambda_1$ and γ is small enough, the problem

$$(Pa) \quad \begin{cases} w^{(iv)} + g(x, a + w(x)) \in Y \\ w''(0) = 0 \\ w'''(0) = -f(a + w(0)) \end{cases}$$

has a unique solution $w_a \in W$. Moreover, denoting $c_a = w_a^{(iv)} + g(x, a + w_a)$, we have

$$c_a = \int_0^1 g(x, a + w_a(x))\, dx + 2f(a + w_a(0))\ . \tag{2.9}$$

In particular, if for some $a \in Y$, $c_a = 0$, then problem (P) has a solution.

Proof: For a fixed $a \in Y$ we set $\Phi(w) := J(a+w)$ for all $w \in W$. Then from (2.7) and (2.8) $\Phi\colon W \to \mathbb{R}$ is a strictly convex and coercive functional. In fact denoting by $\nabla\Phi(u)$ the unique element of W such that $(\nabla\Phi(u), v) = \Phi'(u)v$ for all $v \in W$, $u \in W$, it follows that

$$\begin{aligned}\Big(\nabla\Phi(w_1) - \nabla\Phi(w_2), w_1 - w_2\Big) &\geq \|w_1 - w_2\|_E^2 - \beta\|w_1 - w_2\|_2^2 - 2\gamma\|w_1 - w_2\|_\infty^2 \\ &\geq \left(1 - \frac{\beta}{\lambda_1}\right)\|w_1 - w_2\|_E^2 - 2\,\gamma\,\sigma^2\,\|w_1 - w_2\|_E^2 \;,\end{aligned}$$

where σ is the constant for $\|w\|_\infty \leq \sigma\|w\|_E$. As $\beta < \lambda_1$, then for γ small enough we get a constant $m > 0$ such that

$$\Big(\nabla\Phi(w_1) - \nabla\Phi(w_2), w_1 - w_2\Big) \geq m\|w_1 - w_2\|_E^2$$

for all $w_1, w_2 \in W$. Using the above mentioned theorem [4] it follows that there exists a unique $w_a \in W$ such that

$$\Phi(w_a) = \min_{w\in W} \Phi(w) = \min_{w\in W} J(a+w)$$

and that the mapping $a \mapsto w_a$ is continuous. In particular, $\Phi'(w_a) = 0$, that is, for all $\phi \in W$,

$$\int_0^1 w_a''(x)\,\phi''(x)\,dx + \int_0^1 g(x, a + w_a(x))\,\phi(x)\,dx + 2f(a + w_a(0))\,\phi(0) = 0 \;.$$

It follows that for all $\phi \in E$

$$\int_0^1 w_a''\phi''\,dx + \int_0^1 g(x, a + w_a)\phi\,dx + 2f(a + w_a(0))\,\phi(0) = \int_0^1 c_a\,\phi\,dx \;,$$

where $c_a = \int_0^1 g(x, a + w_a(x))\,dx + 2f(a + w_a(0))$. This ends the proof. ∎

In order to obtain conditions that imply $c_a = 0$ we begin with a lemma that provides an a priori estimate for solutions of (Pa).

Lemma 4. *Under the hypotheses of Proposition 3, if $w_a \in W$ is a solution of (Pa) then the following estimate holds.*

$$\|w_a\|_\infty \leq C\Big(\|g(\cdot, a)\|_1 + |f(a)|\Big) \tag{2.10}$$

for some constant $C > 0$.

Proof: By integrating $(w_a^{(iv)} + g(x, a + w_a))w_a = c_a w_a$ in $(0,1)$ and using the boundary conditions of problem (Pa) we have

$$\int_0^1 w_a''^2\,dx + \int_0^1 g(x, a + w_a)\,w_a\,dx + 2f(a + w_a(0))\,w_a(0) = 0$$

since $\int_0^1 w_a = 0$. Now using (2.7) and (2.8) it follows that

$$\begin{aligned} m\|w_a\|_E^2 &\le -\int_0^1 g(x,a)\,w_a\,dx - 2f(a)\,w_a(0) \\ &\le \|g(\cdot,a)\|_1\,\|w_a\|_\infty + 2|f(a)|\,\|w\|_\infty\ . \end{aligned}$$

Therefore

$$\frac{m}{\sigma}\,\|w_a\|_\infty \le \Big(\|g(\cdot,a)\|_1 + 2|f(a)|\Big)\ . \ \blacksquare$$

Theorem 5. *Let the assumptions of Proposition 3 hold. Suppose in addition that for all $x \in (0,1)$*

$$\frac{g(x,s)}{s} \to 0 \quad \text{as } |s| \to \infty \text{ uniformly in } x\ , \tag{2.11}$$

$$\frac{f(s)}{s} \to 0 \quad \text{as } |s| \to \infty\ , \tag{2.12}$$

and that one of the following hypotheses holds

(i) *$g(x,s) \to \pm\infty$ as $s \to \pm\infty$, uniformly in x, and f is bounded below;*

(ii) *$f(s) \to \pm\infty$ as $s \to \pm\infty$ and $g(x,s)$ is bounded below.*

Then problem (P) has a solution.

Proof: Using the estimate (2.10), we see by (2.11) and (2.12) that

$$\frac{\|w_s\|_\infty}{s} \le C\left(\frac{\|g(\cdot,s)\|_1}{s} + \frac{|f(s)|}{s}\right) \to 0 \quad \text{as } s \to +\infty$$

and then

$$s + w_s(x) = s\left(1 + \frac{w_s(x)}{s}\right) \to +\infty \quad \text{as } s \to +\infty\ .$$

Therefore, in case (i), $g(x, s + w_s(x)) \to +\infty$ as $s \to +\infty$ and since f is bounded below we have

$$c(s) = \int_0^1 g(x, s + w_s(x))\,dx + 2f(s + w_s(0)) \to +\infty \quad \text{as } s \to +\infty\ . \tag{2.13}$$

Likewise, we have $c(s) \to -\infty$ as $s \to -\infty$. Then the continuity of the function $c(s)$ guarantees the existence of $a \in Y$ such that $c(a) = c_a = 0$. In case (ii) the result comes in an analogous way. ∎

We end this section with two existence results for problem (P) that work for problem (1.1)–(1.3) as well. Here we use standard minimization procedure, by combining the growth of the primitives F and G.

Theorem 6. *Suppose that there exist constants $C_1, C_2 \in \mathbb{R}^+$ and $\epsilon \geq 0$ such that for all $s \in \mathbb{R}$, either*

$$G(x,u) \geq C_1|u|^2 - C_2 \quad \text{and} \quad F(s) \geq -\epsilon|s|^p - C_2 \tag{2.14}$$

or

$$G(x,u) \geq -\epsilon\,|u|^p - C_2 \quad \text{and} \quad F(s) \geq C_1|s|^2 - C_2\ . \tag{2.15}$$

Then the problem (P) has a solution if $1 < p < 2$, or $p = 2$ and ϵ is small enough.

Proof: Under hypothesis (2.14) we have

$$\begin{aligned} J(u) &= \frac{1}{2}\int_0^1 u''^2\,dx + \int_0^1 G(x,u)\,dx + 2F(u(0)) \\ &\geq \frac{1}{2}\,\|u''\|_2^2 + C_1\|u\|_2^2 - 2\epsilon\,|u(0)|^p - 2C_2 \\ &\geq \min\{\tfrac{1}{2}, C_1\}\,\|u\|_E^2 - 2\epsilon\sigma^p\,\|u\|_E^p - 2C_2\ , \end{aligned}$$

where σ is the Sobolev constant used earlier. Then, if $p < 2$ or $p = 2$ and $\epsilon \geq 0$ is small enough, J is coercive and therefore it has a minimum in E.

In case of the condition (2.15) the proof is as before if we remark that $\|u\|_*^2 = \|u''\|_2^2 + |u(0)|^2$ provides a norm in E equivalent to $\|\cdot\|_E$. ∎

3 – The singular problem

In this section we consider the case when f is not defined on the whole of $\mathbb{R}$. More precisely we assume that $f\colon (a,b) \to \mathbb{R}$, $-\infty < a < 0 < b < +\infty$, is continuous, strictly monotone and onto. As for $g\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ it is a continuos function, strictly monotone in its second variable. These hypotheses will be supposed to hold in all that follows. We shall use the dual action method (see [5]).

Suppose that f and $g(x,\cdot)$ are strictly increasing. Then we have that

$$F(t) = \int_0^t f(s)\,ds \quad \text{and} \quad G(x,u) = \int_0^1 g(x,t)\,dt$$

are strictly convex functions. If we consider the respective Fenchel-Legendre transforms, F^* and $G^*(x,\cdot)$, it is known that they satisfy the following relations:

$$\forall x \in (0,1) \quad G^*(x,v) = v\,u - G(x,u) \quad \text{if } \; v = g(x,u) \;, \tag{3.1}$$

$$F^*(s) = s\,t - F(t) \quad \text{if } \; s = f(t) \;. \tag{3.2}$$

We shall denote by $g^*(x,\cdot)$ and f^* the inverse functions of $g(x,\cdot)$ and f, respectively. Then is well known that $F^*(s) = \int_0^s f^*(t)\,dt$ and $G^*(x,v) = \int_0^v g^*(x,t)\,dt$. For a basic reference on duality methods and Fenchel-Legendre transform, we refer the reader to the book of Mawhin and Willem [11].

Lemma 7. *Let $p \geq 1$. Given $v \in L^p_s(0,1)$, there exists an operator $K : L^p_s(0,1) \to H^2_s(0,1)$ such that Kv is the solution of problem (W) where $\alpha = -\frac{1}{2}\int v(x)\,dx$ with $(Kv)(0) = (Kv)(1) = 0$. Moreover*

- **(i)** *There exists $k > 0$ such that $|Kv(x)| \leq k\|v\|_p$ for all $v \in L^p_s(0,1)$ and $x \in (0,1)$;*
- **(ii)** $0 \leq \int_0^1 (Kv)\,v\,dx \leq k\|v\|_p^2$;
- **(iii)** *K is compact.*

Proof: Given $v \in L^p_s(0,1)$ put $\alpha = -\frac{1}{2}\int v(x)\,dx$. Suppose that z_1 and z_2 are two symmetric solutions of (W). Then it follows that $z_1(x) = z_2(x) + cx + d$ for some constants c and d. But as z_1 and z_2 are symmetric, we derive that $c = 0$. So the general symmetric solution for (W) is $z = \overline{z} + d$, where $\overline{z}$ is a symmetric solution and d is a constant. Therefore there is a unique $z \in W^{4,p}_s(0,1)$ solution of (W) with $z(0) = z(1) = 0$ and we define K as $Kv = z$. For the other statements, we note that (i) is a direct consequence of definition of K (see explicit formula in the proof of Lemma 1). (ii) follows from (i) and the fact that $\int (Kv)v = \int (Kv)''^2$. As for (iii), it is a straightforward consequence of the fact that K is continuous with values in $W^{4,p}_s(0,1)$. ∎

Next we define the functional J^* in $L^p_s(0,1)$ as follows

$$J^*(v) = \frac{1}{2}\int_0^1 (Kv)\,v\,dx + \int_0^1 G^*(x,v)\,dx + 2F^*(\alpha_v) \;, \tag{3.3}$$

where Kv is defined in Lemma 7 and α_v is the number defined as $\alpha_v = -\frac{1}{2}\int v\,dx$. In our applications, it will be clear that J^* is a C^1 functional and its derivative is given by

$$J^{*\prime}(v)\,\phi = \int_0^1 (Kv)\,\phi\,dx + \int_0^1 g^*(x,v)\,\phi\,dx + 2f^*(\alpha_v)\,\alpha_\phi \tag{3.4}$$

for all $\phi \in L^p_s(0,1)$.

Lemma 8. *Let v be a critical point of J^*. Then there is a constant c such that $u = c - Kv$ is a solution of (P).*

Proof: Let v be a critical point of J^* and take $\phi \neq 0$ such that $\int \phi = 0$. Then $\alpha_\phi = 0$, and from (3.4),

$$\int_0^1 \Big(Kv + g^*(x,v)\Big)\phi(x)\,dx = 0 \quad \text{for every} \quad \phi \in \Big\{u \in L^p_s(0,1);\ \int_0^1 u\,dx = 0\Big\}\ .$$

Thus by symmetry the same is true for any $\phi \in L^p(0,1)$ such that $\int \phi\,dx = 0$ and then $Kv + g^*(\cdot,v) = c$ for some $c \in Y$. Put $u := c - Kv = g^*(\cdot,v)$. Then it follows easily that u satisfies $u''(0) = 0$ and $u^{(iv)} + g(x,u) = 0$ by using Lemma 1 and (3.1). Now we take $\phi \in L^p_s(0,1)$ such that $-\frac{1}{2}\int \phi\,dx = \alpha_\phi \neq 0$. Then

$$\begin{aligned} 0 &= \int_0^1 \Big(Kv + g^*(x,v)\Big)\phi\,dx + 2f^*(\alpha_v)\,\alpha_\phi \\ &= c\int_0^1 \phi(x)\,dx + 2f^*(\alpha_v)\,\alpha_\phi \\ &= (c - f^*(\alpha_v))\int_0^1 \phi(x)\,dx \end{aligned}$$

that is $f^*(\alpha_v) = c$ and therefore $f(c) = \alpha_v$. But

$$u'''(0) = (c - Kv)''' = -\alpha_v = -f(c)\ .$$

Thus, $u'''(0) = -f(u(0))$. This ends the proof. ∎

Theorem 9. *Suppose that $g(x,\cdot)$ is an increasing function such that*

$$\frac{C_1}{p}\,|u|^p - D_1 \le G(x,u) \le \frac{C_2}{p}\,|u|^p + D_2$$

for $x \in (0,1)$, $u \in \mathbb{R}$, where C_1, C_2, D_1, D_2 are positive constants and $p > 1$. Then:

(i) *If $f\colon (a,b) \to \mathbb{R}$ is increasing, then (P) has a solution.*

(ii) *If $f\colon (a,b) \to \mathbb{R}$ is decreasing, then (P) has a solution if $a = -b$ and f is odd.*

Proof: **(i)** We remark first that $G^*(x,\cdot)$ and F^* are convex. By standard Fenchel-Legendre transform properties, there are constants

$$C_1' = \frac{1}{p'}\,C_1^{\frac{-p}{p'}} \quad \text{and} \quad C_2' = \frac{1}{p'}\,C_2^{\frac{-p}{p'}} \tag{3.5}$$

such that, for $v \in \mathbb{R}$ and $x \in (0,1)$,

$$C_2' \, |v|^{p'} - D_2 \le G^*(x,v) \le C_1' \, |v|^{p'} + D_1 \ . \tag{3.6}$$

Then, since $F^* \ge 0$,

$$J^*(v) \ge C_2' \, \|v\|_{p'}^{p'} - D_2 + 2F^*(\alpha_v) \ge C_2' \, \|v\|_{p'}^{p'} - D_2 \ .$$

Thus, J^* is coercive in $L_s^{p'}(0,1)$. To finish the proof, we note that by compactness of K, convexity of G^* and continuity of F^* it follows that J^* is also weakly lower semicontinuous. So, J^* has a critical point in $L_s^{p'}(0,1)$.

(ii) Put $\overline{f} = -f$, then $\overline{F}$ is strictly convex. We consider the C^1 functional

$$J_1^*(v) = \frac{1}{2} \int_0^1 (Kv)\, v \, dx + \int_0^1 G^*(x,v) \, dx - 2\overline{F}^*(\alpha_v) \ .$$

If $v \in L_s^{p'}(0,1)$ is a critical point of J_1^* then, as in Lemma 8, there is a constant c such that $u = c - Kv$ is a solution of $u^{(iv)} + g(x,u) = 0$, $u''(0) = 0$ but with $u'''(0) = f(-u(0))$. So if f is odd, we have a solution for (P). The existence of a critical point of J_1^* is verified as in (i) replacing F^* by $\overline{F}^*$ and noting that, as $\overline{f}^*$ is a bounded function, we have for $C_3, C_4 > 0$

$$|\overline{F}^*(s)| \le C_3 \, |s| + C_4 \qquad \text{for all} \ \ s \in \mathbb{R} \ . \ \blacksquare \tag{3.7}$$

Theorem 10. *Suppose that $g(x,\cdot)$ is a decreasing function and that*

$$\frac{C_1}{p} \, |u|^p - D_1 \le -G(x,u) \le \frac{C_2}{p} \, |u|^p + D_2 \tag{3.8}$$

for $x \in (0,1)$, $u \in \mathbb{R}$, where C_1, C_2, D_1, D_2 are positive constants, $1 < p < 2$ and, in case $p = 2$, $C_2 < \frac{1}{2k}$ (where k is the constant given by Lemma 7 (ii)). Then:

- **(i)** *If $f\colon (a,b) \to \mathbb{R}$ is increasing, then (P) has a solution if $a = -b$ and f is odd;*
- **(ii)** *If $f\colon (a,b) \to \mathbb{R}$ is decreasing, then (P) has a solution.*

Proof: Let $\overline{g} = -g$, then $\overline{G}$ is strictly convex and from (3.8)

$$\frac{C_1}{p} \, |u|^p - D_1 \le \overline{G}(x,u) \le \frac{C_2}{p} \, |u|^p + D_2 \ .$$

Moreover, (3.6) holds for G^* replaced by $\overline{G}^*$. To prove (i), we consider the C^1 functional $J_2^*\colon L_s^{p'}(0,1) \to \mathbb{R}$ defined by

$$J_2^*(v) = \frac{1}{2} \int_0^1 (Kv)\, v \, dx - \int_0^1 \overline{G}^*(x,v) \, dx + 2F^*(\alpha_v) \ .$$

Proceeding in the same way as in Lemma 8, we see that if $v \in L_s^{p'}(0,1)$ is a critical point of J_2^* then there exists a constant c such that $u = Kv - c$ satisfies $u^{(iv)} + g(x,u) = 0$, $u''(0) = 0$ but with $u'''(0) = f(-u(0))$. So, if f is odd we have a solution of problem (P). Using lemma 7, (3.6) and (3.7) with G^* and $\overline{F}^*$ replaced by $\overline{G}^*$ and F^*, respectively, we have

$$\begin{aligned} -J_2^*(v) &\geq -k\,\|v\|_{p'}^2 + C_2'\,\|v\|_{p'}^{p'} - D_2 - 2F^*(\alpha_v) \\ &\geq -k\,\|v\|_{p'}^2 + C_2'\,\|v\|_{p'}^{p'} - 2C_3\,\|v\|_1 - C \end{aligned}$$

for some $C > 0$. Since $p' > 2$, or because of our hypothesis that, in case $p = 2$, implies $C_2' > k$, $-J_2^*$ is coercive in $L_s^{p'}(0,1)$. Note that $-J_2^*$ is also weakly lower semicontinuous and so it has a critical point. The proof of (ii) is of the same nature of (i). We only note that one must consider the functional

$$J_3^*(v) = \frac{1}{2}\int_0^1 (Kv)\,v\,dx - \int_0^1 \overline{G}^*(x,v)\,dx - 2\overline{F}^*(\alpha_v)$$

where $\overline{F}(s) = \int_0^s \overline{f}(t)\,dt$ with $\overline{f} = -f$. ∎

Our next result gives a nontrivial solution for (P). The assumptions are such that $g(x,0) \equiv 0$. The main tool used here is the well known mountain pass lemma of Ambrosetti and Rabinowitz (see [2] , [12] or [3]).

We use the following *Palais–Smale condition* $(PS)_c$: given $c \in \mathbb{R}$, a functional $J \in C^1(E,\mathbb{R})$ is said to satisfy $(PS)_c$ if whenever $u_n \in E$ is such that $J(u_n) \to c$ and $J'(u_n) \to 0$ then c is a critical value of J.

Theorem (mountain pass lemma). *Let E be a Banach space and $J\colon E \to \mathbb{R}$ a C^1 functional such that*

(i) $J(0) = 0$;

(ii) $\exists \rho > 0$ *and* $r > 0$ *such that* $J(u) \geq \rho$ *if* $\|u\| = r$;

(iii) $\exists z \in E$ *and* $R > r$ *such that* $\|z\| = R$ *and* $J(z) < \rho$.

If, in addition, J satisfies the Palais-Smale condition $(PS)_c$ for every $c \in \mathbb{R}$ then J has a critical value c characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \geq \rho$$

where

$$\Gamma = \left\{ \gamma \in C([0,1],E);\ \gamma(0) = 0 \ \text{ and } \ \gamma(1) = z \right\}.$$

Theorem 11. *Suppose that $g(x,\cdot)$ is a decreasing function and that for some $p>2$*

$$\frac{C_1}{p}\,|u|^p \le -G(x,u) \le \frac{C_2}{p}\,|u|^p \tag{3.9}$$

for $x\in(0,1)$, $u\in\mathbb{R}$, where C_1 and C_2 are positive constants,

$$p\,G(x,u) \ge g(x,u)\,u - C_3 \tag{3.10}$$

for $x\in(0,1)$, $u\in\mathbb{R}$ and some $C_3>0$. Assume that the function f is increasing and

$$\lim_{s\to 0}\frac{s}{f(s)} = 0\ . \tag{3.11}$$

Then problem (P) has a nontrivial solution.

Proof: Let $\overline{g}=-g$ as in Theorem 10. We are going to show that the functional $-J_2^*$ satisfies the statements (i), (ii), (iii) of mountain pass lemma and the $(PS)_c$ condition. Of course $J_2^*(0)=0$. From (3.9) we derive that (using (3.5))

$$C_2'\,|v|^{p'} \le \overline{G}^*(x,v) \le C_1'\,|v|^{p'}\ . \tag{3.12}$$

By (3.11), $\lim_{s\to 0}\frac{f^*(s)}{s}=0$, thus given $\epsilon>0$ there is $\delta>0$ such that if $|s|<\delta$ then $|f^*(s)|<\epsilon|s|$. We have consequently for $|\alpha_v|\le\|v\|_1\le\delta$,

$$F^*(\alpha_v) \le \frac{\epsilon}{2}\,\|v\|_1^2\ .$$

This fact together with (3.12) implies

$$-J_2^*(v) \ge -k\|v\|_{p'}^2 + C_2'\|v\|_{p'}^{p'} - \epsilon\|v\|_1^2 \ge -(k+\epsilon)\|v\|_{p'}^2 + C_2'\|v\|_{p'}^{p'}\ .$$

As $p'<2$, we conclude that $-J_2^*(v)\ge\rho>0$ if $\|v\|_{p'}=r$ with ρ and r small enough. Thus (ii) is verified. Now we observe that $\int_0^1(K1)(x)\,dx>0$; then using (3.12) and the fact that $F^*\ge 0$, for $m\in\mathbb{R}$

$$\begin{aligned}-J_2^*(m) &\le -\frac{1}{2}\int_0^1 m^2(K1)\,dx + \int_0^1 \overline{G}^*(x,m)\,dx - 2F^*\Big(-\frac{m}{2}\Big)\\ &\le \frac{-m^2}{2}\int_0^1 (K1)\,dx + C_1'|m|^{p'}\ .\end{aligned}$$

This shows that $-J_2^* \to -\infty$ if $m \to +\infty$, so (iii) holds. It remains to prove the $(PS)_c$ condition. Let $v_n \in L_s^{p'}(0,1)$ be such that $J_2^*(v_n) \to c$ and $J_2^{*\prime}(v_n) \to 0$. Now denoting $\alpha_n = \alpha_{v_n}$ and M_i constants independent of n,

$$\begin{aligned} M_1 + M_2\|v_n\|_{p'} &\geq -J_2^*(v_n) + \frac{1}{2}\, J_2^{*\prime}(v_n)\, v_n \\ &= \int_0^1 \Big(\overline{G}^*(x,v_n) - \frac{1}{2}\,\overline{g}^*(x,v_n)v_n\Big)dx + \Big(2F^*(\alpha_n) - f^*(\alpha_n)\alpha_n\Big) \ . \end{aligned}$$

From (3.10) and Fenchel-Legendre transform properties, see (3.1), we have

$$p'\,\overline{G}^*(x,v) - \overline{g}^*(x,v)\, v \geq -C_3', \qquad p' < 2 \ .$$

Moreover, the boundedness of f^* and the fact that $F^* \geq 0$ give

$$\Big(1 - \frac{p'}{2}\Big) \int_0^1 \overline{G}^*(x,v_n)\, dx \leq M_3 + M_4\, \|v_n\|_{p'} \ .$$

Since $1 < p' < 2$ the above inequality and (3.12) show that (v_n) is bounded and therefore that there is $v \in L_s^{p'}$ such that (for a subsequence if necessary) $v_n \rightharpoonup v$ weakly in $L^{p'}$ and $\alpha_n \to \alpha$ and by Lemma 7

$$K\, v_n \to K\, v \quad \text{ in } \ C^0 \ .$$

In addition, since given $\phi \in L_s^{p'}(0,1)$, $\alpha_\phi = -\frac{1}{2}\int \phi(x)\, dx$,

$$\int_0^1 \overline{g}^*(x,v_n)\, \phi\, dx = -J_2^{*\prime}(v_n)\, \phi + \int_0^1 (Kv_n)\, \phi\, dx + 2f^*(\alpha_n)\, \alpha_\phi$$

and thus $\overline{g}^*(x,v_n)$ is weakly convergent. We shall show by a standard argument (which we include for completeness) that v is a critical point of J_2^* with $J_2^*(v) = c = \lim J_2^*(v_n)$. Let $\phi \in L^{p'}(0,1)$. Since v_n is bounded,

$$J_2^{*\prime}(v_n)(v_n - \phi) \to 0$$

and by monotonicity of $\overline{g}^*$

$$\begin{aligned} &-J_2^{*\prime}(v_n)\,(v_n - \phi) = \\ &= -\int_0^1 Kv_n(v_n-\phi)\, dx + \int_0^1 \overline{g}^*(x,v_n)\,(v_n-\phi)\, dx - 2f^*(\alpha_n)\,(\alpha_n - \alpha_\phi) \\ &= -\int_0^1 Kv_n(v_n-\phi)\, dx + \int_0^1 \Big[\overline{g}^*(x,v_n) - \overline{g}^*(x,\phi)\Big]\,(v_n-\phi)\, dx \\ &\quad + \int_0^1 \overline{g}^*(x,\phi)\,(v_n-\phi)\, dx - 2f^*(\alpha_n)\,(\alpha_n - \alpha_\phi) \\ &\geq -\int_0^1 Kv_n(v_n-\phi)\, dx + \int_0^1 \overline{g}^*(x,v)\,(v_n-\phi)\, dx - 2f^*(\alpha_n)\,(\alpha_n - \alpha_\phi) \end{aligned}$$

and passing to the limit we get

$$0 \geq -\int_0^1 \Big[Kv(v-\phi) - \overline{g}^*(x,\phi)\,(v-\phi)\Big]\,dx - 2f^*(\alpha_v)\,(\alpha_v - \alpha_\phi)\ .$$

If we make $\phi = v + \lambda w$ in the above inequality ($\lambda > 0$),

$$0 \geq -\int_0^1 \Big[Kv(-\lambda w) - \overline{g}^*(x, v+\lambda w)\,(-\lambda w)\Big]\,dx - 2f^*(\alpha_v)\,(-\lambda\alpha_w)$$

and if we divide by $-\lambda$

$$0 \leq -\int_0^1 \Big[(Kv)\,w - \overline{g}^*(x, v+\lambda w)\,w\Big]\,dx - 2f^*(\alpha_v)\,\alpha_w$$

for all $w \in L_s^{p'}$. Letting $\lambda \to 0$ and using Lebesgue's Theorem we get

$$-\int_0^1 \Big[(Kv)\,w - \overline{g}^*(x,v)\,w\Big]\,dx - 2f^*(\alpha_v)\,\alpha_w \geq 0$$

for all $w \in L_s^{p'}(0,1)$. Replacing w by $-w$, we easily conclude that

$$-\int_0^1 \Big[(Kv)\,w - \overline{g}^*(x,v)\,w\Big]\,dx - 2f^*(\alpha_v)\,\alpha_w \leq 0$$

for all $w \in L_s^{p'}(0,1)$ and therefore,

$$J_2^{*\prime}(v) = 0\ .$$

On the other hand, the convexity of $\overline{G}^*$ implies

$$\int_0^1 \Big[\overline{G}^*(x,v_n) - \overline{G}^*(x,v)\Big]\,dx \geq \int_0^1 \overline{g}^*(x,v)\,(v_n - v)\,dx \longrightarrow 0$$

thus

$$\liminf \int_0^1 \overline{G}^*(x,v_n)\,dx \geq \int_0^1 \overline{G}^*(x,v)\,dx\ .$$

Also,

$$\int_0^1 \Big[\overline{G}^*(x,v) - \overline{G}^*(x,v_n)\Big]\,dx \geq \int_0^1 \overline{g}^*(x,v_n)\,(v-v_n)\,dx =$$
$$= -J_2^{*\prime}(v_n)\,(v-v_n) + \int_0^1 (Kv_n)\,(v-v_n)\,dx + 2f^*(\alpha_n)\,(\alpha_v - \alpha_n)$$

and since the right-hand side tends to zero we have

$$\limsup \int_0^1 \overline{G}^*(x,v_n)\,dx \leq \int_0^1 \overline{G}^*(x,v)\,dx\ .$$

Therefore

$$\int_0^1 \overline{G}^*(x,v_n)\,dx \longrightarrow \int_0^1 \overline{G}^*(x,v)\,dx$$

and then it turns out that $J_2^*(v_n) \to J_2^*(v) = c$. This completes the proof. ∎

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