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THE EULER EQUATION FOR A CLASS OF NONCONVEX PROBLEMS

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Abstract. We study the Euler equation for Neumann and Dirichlet problems associated to nonconvex functionals defined on the space of functions with bounded variation and satisfying a *safe load condition*.

1 – Introduction

Recently, much attention has been devoted to nonconvex variational functionals defined on spaces of discontinuous functions (see References). The reason is that in several models in mathematical physics and engineering (e.g. fracture mechanics, computer vision, liquid crystals) the admissible function variables may have "jumps"; therefore, the function space which seems suitable for this kind of problems is the space of functions with bounded variation.

As a consequence, functionals defined on the space of vector-valued measures with finite total variation have been studied (see for instance Bouchitté & Buttazzo [7], [8], [9] and [1], [2], [3], [6]), together with their lower semicontinuity properties, in order to apply the direct method of the calculus of variations. A complete characterization is now available, and we know that for functionals of the form

$$F(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega\setminus A_\lambda} f^\infty(x, \lambda^s) + \int_{A_\lambda} g\,(x, \lambda(x))\; d\# \;, \tag{1}$$

whose precise meaning will be recalled below, the lower semicontinuity with respect to the weak* convergence in $\mathcal{M}(\Omega; \mathbb{R}^n)$ occurs whenever conditions (5.a), (5.d), (5.e), (5.f) are fulfilled. It is important to remark that the above mentioned conditions do not imply the convexity of F, as it is immediate to see by taking

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$f(x,s) = |s|^2$ and $g(x,s) = |s|^{1/2}$; therefore tools of convex analysis cannot be used in the study of problems where functionals like (1) are involved.

The limit analysis problem of determining the real numbers γ for which the minimum

$$\min\Big\{F(\lambda) - \gamma \int_\Omega H\,d\lambda\colon\ \lambda \in \mathcal{M}(\Omega;\mathbb{R}^n)\Big\} \tag{2}$$

is achieved ($H \in C_0(\Omega;\mathbb{R}^n)$ is a given function) has been studied in Buttazzo & Faina [12], where the existence for (2) is proved for every $\gamma \in]\gamma_*,\gamma^*[$ ("safe load condition") being

$$\gamma_* = -\inf\Big\{F_\infty(\lambda)\colon\ \int_\Omega H\,d\lambda = -1\Big\}\ ,$$

$$\gamma^* = \inf\Big\{F_\infty(\lambda)\colon\ \int_\Omega H\,d\lambda = 1\Big\}\ ,$$

and where F_∞ is the topological recession functional introduced in Baiocchi et al. [5].

In this paper we study the Euler-Lagrange equation for problem (2) under the safe load condition $\gamma_* < \gamma < \gamma^*$ and we use it to find necessary conditions of optimality for variational problems defined on the space BV. The Neumann and the Dirichlet cases are separately considered, and an example in which these conditions are not sufficient is shown.

2 – Notation and position of the Problem

Consider an Hausdorff topological vector space (X,σ) and a functional $G\colon X \to]-\infty,+\infty]$. As usual, set

$$\operatorname{dom} G = \Big\{x \in X\colon\ G(x) < +\infty\Big\}\ ;$$

if $\operatorname{dom} G \neq \emptyset$, the functional G is said to be proper.

If G is proper, its behaviour at infinity can be described in terms of the topological recession function defined by (see Baiocchi & al. [5])

$$G_\infty(x) = \liminf_{(t,y)\to(+\infty,x)} \frac{G(x_0+ty)}{t}\ , \qquad x \in X\ , \tag{3}$$

where x_0 is any element of X.

The function G_∞ is $\sigma-$l.s.c. and positively homogeneous of degree 1. Moreover, it is not difficult to see that the definition of G_∞ does not depend on the choice of $x_0 \in X$.

We use the following notation:

$$\ker G_\infty = \Big\{ x \in X : \; G_\infty(x) = 0 \Big\} \; .$$

The topological recession function can be compared with the classical recession function for convex functionals. In this case, they actually coincide (see Baiocchi & al. [5], Proposition 2.5).

In Buttazzo & Faina [12] we have studied the following limit analysis problem:

Given a functional $F\colon X \to]-\infty, +\infty]$ and a linear σ- continuous functional $L\colon X \to \mathbb{R}$, consider the problem

$$\inf_{x \in X} \left(F(x) - \gamma L(x) \right) \; , \tag{4}$$

where γ is a scalar parameter. We looked for the values of γ for which the infimum in (4) is attained.

Mainly, we studied nonconvex functionals defined on measures.

Before stating the main result of Buttazzo & Faina [12], we introduce the notation we shall use in the following, and we refer to Bouchitté & Buttazzo [7], [8], [9] for further details on functionals defined on measures.

From now on, $(\Omega, \mathcal{B}, \mu)$ will denote a measure space, where Ω is a separable locally compact metric space, $\mathcal{B}$ is the σ-algebra of Borel subsets of Ω, and $\mu : \mathcal{B} \to [0, +\infty[$ is a positive, finite, nonatomic measure.

The following spaces will be considered.

$C_0(\Omega; \mathbb{R}^n)$ The space of all continuous functions $u\colon \Omega \to \mathbb{R}^n$ vanishing at the boundary, that is for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that $|u(x)| < \varepsilon$ for all $x \in \Omega \setminus K_\varepsilon$;

$\mathcal{M}(\Omega; \mathbb{R}^n)$ The space of all vector-valued measures $\lambda\colon \mathcal{B} \to \mathbb{R}^n$ with finite variation on Ω.

It is well-known that $\mathcal{M}(\Omega; \mathbb{R}^n)$ can be identified with the dual space of $C_0(\Omega; \mathbb{R}^n)$ by the duality

$$\langle u, \lambda \rangle = \int_\Omega u \, d\lambda \; .$$

The space $\mathcal{M}(\Omega; \mathbb{R}^n)$ is endowed with the weak* topology deriving from the duality between $\mathcal{M}(\Omega; \mathbb{R}^n)$ and $C_0(\Omega; \mathbb{R}^n)$; in particular, a sequence (λ_h) in $\mathcal{M}(\Omega; \mathbb{R}^n)$ is said to w^*-converge to $\lambda \in \mathcal{M}(\Omega; \mathbb{R}^n)$ (and this is indicated by $\lambda_h \to \lambda$) if and only if

$$\langle u, \lambda_h \rangle \to \langle u, \lambda \rangle \qquad \text{for every} \;\; u \in C_0(\Omega; \mathbb{R}^n) \; .$$

The nonconvex functionals defined on $\mathcal{M}(\Omega; \mathbb{R}^n)$ we consider are of the form

$$F(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_\lambda} f^\infty(x, \lambda^s) + \int_{A_\lambda} g(x, \lambda(x)) \, d\#(x) \; , \tag{5}$$

where $f: \Omega \times \mathbb{R}^n \to [0, +\infty]$, $g: \Omega \times \mathbb{R}^n \to [0, +\infty[$ are Borel functions such that,

(5.a) $f(x, \cdot)$ is a proper, convex, l.s.c. function on $\mathbb{R}^n$, and $f(x, 0) = 0$ for μ – a.e. $x \in \Omega$;

(5.b) there is $\alpha_1 > 0$ and $\beta_1 \in \mathbb{R}$ such that $f(x,s) \geq \alpha_1 |s| - \beta_1$ for every $(x,s) \in \Omega \times \mathbb{R}^n$;

(5.c) $f^\infty(x, \cdot)$ is the recession function of $f(x, \cdot)$;

(5.d) $f^\infty(x,s) = \sup\{u(x) \cdot s : u \in C_0(\Omega; \mathbb{R}^n), \int_\Omega f^*(x,u)\, d\mu < +\infty\}$ on $\Omega \times \mathbb{R}^n$, where $f^*(x,s) = \sup\{s \cdot w - f(x,s) : w \in \mathbb{R}^n\}$;

(5.e) g is l.s.c. on $\Omega \times \mathbb{R}^n$, $g(x, \cdot)$ is subadditive on $\mathbb{R}^n$, and $g(x,0) = 0$ for every $x \in \Omega$;

(5.f) $g^0(x,s) = \lim\limits_{t \to 0^+} \dfrac{g(x,ts)}{t} = f^\infty(x,s)$ on $\Omega \times \mathbb{R}^n$;

(5.g) there exists $\alpha_2 > 0$ with $g^0(x,s) \geq \alpha_2 |s|$ for every $(x,s) \in \Omega \times \mathbb{R}^n$;

(5.h) $\lambda = \dfrac{d\lambda}{d\mu} \mu + \lambda^s$ is the Lebesgue-Nikodym decomposition of λ into absolutely continuous and singular parts with respect to μ;

(5.i) A_λ is the set of all atoms of λ;

(5.j) the meaning of the second term in (5) is in the sense of convex functions over measures, that is

$$\int_{\Omega \setminus A_\lambda} f^\infty(x, \lambda^s) = \int_{\Omega \setminus A_\lambda} f^\infty\left(x, \frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s| \ ;$$

(5.k) $\lambda(x)$ is the value $\lambda(\{x\})$;

(5.l) $\#$ is the counting measure.

Functionals of this form have been first consider by Bouchitté & Buttazzo [7], where the sequential weak* lower semicontinuity on $\mathcal{M}(\Omega; \mathbb{R}^n)$ has been proved.

The limit analysis result for functionals of type (5), proved in Buttazzo & Faina [12] is the following:

Theorem 1. *Let $F: \mathcal{M}(\Omega; \mathbb{R}^n) \to [0, +\infty]$ be the functional defined in (5), and let $H \in C_0(\Omega; \mathbb{R}^n)$. Then, setting*

$$\gamma^* = \inf\Big\{F_\infty(\lambda) : \langle H, \lambda \rangle = 1\Big\} \ ,$$

$$\gamma_* = -\inf\Big\{F_\infty(\lambda) : \langle H, \lambda \rangle = -1\Big\} \ ,$$

the functional $F - \gamma\langle H,\cdot\rangle$ admits at least one minimum point on $\mathcal{M}(\Omega;\mathbb{R}^n)$ for every γ such that

$$\gamma_* < \gamma < \gamma^* \ . \tag{6}$$

The aim of this paper is to give necessary and, whenever possible, sufficient conditions for a measure $\lambda_0 \in \mathcal{M}(\Omega;\mathbb{R}^n)$ to be a minimum for $G = F - \gamma\langle H,\cdot\rangle$.

3 – The Euler equation

The main result concerning optimality conditions for solutions of the limit analysis problems associated to the functional (5) is the following.

Theorem 2. *Let $F\colon \mathcal{M}(\Omega;\mathbb{R}^n) \to [0,+\infty]$ be the functional defined in (5), and let $H \in C_0(\Omega;\mathbb{R}^n)$. If $\lambda_0 \in \mathcal{M}(\Omega;\mathbb{R}^n)$ is a minimum for the functional $G = F - \gamma\langle H,\cdot\rangle$, and if the safe load condition (6) is verified, then λ_0 has no singular part with respect to μ, that is*

$$\frac{d\lambda_0}{d\mu}\,\mu = \lambda_0 \ , \tag{7}$$

and

$$\gamma\, H(x) \in \partial_s f\Big(x, \frac{d\lambda_0}{d\mu}(x)\Big) \qquad \text{for } \ \mu\text{-a.e. } x \in \Omega \ , \tag{8}$$

where $\partial_s f(x,\cdot)$ is the subdifferential of the convex function $f(x,\cdot)$. Furthermore, conditions (7) and (8) are also sufficient for λ_0 to be a minimum for $F - \gamma\langle H,\cdot\rangle$.

Proof: Set $\lambda_0 = \frac{d\lambda_0}{d\mu}\,\mu + \lambda_0^s$, let $\lambda_0^s = \lambda_0^c + \lambda_0^{\#}$, where $\lambda_0^{\#}$ is a purely atomic measure on $\mathcal{M}(\Omega;\mathbb{R}^n)$ and λ_0^c is the diffuse part of λ_0^s (called Cantor part of λ_0).

Let $\beta \in \mathcal{M}(\Omega;\mathbb{R}^n)$ be absolutely continuous with respect to μ, that is

$$\beta \ll \mu \ .$$

Since λ_0 is a minimum for the functional G, we have

$$G(\lambda_0) \le G(\beta + \lambda_0^s) \ ,$$

that is

$$\int_\Omega \Big[f\Big(x, \frac{d\lambda_0}{d\mu}\Big) - \gamma\, H \cdot \frac{d\lambda_0}{d\mu}\Big]\, d\mu \le \int_\Omega \Big[f\Big(x, \frac{d\beta}{d\mu}\Big) - \gamma\, H \cdot \frac{d\beta}{d\mu}\Big]\, d\mu \ .$$

This ensures that

$$\frac{d\lambda_0}{d\mu}(x) \in \operatorname*{argmin}_{s\in\mathbb{R}^n}\Big\{f(x,s) - \gamma\, H(x)\cdot s\Big\}\ .$$

Indeed, in virtue of the safe load condition (6), there is a measurable selection $\overline{u}(\cdot)$ of $\operatorname*{argmin}_{s\in\mathbb{R}^n}\{f(\cdot,s) - \gamma\, H(\cdot)\cdot s\}$ (see Appendix, Theorem 7).

Therefore, by Proposition 6 of Appendix for suitable constants $c > 0$ and $D \geq 0$,

$$f\left(x, \frac{d\lambda_0}{d\mu}(x)\right) - \gamma\, H(x)\cdot \frac{d\lambda_0}{d\mu}(x) \geq f(x,\overline{u}(x)) - \gamma H(x)\cdot\overline{u}(x) \geq c\,|\overline{u}(x)| - D\ .$$

Hence, $\overline{u}$ is a μ-integrable function. Clearly, this implies that

$$\text{(9)}\qquad \frac{d\lambda_0}{d\mu}(x) \in \operatorname*{argmin}_{s\in\mathbb{R}^n}\Big\{f(x,s) - \gamma\, H(x)\cdot s\Big\} \quad \text{for } \mu\text{-a.e. } x\in\Omega\ .$$

Now, since $f(x,\cdot) - \gamma H(x)\cdot(\cdot)$ is convex, we get

$$0 \in \partial_s\left[f\left(x, \frac{d\lambda_0}{d\mu}(x)\right) - \gamma\, H(x)\right] \quad \text{for } \mu\text{-a.e. } x\in\Omega\ .$$

Now, let $\beta \in \mathcal{M}(\Omega;\mathbb{R}^n)$ with $\beta \ll \lambda_0^c$. Again, since λ_0 is a minimum for G, we get

$$G(\lambda_0) \leq G\left(\frac{d\lambda_0}{d\mu}\,\mu + \beta + \lambda_0^{\#}\right)\ ,$$

that is,

$$\int_{\Omega\setminus A_{\lambda_0}} f^\infty(x,\lambda_0^c) - \gamma\int_{\Omega\setminus A_{\lambda_0}} H\,d\,|\lambda_0^c| \leq \int_{\Omega\setminus A_{\lambda_0}} f^\infty(x,\beta) - \gamma\int_{\Omega\setminus A_{\lambda_0}} H\,d\beta\ .$$

Reasoning as before, from Proposition 6 (see Appendix), we get for $|\lambda_0^c|$-a.e. $x \in \Omega\setminus A_{\lambda_0}$

$$\frac{d\lambda_0^c}{d|\lambda_0^c|}(x) \;\in\; \operatorname*{argmin}_{s\in\mathbb{R}^n}\Big[f^\infty(x,s) - \gamma\, H(x)\cdot s\Big] \equiv \{0\}\ .$$

Hence,

$$\lambda_0^c = 0\ .$$

Finally, let $\beta \in \mathcal{M}(\Omega;\mathbb{R}^n)$ be purely atomic, with $A_\beta \subset A_{\lambda_0}$. A straightforward calculation gives for every $x \in A_{\lambda_0}$

$$\lambda_0^{\#}(x) \;\in\; \operatorname*{argmin}_{s\in\mathbb{R}^n}\Big[g(x,s) - \gamma\, H(x)\cdot s\Big] \equiv \{0\}\ ,$$

that is

$$\lambda_0^{\#} = 0 \ .$$

The relation (8) follows from (9). Now, the sufficiency of (7) and (8) is an easy calculus. ∎

As we observed in Buttazzo & Faina [12], the result already obtained for functionals defined on measures allows us to derive au Euler equation for a class of nonconvex functionals defined on BV. More precisely, let $I =]a, b[$ be an open interval of $\mathbb{R}$, and assume that f and g are as in hypotheses (5.a-l).

Denote by $BV(I; \mathbb{R}^n)$ the space of all functions $u \in L^1(I; \mathbb{R}^n)$ with distributional derivative $Du \in \mathcal{M}(I; \mathbb{R}^n)$ and consider the nonconvex functional $F\colon BV(I; \mathbb{R}^n) \to [0, +\infty]$ defined by

$$F(u) = \int_I f(x, \nabla u)\, dx + \int_{I \setminus S_u} f^\infty(x, D^s u) + \int_{S_u} g(x, D^s u(x))\, d\#(x) \ , \tag{10}$$

where ∇u and $D^s u$ respectively denote the absolutely continuous and the singular parts of Du with respect to the Lebesgue measure, and S_u is the set of 'jumps 'of u, that is the set of all points $x \in I$ such that the upper and lower approximate limits $u^+(x)$ and $u^-(x)$ do not coincide.

Setting $\lambda = Du$, the functionals of type (10) can be interpreted in terms of functionals of type (5) on $\mathcal{M}(I; \mathbb{R}^n)$.

The Neumann Problem. We deal with functionals G defined on $BV(I; \mathbb{R}^n)$ by

$$G(u) = F(u) - \gamma \langle L, u \rangle$$

where

$$\langle L, u \rangle = \int_I hu\, dx + \int_I \phi\, Du \ ,$$

with $h \in L^1(I; \mathbb{R}^n)$ and $\phi \in C_0(I; \mathbb{R}^n)$.

As a consequence of Theorem 2 (see also the Appendix and Buttazzo & Faina [12]), we get that, setting $H(x) = \int_a^x h(s)\, ds$, under the safe load condition

$$\left[\inf_{x,s} \left\{ \frac{(\phi(x) - H(x)) \cdot s}{g^\infty(s)} \right\} \right]^{-1} < \gamma < \left[\sup_{x,s} \left\{ \frac{(\phi(x) - H(x)) \cdot s}{g^\infty(s)} \right\} \right]^{-1}$$

a function $u_0 \in BV(I; \mathbb{R}^n)$ is a minimum for G if and only if

$$D^s u_0 \equiv 0 \ ,$$

$$\gamma(\phi(x) - H(x)) \in \partial_s f(x, \nabla u_0(x)) \quad \text{for } \mu\text{-a.e. } x \in I \ .$$

Clearly, if $\phi - H \equiv 0$, the safe load condition reads $-\infty < \gamma < \infty$.

The Dirichlet Problem. In order to deal with the Dirichlet problem associated with functionals of the form (10), it is convenient to consider an open interval I_0, containing I, and the space

$$BV_0 = \Big\{ u \in BV(I_0; \mathbb{R}^n);\ u = 0 \ \text{ on } I_0 \setminus I \Big\} .$$

Therefore, given $h \in L^1(I; \mathbb{R}^n)$ and $\Phi \in C(\overline{I}; \mathbb{R}^n)$, and denoting by $\tilde{h} \in L^1(I_0; \mathbb{R}^n)$ and $\tilde{\phi} \in C_0(I_0; \mathbb{R}^n)$ some extensions of h and ϕ to I_0, we may set for every $u \in BV_0$

$$\langle \tilde{L}, u \rangle = \int_{I_0} \tilde{h} u \, dx + \int_{I_0} \tilde{\phi} \, Du = \int_I h u \, dx + \int_{\overline{I}} \phi \, Du$$

and consider the problem

$$\text{(11)} \qquad \min \Big\{ \int_{I_0} f(x, \nabla u) \, dx + \int_{I_0 \setminus S_u} f^\infty(x, D^s u) + \\ + \int_{S_u} g(x, D^s u(x)) \, d\#(x) - \gamma \langle \tilde{L}, u \rangle : \ u \in BV_0 \Big\} ,$$

where S_u denotes now the set of jumps of u on I_0. Following Buttazzo & Faina [12], if $H \in C_0(I_0; \mathbb{R}^n)$ is such that $H' = h$ a.e. in I, then the Dirichlet problem can be written as

$$\text{(12)} \qquad \min \Big\{ \int_\Omega f\Big(x, \frac{d\lambda}{d\mu}\Big) \, d\mu + \int_{\Omega \setminus A_\lambda} f^\infty(x, \lambda^s) + \int_{A_\lambda} g(x, \lambda(x)) \, d\#(x) - \\ - \gamma \langle \phi - H, \lambda \rangle : \ \lambda \in \mathcal{M}(\Omega; \mathbb{R}^n), \ \ \int_\Omega \lambda = 0 \Big\} ,$$

where $\Omega = \overline{I}$, μ is the Lebesgue measure on $\mathbb{R}$, and λ represents the measure $Du \in \mathcal{M}(\Omega; \mathbb{R}^n)$.

For problem (12) we can not derive a necessary condition as an application of Theorem 2, but we must proceed alternatively. For simplicity, we shall assume that

(13) $\quad f(x, \cdot)$ is differentiable on $\mathbb{R}^n$ for μ-a.e. $x \in \Omega$;

(14) $\quad f^\infty(x, \cdot)$ and $g(x, \cdot)$ are differentiable on $\mathbb{R}^n \setminus \{0\}$ for every $x \in \Omega$;

(15) $\quad$ there are $b_1 \in \mathbb{R}^+$, $a_1 \in L^1(\Omega; \mathbb{R}^n)$ such that
$$|\partial_s f(x, s)| \le a_1(x) + b_1 |s| \quad \text{ for } \ \mu\text{-a.e. } x \in \Omega .$$

Theorem 3. *Let $\lambda_0 \in \mathcal{M}(\Omega; \mathbb{R}^n)$ be a minimum for problem (12), and suppose (13), (14), (15) hold.*

Then, there is a constant vector c such that

$$\gamma(\phi(x)-H(x))+c=\partial_s f\Big(x,\frac{d\lambda_0}{d\mu}(x)\Big) \quad \textit{for } \ \mu\text{-a.e. } x\in\Omega\ , \tag{16}$$

$$\gamma(\phi(x)-H(x))+c=\partial_s f^\infty\Big(x,\frac{d\lambda_0^s}{d|\lambda_0^s|}(x)\Big) \quad \textit{for } \ |\lambda_0^s|\text{-a.e. } x\in\Omega\setminus A_{\lambda_0}\ , \tag{17}$$

$$\gamma(\phi(x)-H(x))+c=g_s(x,\lambda_0^s(x)) \quad \textit{for every } \ x\in A_{\lambda_0}\ . \tag{18}$$

Proof: Set $\lambda_0^s=\lambda_0^c+\lambda_0^\#$, where $\lambda_0^\#$ is a purely atomic measure. Let $\beta\in L^\infty_\mu(\Omega;\mathbb{R}^n)$ be such that $\int_\Omega\beta\,d\mu=0$. Since λ_0 is a minimizer for the functional G defined as in (12), for every $\epsilon>0$ we have

$$G(\lambda_0)\le G(\lambda_0+\epsilon\beta\mu)\ ,$$

that is

$$\frac{1}{\epsilon}\int_\Omega f\Big(x,\frac{d\lambda_0}{d\mu}+\epsilon\beta\Big)-f\Big(x,\frac{d\lambda_0}{d\mu}\Big)\,d\mu\ \ge\ \gamma\int_\Omega(\phi-H)\,\beta\,d\mu\ .$$

This ensures, by the standard first variation procedure, that there is a constant vector c_1 with

$$\gamma(\phi(x)-H(x))+c_1=\partial_s f\Big(x,\frac{d\lambda_0}{d\mu}(x)\Big) \quad \text{for } \ \mu\text{-a.e. } x\in\Omega\ .$$

Now let $\beta\in L^\infty_{|\lambda_0^s|}(\Omega;\mathbb{R}^n)$ with $\int_{\Omega\setminus A_{\lambda_0}}\beta\,d|\lambda_0^s|=0$. Again, since λ_0 is a minimum for G, we get

$$\frac{1}{\epsilon}\int_{\Omega\setminus A_{\lambda_0}} f^\infty(x,\lambda_0^s+\epsilon\beta|\lambda_0^s|)\ \ge\ \frac{1}{\epsilon}\int_{\Omega\setminus A_{\lambda_0}} f^\infty(x,\lambda_0^s)+\gamma\int_{\Omega\setminus A_{\lambda_0}}(\phi-H)\,\beta\,d|\lambda_0^s|\ .$$

Reasoning as before we get the existence of a constant vector c_2 with

$$\gamma(\phi(x)-H(x))+c_2=\partial_s f^\infty\Big(x,\frac{d\lambda_0^c}{d|\lambda_0^c|}(x)\Big) \quad \text{for } \ |\lambda_0^c|\text{-a.e. } x\in\Omega\setminus A_{\lambda_0}\ .$$

Now we handle more sophisticated variations for finding out that actually $c_1=c_2$.

Let $\beta=sv\mu-t\delta_{x_0}$, with $v\in L^\infty_\mu(\Omega;\mathbb{R}^n)$, $\int_\Omega v\,d\mu=\dfrac{t}{s}$, and $x_0\in A_{\lambda_0}$. Since

$$G(\lambda_0)\le G(\lambda_0+\beta)\ ,$$

we get

$$\frac{1}{s}\int_\Omega f\Big(x,\frac{d\lambda_0}{d\mu}+sv\Big)-f\Big(x,\frac{d\lambda_0}{d\mu}\Big)\,d\mu-\gamma\int_\Omega(\phi-H)v\,d\mu\,+$$

$$+\int_\Omega\left[\frac{g\Big(x_0,\lambda_0^s(x_0)-s\displaystyle\int_\Omega v\,d\mu\Big)-g(x_0,\lambda_0^s(x_0))}{s\displaystyle\int_\Omega v\,d\mu}+\gamma(\phi(x_0)-H(x_0))\right]v\,d\mu\ \ge\ 0\ ;$$

thus, by usual first variation procedure,

$$g_s(x_0, \lambda_0^s(x_0)) - \gamma(\phi(x_0) - H(x_0)) + \gamma(\phi - H)(x) = \partial_s f\left(x, \frac{d\lambda_0}{d\mu}(x)\right)$$

for μ-a.e. $x \in \Omega$, and therefore

(19) $$c_1 = g_s(x_0, \lambda_0^s(x_0)) - \gamma(\phi(x_0) - H(x_0)) \quad \text{for every} \quad x_0 \in A_{\lambda_0} .$$

Analogously, by taking $\beta = sv\,|\lambda_0^s| - t\,\delta_{x_0}$, with $v \in L^\infty_{|\lambda_0^s|}(\Omega; \mathbb{R}^n)$, $\int_{\Omega \setminus A_{\lambda_0}} v\, d|\lambda_0^s| = \frac{t}{s}$, and $x_0 \in A_{\lambda_0}$, we get

(20) $$c_2 = g_s(x_0, \lambda_0^s(x_0)) - \gamma(\phi(x_0) - H(x_0)) \quad \text{for every} \quad x_0 \in A_{\lambda_0} .$$

Hence, taking (19) and (20) into account, we get $c_1 = c_2$ and the proof is achieved. ∎

Remark 4. In the scalar case $n = 1$, if g has a special form, we can derive easily quantitative properties about the atoms of the solutions of problem (12), which correspond to jumps in (11).

In fact, if g is independent of x, then in many cases any solution has at most two atoms. Indeed, let λ_0 be a solution of problem (12) and denote by $A^+_{\lambda_0} = \{x \in \Omega\colon \lambda_0(x) > 0\}$ and $A^-_{\lambda_0} = \{x \in \Omega\colon \lambda_0(x) < 0\}$. Let $x_0, y_0 \in \Omega$ be such that $\phi(x_0) - H(x_0) = \sup_{x \in \Omega} [\phi(x) - H(x)]$ and $\phi(y_0) - H(y_0) = \inf_{x \in \Omega} [\phi(x) - H(x)]$.

Setting

$$\tilde{\lambda} = \Big[\textstyle\sum_{x \in A^+_{\lambda_0}} \lambda_0(x)\Big]\, \delta_{x_0} + \Big[\textstyle\sum_{x \in A^-_{\lambda_0}} \lambda_0(x)\Big]\, \delta_{y_0} + \lambda_0 - \lambda_0^{\#} \ ,$$

it results

$$G(\tilde{\lambda}) = \int_\Omega f\left(x, \frac{d\lambda_0}{d\mu}\right) d\mu + \int_{\Omega \setminus A_{\lambda_0}} f^\infty(x, \lambda_0^s) + g\Big(\textstyle\sum_{x \in A^+_{\lambda_0}} \lambda_0(x)\Big) +$$
$$+ g\Big(\textstyle\sum_{x \in A^-_{\lambda_0}} \lambda_0(x)\Big) - \gamma(\phi(x_0) - H(x_0)) \cdot \sum_{x \in A^+_{\lambda_0}} \lambda_0(x) - \gamma(\phi(y_0) - H(y_0)) \cdot \sum_{x \in A^-_{\lambda_0}} \lambda_0(x) \ .$$

In force of the subadditivity of g and the definition of x_0 and y_0, we have

$$G(\tilde{\lambda}) < G(\lambda_0) \ ,$$

whenever either g is strictly subadditive, i.e.

$$g(s_1 + s_2) \le g(s_1) + g(s_2) \qquad \forall\, s_1, s_2 \in \mathbb{R}, \quad s_1, s_2 > 0 \ ,$$

or $\phi - H$ has unique minimum and maximum points on Ω. This contradiction proves that λ_0 can not have more than two atoms.

Further, if $g(x,s) = c|s| + M$ for every $(x,s) \in \Omega \times \mathbb{R} \setminus 0$, $g(x,0) = 0$ for every $x \in \Omega$, the function $\phi - H$ has unique minimum and maximum points on Ω, and the following safe load condition holds

$$\text{(21)} \quad \left[\inf\left\{\frac{\langle \phi - H, \lambda\rangle}{\int_\Omega g^\infty(\lambda)} : \int_\Omega \lambda = 0\right\}\right]^{-1} < \gamma < \left[\sup\left\{\frac{\langle \phi - H, \lambda\rangle}{\int_\Omega g^\infty(\lambda)} : \int_\Omega \lambda = 0\right\}\right]^{-1},$$

then any solution of problem (12) has at most one atom. To this end, we assume that λ_0 is a solution of problem (12) with exactly two atoms, at x_1 and x_2, where $\lambda_0(x_1) > 0$ and $\lambda_0(x_2) < 0$. From the safe load condition (21), we derive easily that

$$\text{(22)} \quad |\phi - H|_{C_0(\Omega;\mathbb{R}^n)} < \frac{c}{|\gamma|} ;$$

while, from the Euler equation (18), we have

$$\text{(23)} \quad c\left[\frac{\lambda_0(x_1)}{|\lambda_0(x_1)|} - \frac{\lambda_0(x_2)}{|\lambda_0(x_2)|}\right] = \gamma\Big[(\phi - H)(x_1) - (\phi - H)(x_2)\Big] .$$

Putting together (22) and (23), we get

$$2\,c = \gamma\Big[(\phi - H)(x_1) - (\phi - H)(x_2)\Big] < |\gamma|\,\frac{2c}{|\gamma|} = 2c ,$$

which leads to a contradiction.

Example. We would like to underline that conditions (16), (17), and (18) are not sufficient for a $\lambda_0 \in \mathcal{M}(\Omega;\mathbb{R}^n)$ to be a minumum for G.

Indeed, let $F\colon BV([0,1];\mathbb{R}) \to [0,+\infty]$ be the functional defined by

$$F(u) = \int_0^1 |\nabla u|^2\,dx + \int_{S_u} (1 + |Du^s(x)|)\,d\#(x) ,$$

and consider the problem

$$\text{(24)} \quad \min\Big\{F(u)\colon\ u \in BV([0,1];\mathbb{R}),\ \int_0^1 Du = k\Big\} ,$$

with $k \in \mathbb{N}$.

From a straightforward application of Theorem 3, we get the following Euler equations for problem (24),

$$\text{(25)} \quad 2\,\nabla u(x) = c \quad \text{for a.e. } x \in (0,1)$$

$$\text{(26)} \quad \operatorname{sgn}\Big(\frac{d\,D^s u}{d|D^s u|}(x)\Big) = c \quad \text{for every } x \in S_u ,$$

for a suitable constant c, whereas the Cantor part $D^c u$ is zero due to the superlinear growth of f.

The function $u_0(t) = kt$, $t \in [0,1]$, satisfies the Euler equations (25) and (26), but it is not a solution for problem (24). In fact, let

$$u_1(t) = \begin{cases} \dfrac{t}{2} & \text{if } \ 0 \le t < 1 \\ k & \text{if } \ t = 1 \ ; \end{cases}$$

it results $F(u_0) = k^2 > F(u_1) = \frac{3}{4} + k$, for k sufficiently large ($k > \frac{3}{2}$).

From Remark 1 it is easy to verify that u_1 is actually a solution for problem (24).

4 – Appendix

This Appendix is devoted to the study of some implications of the safe load condition (6). The notations are those of Sections 2 and 3.

We start with a measurable selection theorem that will be useful to determine the Euler equations for functionals of type (5).

Following the proof of Lemma 1.1 and Theorem 1.2 in Ekeland & Temam [17], chapter VIII, we can prove the following selection result.

Theorem 5. *Let $f\colon \Omega \times \mathbb{R}^n \to]-\infty, +\infty]$ be a Borel function such that $f(x,\cdot)$ is l.s.c. for μ-a.e. $x \in \Omega$ and assume that*

$$f(x,s) \ge c|s| - D \ \text{ with } c > 0,\ D \in \mathbb{R}, \text{ for every } (x,s) \in \Omega \times \mathbb{R}^n .$$

Then there is a measurable function $\tilde u\colon \Omega \to \mathbb{R}^n$ such that for μ-a.e. $x \in \Omega$, $\tilde u(x) \in \operatorname{argmin}_{s\in\mathbb{R}^n} f(x,s)$, that is

$$f(x,\tilde u(x)) = \min_{a\in\mathbb{R}^n} \{f(x,a)\} .$$

Now we get some coercivity properties as a consequence of the safe load condition (6).

Proposition 6. *If the safe load condition (6) holds, then there is a $c > 0$ such that*

$$g^\infty(x,s) - \gamma\, H(x)\cdot s \ge c|s| \qquad \text{for every } \ (x,s) \in \Omega \times \mathbb{R}^n \ , \tag{27}$$

where $g^\infty(x,s) = \lim_{t\to+\infty} \dfrac{g(x,ts)}{t}$.

Further, from (27), it follows the existence of two positive constants c_1, c_2 *and a* $D \in \mathbb{R}$ *such that*

(28) $$f^\infty(x,s) - \gamma\, H(x)\cdot s \geq c_1\,|s| \quad \textit{for every} \;\; (x,s)\in\Omega\times\mathbb{R}^n\ ,$$

(29) $$f(x,s) - \gamma\, H(x)\cdot s \geq c_2\,|s| - D \quad \textit{for every} \;\; (x,s)\in\Omega\times\mathbb{R}^n\ .$$

Proof: Following the proof of Theorem 4.4 of Buttazzo & Faina [12], it is easy to verify that

$$F_\infty(\lambda) \leq \int_\Omega g^\infty(x,\lambda) = G^\infty(\lambda) \quad \text{for every} \;\; \lambda\in\mathcal{M}(\Omega;\mathbb{R}^n)\ \ .$$

Therefore,

$$-\inf\Bigl\{\int_\Omega g^\infty(x,\lambda)\colon\ \langle H,\lambda\rangle = -1\Bigr\} < \gamma < \inf\Bigl\{\int_\Omega g^\infty(x,\lambda)\colon\ \langle H,\lambda\rangle = 1\Bigr\}$$

or equivalently,

$$\frac{1}{\inf\{\langle H,\lambda\rangle\colon\ \int_\Omega g^\infty(x,\lambda)=1\}} < \gamma < \frac{1}{\sup\{\langle H,\lambda\rangle\colon\ \int_\Omega g^\infty(x,\lambda)=1\}}\ .$$

By using the definition of polar function, it is easy to see that

$$\frac{1}{\sup\{\langle H,\lambda\rangle\colon\ \int_\Omega g^\infty(x,\lambda)=1\}} = \sup\Bigl\{t\colon\ (G^\infty)^*(tH) = 0\Bigr\}\ .$$

Therefore, being

$$(G^\infty)^*(w) = \begin{cases} 0 & \text{if } [g^\infty(x,\cdot)]^*(w)\equiv 0 \\ +\infty & \text{otherwise}\ , \end{cases}$$

we obtain

$$\frac{1}{\sup\{\langle H,\lambda\rangle\colon\ \int_\Omega g^\infty(x,\lambda)=1\}} = \left[\sup_{x,s}\frac{H(x)\cdot s}{g^\infty(s)}\right]^{-1}.$$

Now relation (27) follows easily. Relation (28) follows directly from (27) since $g^\infty(x,s)\leq g^0(x,s) = f^\infty(x,s)$ for every $(x,s)\in\Omega\times\mathbb{R}^n$ (see Proposition 4.2 in Buttazzo & Faina [12], and Bouchitté & Buttazzo [7]). It is left to prove (29).

One can prove that a sufficient condition for obtaining (29) is the following:

$$\liminf_{|s|\to+\infty}\frac{f(x,s)-\gamma H(x)\cdot s}{1+|s|} > \frac{c_1}{2} \quad \text{for every} \;\; x\in\Omega\ .$$

Assume that there is a $\tilde{x}\in\Omega$ with

$$\liminf_{|s|\to+\infty}\frac{f(\tilde{x},s)-\gamma\, H(\tilde{x})\cdot s}{1+|s|} \leq \frac{c_1}{2}\ .$$

Then, for every $\epsilon > 0$ ($\epsilon < \frac{c_1}{2}$) there is a $s_\epsilon \in \mathbb{R}^n$ with $|s_\epsilon| > \frac{1}{\epsilon}$, such that

$$\frac{f(\tilde{x}, s_\epsilon) - \gamma\, H(\tilde{x}) \cdot s_\epsilon}{1 + |s|} < \frac{c_1}{2} + \epsilon \ .$$

We may assume that $\frac{s_\epsilon}{|s_\epsilon|}$ converges to $w \in \mathbb{R}^n$ with $|w| = 1$. Therefore,

$$0 < c_1 \le f^\infty(\tilde{x}, w) - \gamma H(\tilde{x}) \cdot w \le \liminf_{|s_\epsilon| \to +\infty} \frac{f(\tilde{x}, s_\epsilon) - \gamma H(\tilde{x}) \cdot s_\epsilon}{|s_\epsilon|} \le \frac{c_1}{2} \ ,$$

that leads to a contradiction. ∎

We collect together the results we have obtained and we get,

Theorem 7. *If the safe load condition (6) holds, then there exists a measurable selection of*

$$\mathcal{H}(x) = \operatorname*{argmin}_{s \in \mathbb{R}^n} \Big\{ f(x, s) - \gamma H(x) \cdot s \Big\} \,, \qquad x \in \Omega \ .$$

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REFERENCES

[**1**] Ambrosio, L. – Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.*, 111 (1990), 291–322.

[**2**] Ambrosio, L. and Braides, A. – Functionals defined on partitions in sets of finite perimeter, I and II, *J. Math. Pures et Appl.*, 69 (1990), 285–333.

[**3**] Ambrosio, L. and Dal Maso, G. – On the relaxation in $BV(\Omega; \mathbb{R}^m)$ of quasiconvex integrals, *J. Funct. Anal.,* Trieste, 1991.

[**4**] Anzellotti, G. – The Euler equation for functionals with linear growth, *Trans. Amer. Math. Soc.*, 290 (1985), 483–501.

[**5**] Baiocchi, C., Buttazzo, G., Gastaldi, F. and Tomarelli, F. – General existence theorems for unilateral problems in Continuum Mechanichs, *Arch. Rational Mech. Anal.*, 100 (1988), 149–189.

[**6**] Bouchitté, G., Braides, A. and Buttazzo, G. – Relaxation results for some free discontinuity problems, *J. Reine Angew. Math.* (to appear).

[**7**] Bouchitté, G. and Buttazzo, G. – New lower semicontinuity results for nonconvex functionals defined on measures, *Nonlinear Anal.*, 15 (1990), 679–692.

[8] Bouchitté, G. and Buttazzo, G. – Integral representation of nonconvex functionals defined on measures, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9 (1992), 101–117.
[9] Bouchitté, G. and Buttazzo, G. – Relaxation for a class of nonconvex functionals defined on measures, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* 10 (1993), 345–361.
[10] Bouchitté, G. and Suquet, P. – Equi-coercivity of variational problems: The role of recession functions (to appear).
[11] Buttazzo, G. – Semicontinuity, Relaxation and Integral Representation in the Calculus of Variation, *Pitman Res. Notes Math. Ser.*, 207, Longman, Harlow (1989).
[12] Buttazzo, G. and Faina, L. – Limit analysis for a class of nonconvex problems, *Proc. Royal Soc. Edinburgh* (to appear).
[13] Clarke, F.H. – The Euler-Lagrange differential inclusion, *J. Differential Equations*, 19 (1975), 80–90.
[14] Clarke, F.H. and Winter, R.B. – Regularity properties of solutions to the basic problem in calculus of variations, *Trans. Amer. Mat. Soc.*, 289 (1985), 73–98.
[15] Dal Maso, G. – *An introduction to* Γ*-convergence*, Birkhäuser, Boston (to appear).
[16] De Giorgi, E. and Franzoni, T. – Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.*, 58(8) (1975), 842–850.
[17] Ekeland, I. and Temam, R. – *Convex Analysis and Variational Problems*, North-Holland, New York, 1976.
[18] Goffman, C. and Serrin, J. – Sublinear functions of measures and variational integrals, *Duke Math. J.,* 31 (1964), 159–178.
[19] Rockafellar, R.T. – *Convex Analysis*, Princeton Univ. Press, Princeton, 1970.

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