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EXACT CONTROLLABILITY OF VIBRATIONS OF THIN BODIES

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Abstract: In this paper, we address the problem of exact controllability of the wave equation in three dimensional domains which are thin in one direction. We prove the existence of exact controls and analyze their asymptotic behaviour as thickness parameter goes to zero. We characterize their limit as the solution of an exact controllability problem in two dimension.

1 – Introduction

In this paper, we consider the vibrations of three-dimensional elastic bodies which are thin in one direction say that of the x_3-axis. Let $e > 0$ be the thickness parameter of the body in that direction. We are interested in small values of e. The boundary of the body is divided into three disjoint pieces: the lateral part and the top-bottom surfaces. The system which models the vibrations of this body is described in the next section along with other notations. For now, it suffices to mention that this is an initial boundary value problem with mixed boundary condition. As usual, we impose Dirichlet condition on the lateral part while Neumann condition is taken on top-bottom surfaces. We address the question of exact controllability of these vibrations by acting on the boundary of the body. More precisely, we look for suitable controls acting through the boundary conditions mentioned above and a finite time T such that these vibrations are killed at time T. In this paper, we study the following two aspects:

i) Existence of exact controls and time of controllability for each $e > 0$;

ii) Behaviour of controls as $e \to 0$.

The problem of exact controllability for distributed systems has been studied extensively by Lions [4] in a very general set-up. The method introduced in this work to attack the problem is the so-called Hilbert Uniqueness Method (HUM)

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combined with the method of multipliers. Our plan in this paper is to follow HUM but of course with certain modifications as indicated below.

First, let us say few words about the earlier study of thin three-dimensional bodies. There is a vast literature on the movement of thin elastic bodies under a given force field (cf. Ciarlet and Destuynder [1]). The typical result one gets is the following: when $e \to 0$ the solution converges to that of a two-dimensional problem called plate problem. This is a singular limit in the sense that the limit equation is of order four while the original system of equations of elasticity in three dimensions is of order two. The corresponding result for the vibrating bodies have been obtained by Raoult [6], [7] and Ciarlet and Kesavan [2]. The limit is again singular because we pass from a system of hyperbolic equations in three dimensions to a scalar dispersive equation in two dimensions.

We now discuss the model studied in this paper. We do not consider the system of linear elasticity. We treat, following Lions [5], p. 193, a simpler model namely the classical wave equation in thin domains in three-dimensions. The limit is of course two dimensional wave equation when $e \to 0$. Thus the passage is from hyperbolic to hyperbolic equation and hence regular.

It is now time to comment on the nature of the results obtained and the techniques followed. As mentioned above, we follow the general lines of HUM. However there are modifications. First of all, our domain is not smooth neither convex. Secondly, we have mixed boundary conditions. Thus there is a lack of regularity of solution in such domains. This poses in general serious difficulties in the method of multipliers because the argument involves a regularity result plus a density argument. See Lions [4], p. 179–180. However, in the present case, the geometry of the body and the boundary condition are such that we can apply the regularity results of Grisvard [3] and proceed with HUM. In the sequel, we do this in such a way that the behaviour of the solution as $e \to 0$ is clearly brought out. In particular, we need to choose the multipliers which are more suitable to thin bodies which are of interest to us in this paper. The usual multipliers (Lions [4], p. 29–31), do not seem to serve the purpose. We need to introduce suitable normalizing factor involving e in the nonhomogeneous backward problem. With a view to obtain a two-dimensional problem at the limit, we restrict the usual multiplier used to obtain the so-called inverse inequality. With all these preparations, we are able to prove the existence of exact controls which exist on a part of the lateral boundary of the body as well as on the entire top-bottom surfaces.

Our approach gives not only existence but also estimates on the controls. We use these estimates in analyzing the behaviour as $e \to 0$. First of all, it is established that the minimal time T_e of exact controllability is bounded above independently of e. We fix one time T (independent of e) at which we have exact controllability and work with it subsequently.

Since the problem is linear, weak convergence is enough to pass to the limit. The task is to identify the limit. This requires the use of suitable test functions to thin bodies.

Our results show that the controls on the top-bottom surfaces of the body tend to zero in a suitable sense. The lateral control becomes two-dimensional at the limit. This limiting control is characterized as a boundary control in a problem of exact controllability in two dimensions, in which there is also a control in the entire interior of the domain. This interior control is due to the presence of boundary controls on top-bottom surfaces. (Let us repeat that there were only boundary controls and no interior control in the original problem in three dimensions). Because of the presence of the entire interior control, the limiting problem is exactly controllable for *all* times $\widetilde{T} > 0$ even though the equation is hyperbolic (cf. Lions [4]). This seems to indicate that the minimal times $T_e \to 0$ as $e \to 0$; however we do not quite prove this. As mentioned earlier we fix one time $T > 0$ of exact controllability and show how to pass to the limit. In the case of elasticity systems, identification of the limit will be more difficult as it requires other test functions. We would like to point out one additional phenomenon which does not exist in the case of wave equation. Since the limiting equation is dispersive in the case of elasticity systems, we expect that the minimal times $T_e \to 0$ as $e \to 0$ even without the boundary controls on the top-bottom surfaces in the three-dimensional model. We plan to analyze all these in our subsequent publications.

This article is organized as follows: we introduce the notations and pose the problem in the next section. Following HUM, we consider the associated problem with homogeneous boundary condition in §3. Several estimates in the form of energy inequality, direct and inverse inequalities on this problem are derived in the subsequent sections §4, §5, §6. The exact controllability problem is then solved by the introduction of backward Cauchy problem and the operator Λ. This is done in §7 and §8. The behaviour of its solution as $e \to 0$ is analyzed in later sections. The main result is stated in Theorem 8.5. Summation convention with respect to repeated indices is used unless stated otherwise. Following the standard practice in the analysis of thin bodies, Greek indices α, β, etc. take values in $\{1, 2\}$ and Latin indices i, j, k, etc. take values in $\{1, 2, 3\}$. The same problem has been studied by Yan [8] about which we comment in the last section.

2 – Notations and problem to be studied

The thin three-dimensional body whose vibrations interest us is constructed as follows: let ω be a bounded open set in $\mathbb{R}^2$ whose smooth boundary is denoted as γ. It is not assumed that ω is convex. Given the thickness parameter $e > 0$, we let

$$
\begin{aligned}
\Omega^e &= \widetilde{\omega} \times \,]-\tfrac{e}{2}, \tfrac{e}{2}[\ , \\
\Gamma_0^e &= \gamma \times [-\tfrac{e}{2}, \tfrac{e}{2}]\ , \\
\Gamma_+^e &= \widetilde{\omega} \times \{\tfrac{e}{2}\}\ , \\
\Gamma_-^e &= \omega \times \{-\tfrac{e}{2}\}\ ,
\end{aligned} \tag{2.1}
$$

so that the boundary Γ^e of Ω^e is partitioned into three sets Γ_0^e (lateral part), Γ_+^e (the upper portion) and Γ_-^e (the lower portion):

$$\Gamma^e = \Gamma_+^e \cup \Gamma_-^e \cup \Gamma_0^e\ .$$

As mentioned in the introduction, we take control on a part of the lateral boundary Γ_0^e apart from the entire top-bottom boundaries $\Gamma_\pm^e$. The required part of Γ_0^e is taken as follows because of the existence of good multipliers:

Let x^0 be a point with $x_3^0 = 0$. We define

$$
\begin{aligned}
m(x) &= x - x^0\ , \\
\gamma(x^0) &= \Big\{x \in \gamma\colon\ m(x) \cdot \nu(x) > 0\Big\}\ ,
\end{aligned} \tag{2.2}
$$

where $\nu(x)$ the unit exterior normal to Γ^e. We next set

$$
\begin{aligned}
\gamma_* &= \gamma \backslash \gamma(x^0)\ , \\
\Gamma^e(x^0) &= \gamma(x^0) \times [-\tfrac{e}{2}, \tfrac{e}{2}]\ , \\
\Gamma_*^e &= \Gamma_0^e \backslash \Gamma^e(x^0)\ .
\end{aligned} \tag{2.3}
$$

We take Dirichlet control only on $\Gamma^e(x^0)$ and Neumann control on $\Gamma_\pm^e$. More precisely, we fix $T > 0$ and consider the following initial boundary value problem for the wave equation in Ω^e:

$$\frac{\partial^2 y}{\partial t^2} - \left(\frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2} + \frac{\partial^2 y}{\partial x_3^2}\right) = 0 \quad \text{in } Q^e\ , \tag{2.4a}$$

$$y = v \quad \text{on } \Sigma^e(x^0)\ , \tag{2.4b}$$

$$y = 0 \quad \text{on } \Sigma_*^e\ , \tag{2.4c}$$

$$\frac{\partial y}{\partial \nu} = w_\pm \quad \text{on } \Sigma_\pm^e\ , \tag{2.4d}$$

$$y(0) = y_0,\ \ \frac{\partial y}{\partial t}(0) = y_1 \quad \text{in } \Omega^e\ . \tag{2.4e}$$

Here we have used the following notations:

$$
\begin{aligned}
Q^e &= \Omega^e \times\,]0,T[\ , \\
\Sigma^e &= \Gamma^e \times\,]0,T[\ , \\
\Sigma_0^e &= \Gamma_0^e \times\,]0,T[\ , \\
\Sigma_\pm^e &= \Gamma_\pm^e \times\,]0,T[\ , \\
\Sigma^e(x^0) &= \Gamma^e(x^0) \times\,]0,T[\ , \\
\Sigma_*^e &= \Gamma_*^e \times\,]0,T[\ .
\end{aligned}
\tag{2.5}
$$

We have initial conditions for the wave equation in (2.4e). v is the control on a part of the lateral boundary through Dirichlet action. $w_\pm$ on controls on the top-bottom surfaces through Neumann action. Note that there is no control on the part Γ_*^e.

We ask the following question which is an exact controllability problem: given initial conditions y_0, y_1 in (2.4e), do there exist a time $T > 0$ and controls v, $w_\pm$ in (2.4b), (2.4d) such that the unique solution y of the problem (2.4) satisfies

$$
y(\cdot, T) = 0, \quad \frac{\partial y}{\partial t}(\cdot, T) = 0 \quad \text{in } \Omega^e \ ?
$$

To answer this question, we follow HUM. But first, we transform the problem (2.4) from the variable domain Ω^e to the fixed domain $\Omega = \omega \times\,]-\frac{1}{2}, \frac{1}{2}[$. To this end, we define the following correspondence between points by homothecy:

$$
x = (x_1, x_2, x_3) \to z = (z_1, z_2, z_3)\ , \tag{2.6}
$$

where $z_\alpha = x_\alpha$, $\alpha = 1, 2$, $z_3 = e^{-1} x_3$. We also make the following association of functions f defined on Ω^e with those f^e defined on Ω:

$$
f(x) = f^e(z)\ . \tag{2.7}
$$

With these notations, the transformed problem on Ω can be written as follows:

$$
\frac{\partial^2 y^e}{\partial t^2} - \left(\frac{\partial^2 y^e}{\partial z_1^2} + \frac{\partial^2 y^e}{\partial z_2^2} + e^{-2} \frac{\partial^2 y^e}{\partial z_3^2} \right) = 0 \quad \text{in } Q\ , \tag{2.8a}
$$

$$
y^e = v^e \quad \text{on } \Sigma(z^0)\ , \tag{2.8b}
$$

$$
y^e = 0 \quad \text{on } \Sigma_*\ , \tag{2.8c}
$$

$$
e^{-1} \frac{\partial y^e}{\partial \nu} = w_\pm^e \quad \text{on } \Sigma_\pm\ , \tag{2.8d}
$$

$$
y^e(0) = y_0^e, \quad \frac{\partial y^e}{\partial t}(0) = y_1^e \quad \text{in } \Omega\ . \tag{2.8e}
$$

The notations Q, $\Sigma(z^0)$, $\Sigma_\pm$, Σ_* used above correspond to the domain Ω and are defined as in (2.5). In the sequel, we work with the formulation (2.8) in which the domain is fixed.

We will specify later the spaces in which initial conditions and boundary controls are taken and also the spaces in which the solution is sought. Let us just mention that our aim in this work is to solve the exact controllability problem for (2.8): Given $\{y_0^e, y_1^e\}$, find $\{v^e, w_\pm^e\}$ such that

$$y^e(T) = \frac{\partial y^e}{\partial t}(T) = 0 \quad \text{in } \Omega . \tag{2.8f}$$

3 – Associated homogeneous forward problem

The first step in HUM is to consider the homogeneous problem associated with (2.8), that is we take the boundary controls to be zero. Thus we introduce the following problem:

$$\frac{\partial^2 \phi^e}{\partial t^2} - \left(\frac{\partial^2 \phi^e}{\partial z_1^2} + \frac{\partial^2 \phi^e}{\partial z_2^2} + e^{-2}\,\frac{\partial^2 \phi^e}{\partial z_3^2}\right) = 0 \quad \text{in } Q , \tag{3.1a}$$

$$\phi^e = 0 \quad \text{on } \Sigma_0 , \tag{3.1b}$$

$$\frac{\partial \phi^e}{\partial \nu} = 0 \quad \text{on } \Sigma_\pm , \tag{3.1c}$$

$$\phi^e(0) = \phi_0 \ \text{ and } \ \frac{\partial \phi^e}{\partial t}(0) = \phi_1 \quad \text{in } \Omega . \tag{3.1d}$$

In order to solve this problem, we introduce the spaces

$$V = \Big\{\psi \in H^1(\Omega),\ \psi = 0(\Gamma_0)\Big\}, \qquad V' = \text{dual space of } V .$$

We then have the following classical result (cf. Lions [4], p. 33–37) which provides the existence and uniqueness of solution to (3.1):

Theorem 3.1. *We take the initial conditions $\phi_0 \in V$ and $\phi_1 \in L^2(\Omega)$. Then there exists a unique solution ϕ^e to (3.1) with the following regularity*

$$\phi^e \in C^0([0,T];V) \cap C^1([0,T];L^2(\Omega)) \cap C^2([0,T];V') .$$

Moreover we have the following conservation of energy:

$$E(t) \equiv E(0) , \qquad \forall\, t , \tag{3.3a}$$

where the energy at time t is defined by

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_\Omega \left|\frac{\partial\phi^e}{\partial t}(z,t)\right|^2 dz \\
&+ \frac{1}{2}\int_\Omega \left[\left|\frac{\partial\phi^e}{\partial z_1}(z,t)\right|^2 + \left|\frac{\partial\phi^e}{\partial z_2}(z,t)\right|^2 + e^{-2}\left|\frac{\partial\phi^e}{\partial z_3}(z,t)\right|^2\right] dz\ .
\end{aligned}
\tag{3.3b}
$$

Concerning the regularity of the solution ϕ^e, we will need the following result in the sequel:

Theorem 3.2. *Assume now $\phi_0 \in H^2(\Omega) \cap V$ and $\phi_1 \in V$. Then the unique solution ϕ^e has the following regularity:*

$$\phi^e \in C^0([0,T]; H^s \cap V) \cap C^1([0,T]; V) \cap C^2([0,T]; L^2(\Omega))$$

for some s with $\frac{3}{2} < s < 2$.

Proof: We cannot apply the results of Lions [4], p. 33 directly because Ω is not of class C^2, neither convex. Moreover we have mixed boundary conditions. However using the geometry of Ω in three-dimensions one can prove following Grisvard [3], p. 237 that the solution has the regularity stated in the Theorem. ∎

For later purposes, we shall need regularity results on the following problem in which we have a nonzero source term:

$$\frac{\partial^2\theta^e}{\partial t^2} - \left(\frac{\partial^2\theta^e}{\partial z_1^2} + \frac{\partial^2\theta^e}{\partial z_2^2} + e^{-2}\frac{\partial^2\theta^e}{\partial z_3^2}\right) = f \quad \text{in } Q\ , \tag{3.4a}$$

$$\theta^e = 0 \quad \text{on } \Sigma_0\ , \tag{3.4b}$$

$$\frac{\partial\theta^e}{\partial\nu} = 0 \quad \text{on } \Sigma_\pm\ , \tag{3.4c}$$

$$\theta^e(0) = \theta_0 \ \text{ and } \ \frac{\partial\theta^e}{\partial t}(0) = \theta_1 \quad \text{in } \Omega\ . \tag{3.4d}$$

Theorem 3.3.

a) *Let us take $f \in L^1(0,T; L^2(\Omega))$, $\theta_0 \in V$ and $\theta_1 \in L^2(\Omega)$. Then there exists a unique solution θ^e with the following regularity:*

$$\theta^e \in C^0([0,T]; V) \cap C^1([0,T], L^2(\Omega))\ .$$

We also have the following energy inequality:

$$E(t) \le C_0\Big\{E(0) + \Big(\int_0^t \|f(s)\|_{L^2(\Omega)}\, ds\Big)^2\Big\}\ , \tag{3.5}$$

where $E(t)$ is defined in (3.3) and C_0 is a constant independent of e.

b) *Furthermore, if $f \in L^1(0,T;V)$, $\theta_0 \in H^2 \cap V$ and $\theta_1 \in V$ then the unique solution θ^e enjoys the following regularity properties:*

$$\theta^e \in C^0([0,T]; H^s \cap V) \cap C^1([0,T]; V) \quad \text{for some } s \text{ with } \frac{3}{2} < s < 2 \; .$$

Proof: It is analogous to that of Theorem 3.2 cf. Lions [4] p. 39 and Grisvard [3], p. 237. To get the estimate (3.5), one multiplies (3.4) by $\frac{\partial \theta^e}{\partial t}$ and integrates by parts. ∎

4 – Direct inequality

As is well-known, HUM is based on certain estimates that one derives on the problems (3.1) and (3.4). In this paragraph, we derive what are called direct estimates. These estimates are deduced from an identity valid for solutions with finite energy whose existence has been established in §3. This identity is obtained by the so-called multiplier method. (i.e.) we multiply (3.4) by $m_k(z) \frac{\partial \theta^e}{\partial z_k}$. Thanks to the regularity results in §3, it is possible to establish this identity in our case for arbitrary solutions of finite energy. cf. Theorem 4.1 below.

Once this identity is proved, the classical strategy to derive the direct estimates is to make the following choice of multipliers:

$$\begin{cases} m_k \in W^{1,\infty}(\Omega), \\ m_k(z) = \nu_k(z) \quad \text{on } \Gamma \; , \end{cases} \tag{4.1}$$

where $\nu(z) = (\nu_k(z))$ is the unit outward normal at $z \in \Gamma$. Unfortunately such a choice is not possible because $m_k \in W^{1,\infty}(\Omega) \hookrightarrow C^0(\Gamma)$ but $\nu_k \notin C^0(\Gamma)$.

Hence we turn to other choices. One may think of using Lemma 3.2, p. 31 in Lions [4] but however since we are interested in the limit $e \to 0$, we desire to obtain uniform estimates as $e \to 0$. This motivates us to define the following multipliers:

$$\begin{cases} m_k \in W^{1,\infty}(\Omega), \\ m_1, m_2 \quad \text{are independent of } z_3, \\ m_\alpha = \nu_\alpha \quad \text{on } \Gamma_0, \quad \alpha = 1, 2, \\ m_3 = 0 \quad \text{on } \Gamma \; . \end{cases} \tag{4.2}$$

Another choice is the following one:

$$
\begin{cases}
m_k \in W^{1,\infty}(\Omega), \\
m_1, m_2 \quad \text{are independent of } \; z_3, \\
m_\alpha = 0 \quad \text{on } \; \Gamma_0, \quad \alpha = 1,2, \\
m_3 = \nu_3 \quad \text{on } \; \Gamma_\pm \; .
\end{cases}
\tag{4.3}
$$

Note that such choices are possible; it is enough to work in ω top obtain such functions and then extend them by constancy over "z_3-fibbers". Evidently this choice of multipliers are more adapted to the body Ω under our consideration. With this choice, we derive direct inequalities on the problem (3.4). Cf. Theorem 4.2 below.

Let us therefore consider the problem (3.4). Then we have the identity given by the following result valid for all solutions with finite energy:

Theorem 4.1. *Let $f \in L^1(0,T,L^2(\Omega))$, $\theta_0 \in V$ and $\theta_1 \in L^2(\Omega)$. Then the solution θ^e of (3.4) satisfies*

$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma_0} (m_\alpha \nu_\alpha) \left(\frac{\partial \theta^e}{\partial \nu}\right)^2 d\sigma\, dt + \frac{1}{2}\int_{\Sigma_\pm} (m_3 \nu_3) \left\{ \left(\frac{\partial \theta^e}{\partial t}\right)^2 - \left(\frac{\partial \theta^e}{\partial z_1}\right)^2 - \left(\frac{\partial \theta^e}{\partial z_2}\right)^2 \right\} d\sigma\, dt = \\
&\quad = \left[\int_\Omega \frac{\partial \theta^e}{\partial t}\, m_k\, \frac{\partial \theta^e}{\partial z_k}\, dz\right]_0^T \\
&\quad + \int_Q \frac{\partial m_k}{\partial z_k}\, \frac{1}{2} \left\{ \left(\frac{\partial \theta^e}{\partial t}\right)^2 - \left(\frac{\partial \theta^e}{\partial z_1}\right)^2 - \left(\frac{\partial \theta^e}{\partial z_2}\right)^2 - e^{-2} \left(\frac{\partial \theta^e}{\partial z_3}\right)^2 \right\} dz\, dt \\
&\quad + \int_Q \frac{\partial m_k}{\partial z_\alpha}\, \frac{\partial \theta^e}{\partial z_\alpha}\, \frac{\partial \theta^e}{\partial z_k}\, dz\, dt + \int_Q e^{-2}\, \frac{\partial m_k}{\partial z_3}\, \frac{\partial \theta^e}{\partial z_3}\, \frac{\partial \theta^e}{\partial z_k}\, dz\, dt - \int_Q f\, m_k\, \frac{\partial \theta^e}{\partial z_k}\, dz\, dt \; .
\end{aligned}
$$

Proof: We multiply (3.4) by $m_k \frac{\partial \theta^e}{\partial z_k}$ and integrate by parts following Lions [4], p. 40–43 and p. 185–186. This will establish the above identity in the case of smooth solution. We then complete the proof for general solutions with finite energy by density arguments and by appealing to the regularity result of Theorem (3.3) (b). ∎

As mentioned earlier, we make the choice of the multipliers as given by (4.2) and (4.3) respectively in the identity derived above. This leads to the following result:

Theorem 4.2. *Fix $T^* > 0$. We take again $f \in L^1(0,T,L^2(\Omega))$, $\theta_0 \in V$ and $\theta_1 \in L^2(\Omega)$. Then the solution θ^e of (3.4) admits the following estimates with* *constant C_1 independent of $T \geq T^*$ and $e \to 0$ but depending on T^*.*

$$\int_{\Sigma_0} \left(\frac{\partial\theta^e}{\partial\nu}\right)^2 d\sigma\, dt \leq C_1\, T\Big\{E(0) + \Big(\int_0^T \|f(t)\|_{L^2(\Omega)} dt\Big)^2\Big\}\;, \tag{4.4}$$

$$\left|\int_{\Sigma_\pm}\left[\left(\frac{\partial\theta^e}{\partial t}\right)^2 - \left(\frac{\partial\theta^e}{\partial z_1}\right)^2 - \left(\frac{\partial\theta^e}{\partial z_2}\right)^2\right] d\sigma\, dt\right| \leq \tag{4.5}$$
$$\leq C_1\, T\Big\{E(0) + \Big(\int_0^T \|f(t)\|_{L^2(\Omega)} dt\Big)^2\Big\}\;.$$

Here $E(0)$ is the initial energy as defined in (3.3).

Proof: We apply identity of Theorem 4.1 with the choice (4.2). The nasty terms $\int_Q e^{-2} \frac{\partial m_\alpha}{\partial z_3} \frac{\partial\theta^e}{\partial z_\alpha} \frac{\partial\theta^e}{\partial z_3}\, dz\, dt$ which are not bounded by energy drop out. We obtain indeed

$$\begin{aligned}
\frac{1}{2}\int_{\Sigma_0}\left(\frac{\partial\theta^e}{\partial\nu}\right) d\sigma\, dt &= \left[\int_\Omega \frac{\partial\theta^e}{\partial t}\, m_k\, \frac{\partial\theta^e}{\partial z_k}\, dz\right]_0^T \\
&+ \int_Q \frac{1}{2}\frac{\partial m_k}{\partial z_k}\left\{\left(\frac{\partial\theta^e}{\partial t}\right)^2 - \left(\frac{\partial\theta^e}{\partial z_1}\right)^2 - \left(\frac{\partial\theta^e}{\partial z_2}\right)^2 - \left(\frac{\partial\theta^e}{\partial z_3}\right)^2\right\} dz\, dt \\
&+ \int_Q \frac{\partial m_k}{\partial z_\alpha}\frac{\partial\theta^e}{\partial z_\alpha}\frac{\partial\theta^e}{\partial z_k}\, dz\, dt + \int_Q e^{-2}\,\frac{\partial m_3}{\partial z_3}\left(\frac{\partial\theta^e}{\partial z_3}\right)^2 dz\, dt \\
&- \int_Q f\, m_k\, \frac{\partial\theta^e}{\partial z_k}\, dz\, dt\;.
\end{aligned} \tag{4.6}$$

Let us consider the various terms on the right side of (4.6) except the last one. At each instant of time, they are bounded by $(\max_{1\leq k\leq 3}\|m_k\|_{1,\infty}).E(t)$. We next appeal to the energy estimate (3.5) and this leads to (4.4).

Thus it remains to estimate the last term on the right side of (4.6). This term can be handled as follows:

$$\begin{aligned}
\left|\int_Q f\, m_k\, \frac{\partial\theta^e}{\partial z_k}\, dz\, dt\right| &\leq \|m_k\|_{0,\infty,\Omega}\int_0^T \|f(t)\|_{L^2(\Omega)}\left\|\frac{\partial\theta^e}{\partial z_k}\right\|_{L^2(\Omega)} dt \leq \\
&\leq \|m_k\|_{0,\infty,\Omega}\int_0^T \|f(t)\|_{L^2(\Omega)} E(t)^{\frac{1}{2}} dt \\
&\leq \|m_k\|_{0,\infty,\Omega}\int_0^T \|f(t)\|_{L^2(\Omega)} C_0^{\frac{1}{2}}\Big\{E(0)^{\frac{1}{2}} + \int_0^t \|f(s)\|_{L^2(\Omega)} ds\Big\} dt \leq
\end{aligned}$$

$$
\begin{aligned}
&\le C_0^{\frac12}\|m_k\|_{0,\infty,\Omega}E(0)^{\frac12}\Big(\int_0^T \|f(t)\|_{L^2(\Omega)}dt\Big)\\
&\quad + C_0^{\frac12}\|m_k\|_{0,\infty,\Omega}\Big(\int_0^T \|f(t)\|_{L^2(\Omega)}dt\Big)^2\\
&\le C_0^{\frac12}\|m_k\|_{0,\infty,\Omega}\Big\{\frac12 E(0)+\frac12\Big(\int_0^T \|f(t)\|_{L^2(\Omega)}dt\Big)^2\Big\}\\
&\quad + C_0^{\frac12}\|m_k\|_{0,\infty,\Omega}\Big(\int_0^T \|f(t)\|_{L^2(\Omega)}dt\Big)^2\\
&\le C_0^{\frac12}\|m_k\|_{0,\infty,\Omega}\Big\{\frac12 E(0)+\Big(\int_0^T \|f(t)\|_{L^2(\Omega)}dt\Big)^2\Big\}\\
&\le C_1 T\Big\{E(0)+\Big(\int_0^T \|f(t)\|_{L^2(\Omega)}dt\Big)^2\Big\}
\end{aligned}
$$

where

$$C_1=\frac{C_0^{\frac12}}{T^*}\,\Big(\max_{1\le k\le 3}\|m_k\|_{0,\infty,\Omega}\Big)\ .$$

This establishes the estimate (4.4). Proof of (4.5) is similar; it suffices to use the multipliers given by (4.3). ∎

Remark 4.1. If ∃ no source term f then the last term in (4.6) does not exist. The above proof then shows that the estimates (4.4), (4.5) are valid with a constant C which is independent of T^*. In other words, we have the following estimates for the solution ϕ^e of (3.1) with a constant C_2 independent of $T>0$ and $e\to 0$:

$$\int_{\Sigma_0}\Big(\frac{\partial\phi^e}{\partial\nu}\Big)^2 d\sigma\,dt\le C_2\,T\,E(0)\ , \tag{4.7}$$

$$\left|\int_{\Sigma_\pm}\Big[\Big(\frac{\partial\phi^e}{\partial t}\Big)^2-\Big(\frac{\partial\phi^e}{\partial z_1}\Big)^2-\Big(\frac{\partial\phi^e}{\partial z_2}\Big)^2\Big]d\sigma\,dt\right|\le C_2\,T\,E(0)\ . \tag{4.8}$$

These are regularity properties of solutions with finite energy which are uniform as $e\to 0$.

5 – Inverse inequality

This is again an estimate on the solution ϕ^e of the equation (3.1) which goes in a direction opposite to the direct inequality derived in the previous section. In other words, we now want to estimate the energy. This is something very essential to HUM. Once again we use the identity of Theorem 4.1 with different choice of multipliers. The classical choice is the following:

$$m_k(z) = z_k - z_k^0\,, \qquad k = 1,2,3\ . \tag{5.1}$$

However we restrict the choice z^0 such that $z_3^0 = 0$. This is done with the view to obtain a two dimensional problem of exact controllability at the limit $e \to 0$. With this choice, the portion $\Gamma(z^0)$ of the lateral part of the boundary has a two dimensional structure. Indeed

$$\Gamma(z^0) = \gamma(z^0) \times \left[-\frac{1}{2}, \frac{1}{2}\right]\,,$$

where $\gamma(z^0)$ is defined as in (2.2).

If we use the multiplier (5.1), we obtain the inverse inequality given by the following result:

Theorem 5.1. *Let us consider the solution ϕ^e of the equation (3.1) with initial conditions $\phi_0 \in V$ and $\phi_1 \in L^2(\Omega)$. Then there exists a constant C_3 and $T^* > 0$, both independent of e such that for $T \geq T^*$ we have the following estimate on the solution ϕ^e:*

$$E(0) \leq C_3 \left\{ \int_{\Sigma(z^0)} \left(\frac{\partial\phi^e}{\partial\nu}\right)^2 d\sigma\, dt + \int_{\Sigma_\pm} \left[\left(\frac{\partial\phi^e}{\partial t}\right)^2 + (\phi^e)^2 \right] d\sigma\, dt \right\}\ . \tag{5.2}$$

Proof: We apply the identity of Theorem 4.1 with the choice (5.1). We obtain the following relation without difficulty:

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma_0} (m_\alpha \nu_\alpha) \left(\frac{\partial\phi^e}{\partial\nu}\right)^2 d\sigma\, dt + \frac{1}{2}\int_{\Sigma_\pm} (m_3\,\nu_3) \left\{ \left(\frac{\partial\phi^e}{\partial t}\right)^2 - \left(\frac{\partial\phi^e}{\partial z_1}\right)^2 - \left(\frac{\partial\phi^e}{\partial z_2}\right)^2 d\sigma\, dt \right\} - \\
&- \left[\int_\Omega \frac{\partial\phi^e}{\partial t}\, m_k\, \frac{\partial\phi^e}{\partial z_k}\, dz\right]_0^T \\
&- \int_Q \left\{ \frac{3}{2}\left(\frac{\partial\phi^e}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi^e}{\partial z_1}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi^e}{\partial z_2}\right)^2 - \frac{1}{2}\, e^{-2} \left(\frac{\partial\phi^e}{\partial z_3}\right)^2 \right\} dz\, dt = 0\ .
\end{aligned}$$

We add the following to both sides:

$$\frac{1}{2}\int_Q \left\{ \left(\frac{\partial\phi^e}{\partial t}\right)^2 + \left(\frac{\partial\phi^e}{\partial z_1}\right)^2 + \left(\frac{\partial\phi^e}{\partial z_2}\right)^2 + e^{-2}\left(\frac{\partial\phi^e}{\partial z_3}\right)^2 \right\} dz\, dt\ .$$

We then end up in the following relation:

$$\frac{1}{2}\int_Q \left\{ \left(\frac{\partial\phi^e}{\partial t}\right)^2 + \left(\frac{\partial\phi^e}{\partial z_1}\right)^2 + \left(\frac{\partial\phi^e}{\partial z_2}\right)^2 + e^{-2}\left(\frac{\partial\phi^e}{\partial z_3}\right)^2 \right\} dz\, dt = \tag{5.3}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\Sigma_0} (m_\alpha \nu_\alpha)\Big(\frac{\partial \phi^e}{\partial \nu}\Big)^2 d\sigma\, dt \\
&\quad + \frac{1}{2}\int_{\Sigma_\pm} (m_3\nu_3)\Big\{\Big(\frac{\partial \phi^e}{\partial \nu}\Big)^2 - \Big(\frac{\partial \phi^e}{\partial z_1}\Big)^2 - \Big(\frac{\partial \phi^e}{\partial z_2}\Big)^2\Big\} d\sigma\, dt \\
&\quad - \Big[\int_\Omega \frac{\partial \phi^e}{\partial t}\, m_k\, \frac{\partial \phi^e}{\partial z_k}\, dz\Big]_0^T \\
&\quad - \int_Q \Big\{\Big(\frac{\partial \phi^e}{\partial t}\Big)^2 - \Big(\frac{\partial \phi^e}{\partial z_1}\Big)^2 - \Big(\frac{\partial \phi^e}{\partial z_2}\Big)^2 - e^{-2}\Big(\frac{\partial \phi^e}{\partial z_3}\Big)^2\Big\} dz\, dt\ .
\end{aligned}
$$

On the other hand, by multiplying (3.1) by ϕ^e and integrating by parts, we get

$$
(5.4)\quad \int_Q \Big\{\Big(\frac{\partial \phi^e}{\partial t}\Big)^2 + \Big(\frac{\partial \phi^e}{\partial z_1}\Big)^2 + \Big(\frac{\partial \phi^e}{\partial z_2}\Big)^2 + e^{-2}\Big(\frac{\partial \phi^e}{\partial z_3}\Big)^2\Big\} dz\, dt = \Big[\int_Q \frac{\partial \phi^e}{\partial t}\, \phi^e\, dz\Big]_0^T .
$$

Using (5.4) and the fact that

$$
(5.5)\qquad m_3\, \nu_3 = \frac{1}{2}\,(\Gamma_\pm)\ ,
$$

we get from (5.3) that

$$
\begin{aligned}
(5.6)\quad & \frac{1}{2}\int_Q \Big\{\Big(\frac{\partial \phi^e}{\partial t}\Big)^2 + \Big(\frac{\partial \phi^e}{\partial z_1}\Big)^2 + \Big(\frac{\partial \phi^e}{\partial z_2}\Big)^2 + e^{-2}\Big(\frac{\partial \phi^e}{\partial z_3}\Big)^2\Big\} dz\, dt = \\
&= \frac{1}{2}\int_{\Sigma_0} (m_\alpha \nu_\alpha)\Big(\frac{\partial \phi^e}{\partial \nu}\Big)^2 d\sigma\, dt + \frac{1}{4}\int_{\Sigma_\pm} \Big\{\Big(\frac{\partial \phi^e}{\partial t}\Big)^2 - \Big(\frac{\partial \phi^e}{\partial z_1}\Big)^2 - \Big(\frac{\partial \phi^e}{\partial z_2}\Big)^2\Big\} d\sigma\, dt \\
&\quad - \Big[\int_\Omega \frac{\partial \phi^e}{\partial t}\Big(\phi^e + m_k\, \frac{\partial \phi^e}{\partial z_k}\Big) dz\Big]_0^T .
\end{aligned}
$$

We observe the following points: The left side of (5.6) is equal to $T\,E(0)$ because energy is conserved (cf. (3.3)). The integral over Σ_0 can be split into two parts: one integral over $\Sigma(z^0)$ and another over Σ_*. Recall we have $m_\alpha\nu_\alpha \leq 0$ over Γ_* (cf. (2.2), (2.3)). We can drop the integral over Σ_* at the cost of replacing equality by inequality. For the last term we apply the estimate of the Lemma 5.2 below. Finally we obtain

$$
\begin{aligned}
TE(0) &\leq \frac{1}{2}\int_{\Sigma(z^0)} (m_\alpha \nu_\alpha)\Big(\frac{\partial \phi^e}{\partial \nu}\Big)^2 d\sigma\, dt \\
&\quad + \frac{1}{4}\int_{\Sigma_\pm} \Big\{\Big(\frac{\partial \phi^e}{\partial t}\Big)^2 - \Big(\frac{\partial \phi^e}{\partial z_1}\Big)^2 - \Big(\frac{\partial \phi^e}{\partial z_2}\Big)^2\Big\} d\sigma\, dt \\
&\quad + C_4\Big\{E(0) + \int_{\Sigma_\pm} \Big[\Big(\frac{\partial \phi^e}{\partial t}\Big)^2 + (\phi^e)^2\Big] d\sigma\, dt\Big\}\ .
\end{aligned}
$$

To complete the proof, it is enough to choose

$$
\begin{aligned}
&T^* > C_4 \ ,\\
(5.7)\qquad &C_3 \geq \frac{2}{T-C_4} \max\Big\{\frac{1}{4}, C_4, \frac{1}{2} \max_{\gamma(z^0)} |m_\alpha \nu_\alpha|\Big\} \ . \ \blacksquare
\end{aligned}
$$

Lemma 5.2. *There is a constant C_4 independent of e such that*

$$\left|\left[\int_\Omega \frac{\partial\phi^e}{\partial t}\Big(\phi^e + m_k \frac{\partial\phi^e}{\partial z_k}\Big) dz\right]_0^T\right| \leq C_4\Big\{E(0) + \int_{\Sigma_\pm}\Big[\Big(\frac{\partial\phi^e}{\partial t}\Big)^2 + (\phi^e)^2\Big] d\sigma\, dt\Big\} \ .$$

Proof: We have

$$\left|\int_\Omega \frac{\partial\phi^e}{\partial t}\Big(\phi^e + m_k \frac{\partial\phi^e}{\partial z_k}\Big) dz\right| \leq \frac{1}{2}\int_\Omega \Big(\frac{\partial\phi^e}{\partial t}\Big)^2 dz + \frac{1}{2}\int_\Omega \Big(m_k \frac{\partial\phi^e}{\partial z_k} + \phi^e\Big)^2 dz \ .$$

The second term on the right side is expressed as follows:

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \Big[m_k^2\Big(\frac{\partial\phi^e}{\partial z_k}\Big)^2 + (\phi^e)^2 + 2\, m_k\, \phi^e \frac{\partial\phi^e}{\partial z_k}\Big] dz = \\
&\qquad\qquad = \frac{1}{2}\int_\Omega \Big[m_k^2\Big(\frac{\partial\phi^e}{\partial z_k}\Big)^2 + (\phi^e)^2 + m_k \frac{\partial}{\partial z_k}(\phi^e)^2\Big] dz \ .
\end{aligned}
$$

By doing integration by parts in the last term, we get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega m_k^2\Big(\frac{\partial\phi^e}{\partial z_k}\Big)^2 dz - \int_\Omega (\phi^e)^2 dz + \frac{1}{2}\int_\Gamma (m_k\, \nu_k)(\phi^e)^2\, d\sigma = \\
&\qquad\qquad = \frac{1}{2}\int_\Omega m_k^2\Big(\frac{\partial\phi^e}{\partial z_k}\Big)^2 dz - \int_\Omega (\phi^e)^2\, dz + \frac{1}{4}\int_{\Gamma_\pm} (\phi^e)^2\, d\sigma
\end{aligned}
$$

because $m_k \nu_k = m_3 \nu_3 = \frac{1}{2}$ on $\Gamma_\pm$ and $\phi^e = 0$ on Γ_0. Thus we obtain

$$
\begin{aligned}
(5.8)\qquad &\left|\int_\Omega \frac{\partial\phi^e}{\partial t}\Big(\phi^e + m_k \frac{\partial\phi^e}{\partial z_k}\Big) dz\right| \leq \frac{1}{2}\int_\Omega \Big(\frac{\partial\phi^e}{\partial t}\Big)^2 dz + \frac{1}{2}\int_\Omega m_k^2\Big(\frac{\partial\phi^e}{\partial z_k}\Big)^2 dz \\
&\qquad\qquad - \int_\Omega (\phi^e)^2\, dz + \frac{1}{4}\int_{\Gamma_\pm} (\phi^e)^2\, d\sigma \ .
\end{aligned}
$$

Now we estimate the last term on the right side by using Trace inequality.

$$(5.9)\qquad \int_{\Gamma_\pm} (\phi^e)^2\, d\sigma \leq C_4' \int_{\Sigma_\pm} \Big[(\phi^e)^2 + \Big(\frac{\partial\phi^e}{\partial t}\Big)^2\Big] d\sigma\, dt \ .$$

The estimate in the Lemma follows from (5.8) and (5.9) with

$$C_4 = \max\Big\{1, \frac{1}{4}C'_4, \max_{\substack{z\in\Omega \\ 1\le k\le 3}} m_k^2\Big\} \ . \blacksquare \tag{5.10}$$

6 – The space F and the backward Cauchy problem

The next step in HUM is to introduce the space F and to resolve backward Cauchy problem with data taken from F. To define the space F, we use the inverse inequality derived in the previous section. Recall that this inequality (cf. Theorem 5.1) is valid for T sufficiently large. We fix one such $T > 0$ independent of e, which is possible. We consider the Cauchy problem (3.1) with initial conditions $\{\phi_0, \phi_1\}$. We define the following norm:

$$\|\{\phi_0,\phi_1\}\|_F^2 = \int_{\Sigma(z^0)} \Big(\frac{\partial\phi^e}{\partial\nu}\Big)^2 d\sigma\, dt + \int_{\Sigma_\pm} \Big[\Big(\frac{\partial\phi^e}{\partial t}\Big)^2 + (\phi^e)^2\Big] d\sigma\, dt \ , \tag{6.1}$$

where ϕ^e is the unique solution of (3.1). Inequality (4.7) shows that the first term in the right side of (6.1) is bounded by the energy uniformly on $e \to 0$. However the second term is not so. Hence we take more regular initial conditions, namely $\phi_0 \in H^2(\Omega) \cap V$ and $\phi_1 \in V$. By Theorem 3.2, we then have $\frac{\partial\phi^e}{\partial t} \in C^0([0,T]; V)$ and hence the second term on the right side of (6.1) also makes sense. Hence $\| \ \|_F$ is well defined by (6.1) for initial conditions $\{\phi_0, \phi_1\} \in (H^2 \cap V) \times V$. We define F to be the completion of $(H^2 \cap V) \times V$ under the norm (6.1). Even though the norm (6.1) depends on e, the space F is independent of e. It follows from the inverse inequality that we have the following dense and continuous injection

$$(F, \| \ \|_F) \hookrightarrow (V \times L^2(\Omega), \| \ \|_E) \ , \tag{6.2}$$

where the norm $\| \ \|_E$ is defined as in (3.3) i.e.

$$\|\{\phi_0,\phi_1\}\|_E^2 = \frac{1}{2}\int_\Omega \Big[\Big(\frac{\partial\phi_0}{\partial z_1}\Big)^2 + \Big(\frac{\partial\phi_0}{\partial z_2}\Big)^2 + e^{-2}\Big(\frac{\partial\phi_0}{\partial z_3}\Big)^2\Big] dz + \frac{1}{2}\int_\Omega \phi_1^2\, dz \ . \tag{6.3}$$

Moreover the constant of continuity in (6.2) independent of e: there is a constant C_5 independent of e such that

$$\|\{\phi_0,\phi_1\}\|_E \le C_5\, \|\{\phi_0,\phi_1\}\|_F \ . \tag{6.4}$$

Once the space F is defined, we are now ready to introduce the backward Cauchy problem which can be formally written down as follows:

$$\frac{\partial^2\psi^e}{\partial t^2} - \Big(\frac{\partial^2\psi^e}{\partial z_1^2} + \frac{\partial^2\psi^e}{\partial z_2^2} + e^{-2}\,\frac{\partial^2\psi^e}{\partial z_3^2}\Big) = 0 \quad \text{in } Q \ , \tag{6.5a}$$

$$\psi^e = \frac{\partial \phi^e}{\partial \nu} \quad \text{on } \Sigma(z^0) , \tag{6.5b}$$

$$\psi^e = 0 \quad \text{on } \Sigma_* , \tag{6.5c}$$

$$\frac{\partial \psi^e}{\partial \nu} = e^2 \Big\{ \frac{\partial}{\partial t}\Big(\frac{\partial \phi^e}{\partial t} \Big) - \phi^e \Big\} \quad \text{on } \Sigma_\pm , \tag{6.5d}$$

$$\psi^e(T) = \frac{\partial \psi^e}{\partial t}(T) = 0 \quad \text{in } \Omega , \tag{6.5e}$$

where ϕ^e is the solution of (3.1) with initial condition $\{\phi_0, \phi_1\} \in F$ and $T > 0$ is such that the inverse inequality holds. Here the term $\frac{\partial}{\partial t}(\frac{\partial \phi^e}{\partial t})$ is taken in the following sense

$$\Big\langle \frac{\partial}{\partial t}\Big(\frac{\partial \phi^e}{\partial t} \Big), v \Big\rangle = - \int_{\Sigma_\pm} \frac{\partial \phi^e}{\partial t} \frac{\partial v}{\partial t} \, d\sigma \, dt \qquad \text{for all } \ v \in H^1(0,T; L^2(\Gamma_\pm))$$

(cf. Lions [4], p. 209). A weak formulation of this is obtained if we multiply (6.5) by θ^e where θ^e solves (3.4). By doing integration by parts twice pretending ψ^e is smooth, we arrive at the following relation formally:

$$\begin{aligned} &\int_\Omega \frac{\partial \psi^e}{\partial t}(z,0)\, \theta_0(z)\, dz - \int_\Omega \psi^e(z,0)\, \theta_1(z)\, dz = \\ &\quad = \int_Q \psi^e f \, dz\, dt + \int_{\Sigma(z^0)} \frac{\partial \phi^e}{\partial \nu} \frac{\partial \theta^e}{\partial \nu} \, d\sigma\, dt + \int_{\Sigma_\pm} \Big[\frac{\partial \phi^e}{\partial t} \frac{\partial \theta^e}{\partial t} + \phi^e \theta^e \Big] d\sigma\, dt \, . \end{aligned} \tag{6.6}$$

Thus we are led to propose the following formulation of the problem (6.5):

Find $\psi^e \in L^\infty(0,T;V')$ which satisfies the following condition: there exists $\{\psi_1^e, -\psi_0^e\} \in F'$ such that we have

$$\begin{aligned} {}_{F'}\Big\langle \{\psi_1^e, -\psi_0^e\}, \{\theta_0, \theta_1\} \Big\rangle_F = \int_Q \psi^e f \, dz\, dt &+ \int_{\Sigma(z^0)} \frac{\partial \phi}{\partial \nu} \frac{\partial \theta}{\partial \nu} \, d\sigma\, dt \\ &+ \int_{\Sigma_\pm} \Big[\frac{\partial \phi^e}{\partial t} \frac{\partial \theta^e}{\partial t} + \phi^e \theta^e \Big] d\sigma\, dt \end{aligned} \tag{6.7}$$

for all solutions θ^e of (3.4) where we take $f \in L^1(0,T;V)$ and $\{\theta_0, \theta_1\} \in F$.

The first term on the right side of (6.7) is interpreted as the duality between $L^1(0,T;V)$ and $L^\infty(0,T;V')$. The second term makes sense because of (4.4). The third one has a meaning thanks to Theorem 3.3 b). We remark that if there exists a smooth ψ^e satisfying (6.7) then necessarily $\frac{\partial \psi^e}{\partial t}(0) = \psi_1^e$ and $\psi^e(0) = \psi_0^e$.

Existence and uniqueness of ψ^e, ψ_0^e and ψ_1^e follow immediately by duality arguments. In fact choosing $\theta_0 = \theta_1 = 0$ we see that the integrals over $\Sigma(z^0)$ and Σ_* in (6.7) define continuous linear functionals when $f \in L^1(0,T;V)$. Hence ψ^e exists uniquely as an element of $L^\infty(0,T;V')$. Similar arguments hold for ψ_0^e and

ψ_1^e. We use the backward problem to establish the exact controllability in the next section.

7 – The operator Λ^e and exact controllability

Following HUM, we now define a linear operator Λ^e and prove that it is invertible uniformly with respect to e. This will establish the uniform exact controllability of our problem (2.8).

Λ^e will be bounded linear operator from F onto F' and it is defined as follows: Let us start with $\{\phi_0, \phi_1\} \in F$. We solve first the forward Cauchy problem (3.1) for ϕ^e. Next we solve backward Cauchy problem (6.5) for ψ^e (or rather its weak formulation (6.7)). This gives the element $\{\psi_1^e - \psi_0^e\} \in F'$. We define

$$\begin{aligned} &\Lambda^e \colon F \to F' \ , \\ &\{\phi_0, \phi_1\} \to \{\psi_1^e - \psi_0^e\} \ . \end{aligned} \tag{7.1}$$

Properties of Λ^e are listed in the result below:

Theorem 7.1.

a) *Λ^e is a continuous linear operator whose norm is bounded independent of e.*

b) *Λ^e is an isomorphism onto F'. The norm of its inverse is bounded independent of e.*

Proof:

a) In the weak formulation (6.7), we take $f = 0$.

$$_{F'}\Big\langle \{\psi_1^e, -\psi_0^e\}, \{\theta_0, \theta_1\} \Big\rangle_F = \int_{\Sigma(z^0)} \frac{\partial \phi^e}{\partial \nu} \, \frac{\partial \theta^e}{\partial \nu} \, d\sigma \, dt + \int_{\Sigma_\pm} \left[\frac{\partial \phi^e}{\partial t} \, \frac{\partial \theta^e}{\partial t} + \phi^e \, \theta^e \right] d\sigma \, dt \ .$$

The right side is by definition equal to the inner product $(\{\phi_0, \phi_1\}, \{\theta_0, \theta_1\})$ in the space F. As a consequence, we get

$$\|\{\psi_1^e, -\psi_0^e\}\|_{F'} \le \|\{\phi_0, \phi_1\}\|_F \ , \tag{7.2}$$

which means that $\|\Lambda^e\| \le 1$.

b) We take $\{\theta_0, \theta_1\} = \{\phi_0, \phi_1\}$ and $f = 0$ in the weak formulation (6.7). Then the problem (3.4) is the same thing on (3.1) and hence $\theta^e = \phi^e$. Therefore, we get

$$(7.3) \qquad \text{(i.e.)} \qquad \begin{aligned} &{}_{F'}\Big\langle \{\psi_1^e, -\psi_0^e\}, \{\theta_0, \theta_1\} \Big\rangle_F = \|\{\phi_0, \phi_1\}\|_F^2 \ , \\ &{}_{F'}\Big\langle \Lambda^e \{\phi_0, \phi_1\}, \{\theta_0, \theta_1\} \Big\rangle_F = \|\{\phi_0, \phi_1\}\|_F^2 \ . \end{aligned}$$

(b) follows easily from this property. ∎

Now we are in a position to show the exact controllability of the problem (2.8). We shall also specify the spaces in which the initial conditions y_0^e, y_1^e are taken and in which sense the problem (2.8) is solved. We take $\{y_1^e, -y_0^e\} \in F'$ and the problem (2.8) is understood in a similar to (6.5). We solve

$$(7.4) \qquad \Lambda^e \{\phi_0^e, \phi_1^e\} = \{y_1^e, -y_0^e\} \ ,$$

with $\{\phi_0^e, \phi_1^e\} \in F$. This is possible because $\Lambda^e \colon F \to F'$ is isomorphism. Next we solve (3.1) for ϕ^e with initial conditions $\{\phi_0^e, \phi_1^e\}$. We next take

$$(7.5a) \qquad v^e = \frac{\partial \phi^e}{\partial \nu} \quad \text{on} \ \ \Sigma(z^0) \ ,$$

$$(7.5b) \qquad w_\pm^e = e \left\{ \frac{\partial}{\partial t} \left(\frac{\partial \phi^e}{\partial t} \right) - \phi^e \right\} \quad \text{on} \ \ \Sigma_\pm \ ,$$

as controls in the problem (2.8). With these choices, we observe that problem (2.8) coincides with (6.5) and so $y^e = \psi^e$. In particular,

$$y^e(T) = \frac{\partial y^e}{\partial t}(T) = 0 \quad \text{in} \ \ \Omega \ .$$

We states this as a separate result.

Theorem 7.2. *As mentioned before, we fix $T > 0$ such that the inverse inequality hold. Then the problem (2.8) with initial data $\{y_0^e, y_1^e\} \in F'$ is exactly controllable at time T with controls defined by (7.4) and (7.5). Moreover these controls have the following regularity properties:*

$$(7.6) \qquad \begin{aligned} &v^e \in L^2(\Sigma(z^0)) \ , \\ &w_\pm^e \in \Big[H^1(0, T; L^2(\Gamma_\pm)) \Big]' \ . \ \blacksquare \end{aligned}$$

8 – Behaviour when the thickness parameter is small

In this section, we let the thickness parameter $e \to 0$ and we shall analyze the behaviour of the exact controllability problem which we have solved the in the previous section. We will naturally use the various bounds independent of e established already and pass to the limit in various problems introduced previously. Recall that the minimal time of exact controllability was show to be bounded above as e varies this can be chosen independent of e. We fix one such time $T > 0$ throughout our discussion.

This section is divided into several subsections. In §8.1 and §8.2, we analyze the behaviour of the homogeneous forward problem and nonhomogeneous backward problem respectively. These results are subsequently used to pass to the limit in the exact controllability problem. Indeed in §8.4, it is show that the top bottom controls converge to zero while the lateral control converges to a limit control which is independent of z_3. Indeed there exists a two-dimensional exact controllability problem which involves an interior control apart from a boundary control and this boundary control coincides with the limit of the three-dimensional lateral controls. Our task is to identify this limit control. The subsection 8.3 is devoted to the description and the study the two dimensional exact controllability problem mentioned above.

In this section, adopt a new notation which was used in previous sections. For a section g of three variables (z_1, z_2, z_3) defined on Ω, we denote by $m(g)$, its average with respect to z_3, which is a function of two variables (z_1, z_2) defined on ω:

$$m(g)\,(z_1, z_2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(z_1, z_2, z_3)\, dz_3 \ . \tag{8.1}$$

We add that $m(g)$ can be interpreted as the duality bracket between g and the constant function identically equal to unity in the z_3-variable.

8.1. Behaviour of the Homogeneous Forward problem

Here we Study the behaviour of θ^e as $e \to 0$ and where θ^e is the unique solution of the problem (3.4). With applications in mind, we take the initial conditions in (3.4d) to depend on e. Let us therefore rewrite the system:

$$\frac{\partial^2 \theta^e \theta^e}{\partial t^2} - \left(\frac{\partial^2 \theta^e}{\partial z_1^2} + \frac{\partial^2 \theta^e}{\partial z_2^2} + e^{-2}\,\frac{\partial^2 \theta^e}{\partial z_3^2}\right) = f \quad \text{in } Q \ , \tag{8.2a}$$

$$\theta^e = 0 \quad \text{on } \Sigma_0 \tag{8.2b}$$

$$\frac{\partial \theta^e}{\partial \nu} = 0 \quad \text{on } \Sigma_\pm \ , \tag{8.2c}$$

$$\theta^e(0) = \theta_0^e \ \text{ and } \ \frac{\partial \theta^e}{\partial t}(0) = \theta_1^e \quad \text{in } \Omega \ . \tag{8.2d}$$

We have then the following result:

Theorem 8.1 *Suppose that* $f \in L^1(0,T;L^2(\Omega))$ *and further that*

$$\theta_0^e \to \theta_0^* \quad \text{weakly in } V , \tag{8.3a}$$

$$\left\{ e^{-1} \frac{\partial \theta_0^e}{\partial z_3} \right\} \quad \text{bounded in } L^2(\Omega) , \tag{8.3b}$$

$$\theta_1^e \to \theta_1^* \quad \text{weakly in } L^2(\Omega) . \tag{8.3c}$$

Then

$$\theta^e \to \theta^* \quad \text{weakly in } L^\infty(0,T;V) , \tag{8.4a}$$

$$\frac{\partial \theta^e}{\partial t} \to \frac{\partial \theta^*}{\partial t} \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)) , \tag{8.4b}$$

where θ^* *is the unique solution of the following problem:*

$$\frac{\partial^2 \theta^*}{\partial t^2} - \left(\frac{\partial^2 \theta^*}{\partial z_1^2} + \frac{\partial^2 \theta^*}{\partial z_2^2} \right) = m(f) \quad \text{in } \omega \times (0,T) , \tag{8.5a}$$

$$\theta^* = 0 \quad \text{on } \gamma \times (0,T) , \tag{8.5b}$$

$$\theta^*(0) = m(\theta_0^*), \ \frac{\partial \theta^*}{\partial t}(0) = m(\theta_1^*) \quad \text{in } \omega , \tag{8.5c}$$

$$\theta^* \quad \text{is independent of } z_3 . \tag{8.5d}$$

Proof: We first remark that one can solve the problem (8.5) in a unique manner with the following regularity:

$$\theta^* \in C^0([0,T];\widetilde{V}) \cap C^1([0,T],L^2(\omega)) , \tag{8.6}$$

where

$$\widetilde{V} = \left\{ v \in H^1(\omega), \ v = 0 \ \text{ on } \gamma \right\} , \tag{8.7}$$

because $m(f) \in L^1(0,T;L^2(\omega))$, $m(\theta_0^*) \in \widetilde{V}$ and $m(\theta_1^*) \in L^2(\omega)$.

In order to pass to the limit in (8.2) we establish bounds on θ^e. The energy estimate (3.5) with our hypotheses on f, θ_0^e, θ_1^e imply that

$$\frac{\partial \theta^e}{\partial t} \in \text{bounded in } L^\infty(0,T;L^2(\Omega)) , \tag{8.8a}$$

$$\theta^e \in \text{bounded in } L^\infty(0,T;V) , \tag{8.8b}$$

$$e^{-1} \frac{\partial \theta^e}{\partial z_3} \in \text{bounded in } L^\infty(0,T;L^2(\Omega)) . \tag{8.8c}$$

We conclude therefore that there exists θ^* such that (along a subsequence)

(8.9a) $$\theta^e \to \theta^* \quad \text{in } \ L^\infty(0,T;V)\text{-weak}^* \ ,$$

(8.9a) $$\frac{\partial \theta^e}{\partial t} \to \frac{\partial \theta^*}{\partial t} \quad \text{in } \ L^\infty(0,T;L^2(\Omega))\text{-weak}^* \ ,$$

(8.9c) $$\frac{\partial \theta^*}{\partial z_3} = 0 \ .$$

As a consequence of (8.9c), we see that

(8.10) $$\theta^* \in L^\infty(0,T;\widetilde{V}) \quad \text{and} \quad \frac{\partial \theta^*}{\partial t} \in L^\infty(0,T;L^2(\omega)) \ .$$

Next, to see that this limit θ^* is actually a solution of (8.5), we multiply the problem (8.2) by a test function v which is independent of z_3 and pass to the limit using the weak convergence (8.2). More precisely, we take v in $L^1(0,T;\widetilde{V})\cap H^2(0,T;L^2(\omega))$ and multiply (8.2) and integrate by parts:

(8.11) $$\begin{aligned}\int_Q f\, v\, dz\, dt = \int_Q \theta^e \frac{\partial^2 v}{\partial t^2}\, dz\, dt + \int_Q \left\{ \frac{\partial \theta^e}{\partial z_1}\frac{\partial v}{\partial z_1} + \frac{\partial \theta^e}{\partial z_2}\frac{\partial v}{\partial z_2} \right\} dz\, dt \\ + \left[\int_\Omega \frac{\partial \theta^e}{\partial t}\, v\, dz \right]_0^T - \left[\int_\Omega \theta^e \frac{\partial v}{\partial t}\, dz \right]_0^T .\end{aligned}$$

We remark that there are no terms involving the derivatives with respect to z_3 because v is independent of z_3. We further restrict v such that

$$v(T) = \frac{\partial v}{\partial t}(T) = 0 \quad \text{in } \ \Omega \ .$$

We can obviously pass to the limit in the above relation (8.11) as $e \to 0$. In the resulting relation, we integrate with respect to z_3 keeping z_1, z_2 fixed. We obtain

(8.12)
$$\begin{aligned}\int_{\omega\times(0,T)} m(f)\, v\, d\widetilde{z}\, dt = \int_{\omega\times(0,T)} \theta^* \frac{\partial^2 v}{\partial t^2}\, d\widetilde{z}\, dt + \int_{\omega\times(0,T)} \left[\frac{\partial \theta^*}{\partial z_1}\frac{\partial v}{\partial z_1} + \frac{\partial \theta^*}{\partial z_2}\frac{\partial v}{\partial z_2} \right] d\widetilde{z}\, dt \\ + \left[\int_\omega m(\theta_1^*)\, v\, d\widetilde{z} \right]_0^T - \left[\int_\omega m(\theta_0^*) \frac{\partial v}{\partial t}\, d\widetilde{z} \right]_0^T .\end{aligned}$$

This relation is obviously a weak form of (8.5) since the limit is unique we get the convergence of the entire sequence and hence the theorem. ∎

As a corollary of the previous theorem, we can deduce the following convergence result on the problem (3.1) where the initial conditions also depend on e.

Theorem 8.2 *Let ϕ^e be the solution of the following system:*

(8.13a) $$\frac{\partial^2\phi^e}{\partial t^2} - \left(\frac{\partial^2\phi^e}{\partial z_1^2} + \frac{\partial^2\phi^e}{\partial z_2^2} + e^{-2}\frac{\partial^2\phi^e}{\partial z_3^2}\right) = 0 \quad in \ \ Q \ ,$$

(8.13b) $$\phi^e = 0 \quad on \ \ \Sigma_0 \ ,$$

(8.13c) $$\frac{\partial\phi^e}{\partial\nu} = 0 \quad on \ \ \Sigma_\pm \ ,$$

(8.13d) $$\phi^e(0) = \phi_0^e, \ \ \frac{\partial\phi^e}{\partial t}(0) = \phi_1^e \quad in \ \ \Omega \ .$$

Suppose that

(8.14a) $$\phi_0^e \to \phi_0^* \quad in \ \ V \ \ weakly \ ,$$

(8.14b) $$\left\{e^{-1}\frac{\partial\phi_0^e}{\partial z_3}\right\} \quad bounded \ in \ \ L^2(\Omega) \ ,$$

(8.14c) $$\phi_1^e \to \phi_1^* \quad in \ \ L^2(\Omega) \ weakly \ .$$

Then

(8.15a) $$\phi^e \to \phi^* \quad in \ \ L^\infty(0,T;V) \ \ weakly* \ ,$$

(8.15b) $$\frac{\partial\phi^e}{\partial t} \to \frac{\partial\phi^*}{\partial t} \quad in \ \ L^\infty(0,T;L^2(\Omega)) \ \ weakly* \ ,$$

where ϕ^* is the unique solution characterized by

(8.16a) $$\phi^* \quad indepednt \ of \ \ z_3 \ ,$$

(8.16b) $$\frac{\partial^2\phi^*}{\partial t^2} - \left(\frac{\partial^2\phi^*}{\partial z_1^2} + \frac{\partial^2\phi^*}{\partial z_2^2}\right) = 0 \quad in \ \ \omega\times(0,T) \ ,$$

(8.16c) $$\phi^* = 0 \quad on \ \ \gamma\times(0,T) \ ,$$

(8.16d) $$\phi^*(0) = m(\phi_0^*), \ \ \frac{\partial\phi^*}{\partial t}(0) = m(\phi_1^*) \quad in \ \ \omega \ . \ \blacksquare$$

8.2. Behaviour of the Nonhomogeneous backward problem

Our next aim is to pass to the limit in the backward Cauchy problem (6.5). This is done in our next result. We start with $\{\phi_0^e, \phi_1^e\}$ such that

(8.17) $$\|\{\phi_0^e, \phi_1^e\}\|_F \le c_6 \quad \text{independent of} \ \ e \ .$$

We then have, for a subsequence

(8.18) $$\{\phi_0^e, \phi_1^e\} \to \{\phi_0^*, \phi_1^*\} \quad \text{in } F \text{ weak} .$$

Since the norm in F dominates the energy norm $\| \ \|_F$ uniformly with respect to e (cf. (6.4)), it follows that the hypotheses of Theorem 8.2 are satisfied. Thus we have the convergence given by (8.15), (8.16). However, our stronger assumption (8.18) is equivalent to

(8.19a) $$\frac{\partial \phi^e}{\partial t} \to \frac{\partial \phi^*}{\partial t} \quad \text{in } L^2(\Sigma_\pm) \text{ weak} ,$$

(8.19b) $$\frac{\partial \phi^e}{\partial \nu} \to \frac{\partial \phi^*}{\partial \nu} \quad \text{in } L^2(\Sigma(z^0)) \text{ weak} ,$$

where ϕ^* is the unique solution of (8.16).

We are now ready to examine the consequence of the convergence (8.19) on the solution ψ^e of (6.5). More precisely, we have the following result:

Theorem 8.3 *Let us start with $\{\phi_0^e, \phi_1^e\}$ satisfying (8.18). We solve (8.13) for ϕ^e and then solve (6.5) for $\psi^e \in L^\infty(0,T;V')$. Then we have*

(8.20a) $$m(\psi^e) \quad \textit{makes sense as an element of } L^\infty(0,T;L^2(\omega)) ,$$

(8.20b) $$m(\psi^e) \to \psi^* \quad \textit{weak* in } L^\infty(0,T;L^2(\omega)) ,$$

where ψ^* is the unique solution of

(8.21b) $$\psi^* \quad \textit{is independent of } z_3 ,$$

(8.21b) $$\psi^* \in L^\infty(0,T;L^2(\omega)) ,$$

(8.21c) $$\frac{\partial^2 \psi^*}{\partial t^2} - \Big(\frac{\partial^2 \psi^*}{\partial z_1^2} + \frac{\partial^2 \psi^*}{\partial z_2^2}\Big) = 2\Big(\frac{\partial^2 \phi^*}{\partial t^2} - \phi^*\Big) \quad \textit{in } \omega \times (0,T) ,$$

(8.21d) $$\psi^* = \frac{\partial \phi^*}{\partial \nu} \quad \textit{on } \gamma(z^0) \times (0,T) ,$$

(8.21e) $$\psi^* = 0 \quad \textit{on } \gamma_* \times (0,T) ,$$

(8.21f) $$\psi^*(T) = \frac{\partial \psi^*}{\partial t}(T) = 0 \quad \textit{in } \omega ,$$

with ϕ^* being the solution of (8.16).

Proof: In order to pass to the limit in the backward problem (6.5), we consider its weak formulation (6.7). The idea is to take

(8.22) $$\theta_0,\ \theta_1 \text{ and } f \quad \text{independent of } z_3 .$$

With this choice, it is easily seen that the unique solution θ^e of (3.4) does not depend on z_3 neither on e

$$\text{(i.e.)} \quad \theta^e = \theta^* \ , \tag{8.23}$$

where θ^* is the solution of (8.5) Thus the weak converges (8.19) is sufficient to pass to the limit in (6.7).

Let us make this idea more precise. Recall that the weak formulation (6.7) is valid for $f \in L^1(0,T;V)$ and $\{\theta_0, \theta_1\} \in F$. However we observe that if we impose the condition (8.22) then the weak formulation (6.7) is available to us with

$$f \in L^1(0,T;L^2(\omega))\,, \quad \theta_0 \in \widetilde{V}, \quad \theta_1 \in L^2(\omega) \ . \tag{8.24}$$

Indeed, one has automatically that (without $\{\theta_0, \theta_1\} \in F$)

$$\frac{\partial \theta^e}{\partial \nu} \in L^2(\Sigma(z^0)) \ , \tag{8.25a}$$

$$\frac{\partial \theta^e}{\partial t} \in L^2(\Sigma_\pm) \ , \tag{8.25b}$$

for the following reasons:

i) According to (4.4), we have $\frac{\partial\theta^e}{\partial\nu} \in L^2(\Sigma(z^0))$ for any solution with finite energy and in particular the same is valid under (8.24). Indeed

$$\int_{\Sigma(z^0)} \Big(\frac{\partial \theta^e}{\partial \nu}\Big)^2 d\sigma\, dt \le C_1 T \Big\{ \|\theta_0\|^2_{\widetilde{V}} + \|\theta_1\|^2_{L^2(\omega)} + \Big(\int_0^T \|f(s)\|_{L^2(\omega)}\, ds\Big)^2 \Big\} \ . \tag{8.26}$$

ii) With (8.23) we see that θ^e is independent of z_3 and hence

$$\begin{aligned} \int_{\Sigma_\pm} \Big(\frac{\partial \theta^e}{\partial t}\Big)^2 d\sigma\, dt &= \int_{\omega\times(0,T)} \Big(\frac{\partial \theta^e}{\partial t}\Big)^2 d\tilde{z}\, dt = \int_Q \Big(\frac{\partial \theta^e}{\partial t}\Big)^2 dz\, dt \\ &\le C_0 \Big\{ \|\theta_0\|^2_{\widetilde{V}} + \|\theta_1\|^2_{L^2(\omega)} + \Big(\int_0^T \|f(s)\|_{L^2(\omega)} ds\Big)^2 \Big\} \end{aligned} \tag{8.27}$$

using (3.5).

We now choose $\theta_0 = \theta_1 = 0$ and f respecting the conditions (8.22), (8.24). Using (6.7), (8.26) and (8.27) we see that

$$\Big|\int_Q \psi^e f\, dz\, dt\Big| \le C_7 \Big(\int_0^T \|f(s)\|_{L^2(\omega)} ds\Big) \ , \tag{8.28}$$

with a constant C_7 independent of e.

This means that

$$m(\psi^e) \;\in\; \text{bounded in } L^\infty(0,T;L^2(\omega)) \;.$$

As usual by passing to a subsequence, we assume that

$$m(\psi^e) \to \psi^* \quad \text{in} \;\; L^\infty(0,T;L^2(\omega)) \text{ weak} * \;. \tag{8.29}$$

As the next step, we take $f \equiv 0$ and $\{\theta_0, \theta_1\}$ satisfying (8.22), (8.24). We see, by above argument, that $m(\psi_0^e)$ and $m(\psi_1^e)$ make sense and

$$\begin{aligned} m(\psi_0^e) &\quad \text{is bounded in} \;\; L^2(\omega) \;, \\ m(\psi_1^e) &\quad \text{is bounded in} \;\; \widetilde{V}' \;. \end{aligned}$$

As usual, we suppose that

$$m(\psi_0^e) \to \psi_0^* \quad \text{weakly in} \;\; L^2(\omega) \;, \tag{8.30a}$$

$$m(\psi_1^e) \to \psi_1^* \quad \text{weakly in} \;\; \widetilde{V}' \;. \tag{8.30b}$$

We are now in a position to pass to the limit in (6.7) and we obtain
(8.31)

$$\begin{aligned} {}_{\widetilde{V}'}\langle \psi_1^*, \theta_0 \rangle_{\widetilde{V}} - {}_{L^2(\omega)}\langle \psi_0^*, \theta_1 \rangle_{L^2(\omega)} &= \int_{\omega\times(0,T)} \psi^* \, f \, d\widetilde{z} \, dt + \int_{\gamma(z^0)\times(0,T)} \frac{\partial \phi^*}{\partial \nu} \, \frac{\partial \theta^*}{\partial \nu} \, d\widetilde{\sigma} \, dt \\ &\quad + \int_{\omega\times(0,T)} \left[\frac{\partial \phi^*}{\partial t} \, \frac{\partial \theta^*}{\partial t} + \phi^* \, \theta^* \right] d\widetilde{z} \, dt \;. \end{aligned}$$

This remains true for all f, θ_0, θ_1 satisfying (8.24). To finish the proof, it remains to observe that (8.31) is nothing but a weak formulation of (8.21). ∎

Let us note that the above result gives the convergence of the average $m(\psi^e)$ and not of ψ^e.

8.3. The Limiting two-dimensional exact controllability problem

In this section, we introduce and analyze the exact controllability problem which will be the limit of the three dimensional problems (2.8) as $e \to 0$. This limit problem is two dimensional and posed in ω. Our task here is to define it and examine its well posedness via Hum. We could have this earlier but it could look very artificial. Having seen the behaviour of the forward and backward Cauchy problems in the previous sections, we feel that this is the appropriate place to introduce the limit problem. As mentioned earlier and as is evident from (821c, d), this limit problem will have an anterior control apart from a boundary one. The analysis of this problem parallels that of the three-dimensional problem presented in sections 2–7 and so we will be brief. The question of convergence of the three-dimensional problem (2.8) to the limit problem described here will be taken up in the next section.

Let us fix one interior control $\widetilde{w}$ and one boundary control $\widetilde{v}$ and consider the following problem:

$$\frac{\partial^2 \widetilde{y}}{\partial t^2} - \left(\frac{\partial^2 \widetilde{y}}{\partial z_1^2} + \frac{\partial^2 \widetilde{y}}{\partial z_2^2} \right) = \widetilde{w} \quad \text{in } \omega \times (0, \widetilde{T}) \,, \tag{8.32a}$$

$$\widetilde{y} = \widetilde{v} \quad \text{on } \gamma(z^0) \times (0, \widetilde{T}) \,, \tag{8.32b}$$

$$\widetilde{y} = 0 \quad \text{on } \gamma_* \times (0, \widetilde{T}) \,, \tag{8.32c}$$

$$\widetilde{y}(0) = \widetilde{y}_0 \ \text{ and } \ \frac{\partial \widetilde{y}}{\partial t}(0) = \widetilde{y}_1 \quad \text{in } \omega \,. \tag{8.32d}$$

For the moment $\widetilde{T} > 0$ is fixed. The problem is to suitable controls $\widetilde{v}$, $\widetilde{w}$ and time $\widetilde{T}$ such that the state of system (9.32) is driven to rest at time $\widetilde{T}$:

$$\widetilde{y}(\widetilde{T}) = \frac{\partial \widetilde{y}}{\partial t}(\widetilde{T}) = 0 \quad \text{in } \omega \,. \tag{8.33}$$

To show that this problem admits a solution via Hum, we introduce, as usual, the associated homogeneous forward problem: given initial conditions $\{\widetilde{\phi}_0, \widetilde{\phi}_1\} \in \widetilde{V} \times L^2(\omega)$, we seek $\widetilde{\phi} \in C^0([0,\widetilde{T}], \widetilde{V}) \cap C^1([0,\widetilde{T}], L^2(\omega))$ such that

$$\frac{\partial^2 \widetilde{\phi}}{\partial t^2} - \left(\frac{\partial^2 \widetilde{\phi}}{\partial z_1^2} + \frac{\partial^2 \widetilde{\phi}}{\partial z_2^2} \right) = 0 \quad \text{in } \omega \times (0, \widetilde{T}) \,, \tag{8.34a}$$

$$\widetilde{\phi} = 0 \quad \text{on } \gamma \times (0, \widetilde{T}) \,, \tag{8.34b}$$

$$\widetilde{\phi}(0) = \widetilde{\phi}_0, \ \ \frac{\partial \widetilde{\phi}}{\partial t}(0) = \widetilde{\phi}_1 \quad \text{in } \omega \,. \tag{8.34c}$$

This problem is seen to admit a unique solution via a unitary group which preserves the following energy functional:

$$\widetilde{E}(t) = \int_\omega \left\{ |\nabla \widetilde{\phi}(\widetilde{z}, t)|^2 + \frac{\partial \widetilde{\phi}}{\partial t}(\widetilde{z}, t)^2 \right\} d\widetilde{z} \,. \tag{8.35}$$

We have in fact that

$$\widetilde{E}(t) \equiv \widetilde{E}(0) \,, \quad \forall t \,, \tag{8.36}$$

$$\widetilde{E}(0) = \int_\omega \left[|\nabla \widetilde{\phi}_0|^2 + |\widetilde{\phi}_1|^2 \right] d\widetilde{z} \ \ (\equiv \|\{\widetilde{\phi}_0, \widetilde{\phi}_1\}\|^2_{\widetilde{E}}) \,. \tag{8.37}$$

One establishes, next, by the method of multipliers the following direct and inverse inequalities for the problem (8.34). (cf. Lions [4], p. 401, p. 55)

$$\int_{\gamma(z^0)\times(0,\widetilde{T})} \left(\frac{\partial\widetilde{\phi}}{\partial\nu}\right)^2 d\sigma\, dt \le C_8 \widetilde{T}\, \|\{\widetilde{\phi}_0,\widetilde{\phi}_1\}\|^2_{\widetilde{E}} \ , \tag{8.38a}$$

$$\|\{\widetilde{\phi}_0,\widetilde{\phi}_1\}\|^2_{\widetilde{E}} \le C_8 \int_{\gamma(z^0)\times(0,\widetilde{T})} \left(\frac{\partial\widetilde{\phi}}{\partial\nu}\right)^2 d\sigma\, dt \ , \tag{8.38b}$$

$$\|\{\widetilde{\phi}_0,\widetilde{\phi}_1\}\|^2_{\widetilde{E}} \le C_8 \int_{\omega\times(0,\widetilde{T})} \left(\frac{\partial\widetilde{\phi}}{\partial t}\right)^2 d\widetilde{z}\, dt \ . \tag{8.38c}$$

We remark that while (8.33a) and (8.38c) are valid for all times $\widetilde{T} > 0$, the inequality (8.38b) holds for time $\widetilde{T}$ sufficiently large.

Next we associate a space $\widetilde{F}$ to the exact controllability problem (8.32). Define $\widetilde{F}$ to be the completion of smooth functions for the norm defined below:

$$\|\{\widetilde{\phi}_0,\widetilde{\phi}_1\}\|^2_{\widetilde{F}} = \int_{\gamma(z^0)\times(0,\widetilde{T})} \left(\frac{\partial\widetilde{\phi}}{\partial\nu}\right)^2 d\sigma\, dt + 2\int_{\omega\times(0,\widetilde{T})} \left[\left(\frac{\partial\widetilde{\phi}}{\partial t}\right)^2 + (\widetilde{\phi}^2)\right] d\widetilde{z}\, dt \ . \tag{8.39}$$

It follows immediately from (8.36)–(8.38) that the norms $\|\ \|_{\widetilde{F}}$ and $\|\ \|_{\widetilde{E}}$ are equivalent $\forall\, \widetilde{T} > 0$ and hence

$$\widetilde{F} = \widetilde{V} \times L^2(\omega) \ ; \tag{8.40a}$$

consequently, we obtain

$$\widetilde{F}' = \widetilde{V}' \times L^2(\omega) \ . \tag{8.40b}$$

Remark: Even though we have the equivalence of the norms $\|\ \|_{\widetilde{E}}$ and $\|\ \|_{\widetilde{F}}$ for all $\widetilde{T} > 0$, we are not going to use it in the sequel. We will use it only for $\widetilde{T}$ sufficiently large. This is because, for $e > 0$, the inequality and exact controllability for the problem (2.8) were shown only for T sufficiently large. (cf. Theorem 5.1). Hence to prove the convergence results in the next section, we are forced to take $\widetilde{T} = T$ so that both the problems (2.8) and (8.32) are exactly controllable at the same time.

The next step in HUM is to define the corresponding nonhomogeneous backward Cauchy problem:

Find $\widetilde{\psi} \in L^\infty(0,\widetilde{T}; L^2(\omega))$ such that

$$\frac{\partial^2\widetilde{\psi}}{\partial t^2} - \left(\frac{\partial^2\widetilde{\psi}}{\partial z_1^2} + \frac{\partial^2\widetilde{\psi}}{\partial z_2^2}\right) = 2\left(\frac{\partial^2\widetilde{\phi}}{\partial t^2} - \widetilde{\phi}\right) \quad \text{in } \omega\times(0,\widetilde{T}) \ , \tag{8.41a}$$

$$\widetilde{\psi} = \frac{\partial \widetilde{\phi}}{\partial \nu} \quad \text{on} \;\; \gamma(z^0) \times (0, \widetilde{T}) \;, \tag{8.41b}$$

$$\widetilde{\psi} = 0 \quad \text{on} \;\; \gamma_* \times (0, \widetilde{T}) \;, \tag{8.41c}$$

$$\widetilde{\psi}(\widetilde{T}) = \frac{\partial \widetilde{\psi}}{\partial t}(\widetilde{T}) = 0 \quad \text{in} \;\; \omega \;. \tag{8.41d}$$

Here $\widetilde{\phi}$ is the solution of the problem (8.34).

To obtain the required weak formulation of (8.41), we take

$$\widetilde{f} \in L^1(0, \widetilde{T}, L^2(\omega)) \,, \qquad \widetilde{\theta}_0 \in \widetilde{V}, \quad \widetilde{\theta}_1 \in L^2(\omega) \;. \tag{8.42}$$

Let us solve uniquely the following problem:

$$\frac{\partial^2 \widetilde{\theta}}{\partial t^2} - \left(\frac{\partial^2 \widetilde{\theta}}{\partial z_1^2} + \frac{\partial^2 \widetilde{\theta}}{\partial z_2^2} \right) = \widetilde{f} \quad \text{in} \;\; \omega \times (0, \widetilde{T}) \;, \tag{8.43a}$$

$$\widetilde{\theta} = 0 \quad \text{in} \;\; \gamma \times (0, \widetilde{T}) \;, \tag{8.43b}$$

$$\widetilde{\theta}(0) = \widetilde{\theta}_0, \quad \frac{\partial \widetilde{\theta}}{\partial t}(0) = \widetilde{\theta}_1 \quad \text{in} \;\; \omega \;, \tag{8.43c}$$

$$\widetilde{\theta} \in C^0([0, \widetilde{T}]; \widetilde{V}) \cap C^1([0, \widetilde{T}]; L^2(\omega)) \;. \tag{8.43d}$$

We multiply (8.41) by $\widetilde{\theta}$ and integrate by parts twice to obtain the following weak formulation of the problem (8.41):

There exist $\{\widetilde{\psi}_0, \widetilde{\psi}_1\} \in L^2(\omega) \times \widetilde{V}'$ and $\widetilde{\psi} \in L^\infty(0, \widetilde{T}; L^2(\omega))$ such that the relation

(8.44)

$$\begin{aligned} {}_{\widetilde{V}'}\langle \widetilde{\psi}_1, \widetilde{\theta}_0 \rangle_{\widetilde{V}} - {}_{L^2(\omega)}\langle \widetilde{\psi}_0, \widetilde{\theta}_1 \rangle_{L^2(\omega)} = \int_{\omega \times (0,\widetilde{T})} \psi^* \, \widetilde{f} \, d\widetilde{z} \, dt + \int_{\gamma(z^0) \times (0,\widetilde{T})} \frac{\partial \widetilde{\phi}}{\partial \nu} \, \frac{\partial \widetilde{\theta}}{\partial \nu} \, d\widetilde{\sigma} \, dt \\ + 2 \int_{\omega \times (0,\widetilde{T})} \left(\frac{\partial \widetilde{\phi}}{\partial t} \, \frac{\partial \widetilde{\theta}}{\partial t} + \widetilde{\phi} \, \widetilde{\theta} \right) d\widetilde{z} \, dt \end{aligned}$$

holds for all $\widetilde{f}$, $\widetilde{\theta}_0$, $\widetilde{\theta}_1$ satisfying (8.42).

It is easily seen that (8.44) admits a unique solution.

The final step in the preparation towards solving the exact controllability problem is to introduce the linear operator $\widetilde{\Lambda}$ as follows:

$$\begin{aligned} &\widetilde{\Lambda} \colon \, \widetilde{F} \to \widetilde{F}' \;, \\ &\widetilde{\Lambda}\{\widetilde{\phi}_0, \widetilde{\phi}_1\} = \{\widetilde{\psi}_1, -\widetilde{\psi}_0\} \;. \end{aligned} \tag{8.45}$$

By taking $\widetilde{f} = 0$, $\widetilde{\theta}_0 = \widetilde{\phi}_0$ and $\widetilde{\theta}_1 = \widetilde{\phi}_1$ in (8.44) we see that

$$ {}_{\widetilde{F}'}\Big\langle \widetilde{\Lambda}\{\widetilde{\phi}_0, \widetilde{\phi}_1\}, \{\widetilde{\phi}_0, \widetilde{\phi}_1\} \Big\rangle_{\widetilde{F}} = \|\{\widetilde{\phi}_0, \widetilde{\phi}_1\}\|^2_{\widetilde{F}} \;. \tag{8.46}$$

This proves that $\widetilde{\Lambda}$ is a isomorphism.

Now that all the preparatory material out of our way, it is a simple matter to solve the exact controllability problem (8.32) and (8.33). In fact the problem is equivalent to finding $\{\widetilde{\phi}_0, \widetilde{\phi}_1\} \in \widetilde{F}$ such that

$$\widetilde{\Lambda}\{\widetilde{\phi}_0, \widetilde{\phi}_1\} = \{\widetilde{y}_1, -\widetilde{y}_0\} \ \in \ \widetilde{F}' \tag{8.47}$$

and then taking

$$\widetilde{w} = 2\left(\frac{\partial^2 \widetilde{\phi}}{\partial t^2} - \widetilde{\phi}\right) \quad \text{in } \ \omega \times (0, \widetilde{T}) \ , \tag{8.48a}$$

$$\widetilde{v} = \frac{\partial \widetilde{\phi}}{\partial \nu} \quad \text{on } \ \gamma(z^0) \times (0, \widetilde{T}) \ , \tag{8.48b}$$

where $\widetilde{\phi}$ solves (8.34). In particular, we see that the controls have the following properties:

$$\widetilde{w} \in C^0([0, \widetilde{T}]; H^{-1}(\omega)) \ , \tag{8.49a}$$

$$\widetilde{v} \in L^2(\gamma(z^0)) \ . \tag{8.49b}$$

Thus we have analyzed completely, in this section, the two dimensional exact controllability problem (8.32) (8.33). We group these results in our next proposition.

Theorem 8.4 *We consider the two-dimensional exact controllability problem (8.32) (8.33). We take the initial conditions $\{\widetilde{y}_1, -\widetilde{y}_0\} \in \widetilde{V}' \times L^2(\omega)$. We also suppose that $\widetilde{T} = T$, the time of exact controllability fixed in Theorem 7.2. Then the problem is exactly controllable with exact controls given by (8.47), (8.48), (8.34).*

8.4. Behaviour of the exact controllability problem

We have shown in Sections 8.1, 8.2 how to pass to the limit in the forward and backward Cauchy problems under certain hypotheses. We are now in a position to pass to the limit in the three-dimensional exact controllability problem (2.8). Recall that we proved the exact controllability with the controls given by (7.4) and (7.5). The aim here is to describe the behaviour of these controls as $e \to 0$. According to our results below, the controls $w^e_\pm$ on the top-bottom surfaces $\Sigma_\pm$ go to zero in a suitable sense. On the other hand, the lateral control v^e on $\Sigma(z^0)$ approaches the boundary control $\widetilde{v}$ in the two dimensional problem (8.32) (8.33). Recall that there is an interior control also in the problem (8.32) (8.33). These two controls are not independent as (8.48) shows and they cannot be separated in general. We achieve these results by making some natural hypotheses on the initial data of the problem (2.8). In fact, we suppose that

$$\|\{y_1^e, -y_0^e\}\|_{F'} \le C_9, \quad \text{independent of } e \ , \tag{8.50a}$$

$$m(y_0^e) \to y_0^* \quad \text{in } \ L^2(\omega) \text{ weak} \ , \tag{8.50b}$$

$$m(y_1^e) \to y_1^* \quad \text{in } \ \widetilde{V}' \text{ weak} \ . \tag{8.50c}$$

Thanks to (6.4), one can give sufficient conditions which guarantee (8.50). For example, one can take $\{y_1^e, -y_0^e\}$ such that

$$\{y_0^e\} \quad \text{bounded in } \ L^2(\Omega) \ , \tag{8.51a}$$

$$m(y_0^e) \to y_0^* \quad \text{in } \ L^2(\omega) \text{ weak} \ , \tag{8.51b}$$

$$y_1^e = \frac{\partial g_1^e}{\partial z_1} + \frac{\partial g_2^e}{\partial z_2} \quad \text{in } \ \Omega \ , \tag{8.51c}$$

$$\{g_1^e\} \ \text{ and } \ \{g_2^e\} \quad \text{bounded in } \ L^2(\Omega) \ , \tag{8.51d}$$

$$m(g_1^e) \to g_1^* \quad \text{in } \ L^2(\omega) \text{ weak} \ , \tag{8.51e}$$

$$m(g_2^e) \to g_2^* \quad \text{in } \ L^2(\omega) \text{ weak} \ . \tag{8.51f}$$

The importance of the hypothesis (8.50) is that it implies in conjunction with Theorem 7.1 (b) that

$$\|\{\phi_0^e, \phi_1^e\}\|_F \le C_{10} \ , \tag{8.52}$$

where $\{\phi_0^e, \phi_1^e\}$ is the solution of (7.4) and C_{10} is a constant independent of e. It follows then that (cf. (6.4))

$$\|\{\phi_0^e, \phi_1^e\}\|_E \le C(10)\, C_5 \ . \tag{8.53}$$

Hence for a subsequence of $e \to 0$, we will have the convergence results described by (8.14) and (8.19). In particular, we see that the hypotheses of Theorem 8.2, 8.3 are satisfied and so we have convergence properties described by Theorem 8.2 and Theorem 8.3 for the subsequence under consideration. We now show that the convergence takes place for the entire sequence. To this end, let us first observe that (8.14a) (8.14b) imply that ϕ_0^* is independent of z_3:

$$\text{(i.e.)} \quad m(\phi_0^*) = \phi_0^* \ . \tag{8.54}$$

Secondly, we observe that it is enough to show that

$$m(\phi_0^*), \ m(\phi_1^*) \ \text{ are uniquely determined} \ . \tag{8.55}$$

Indeed this will imply that the solution ϕ^* of (8.16) and therefore the solution ψ^* of (8.21) are uniquely determined. Hence the convergences (8.14a) (8.15), (8.19), (8.20b) and (8.30) take the entire sequence . The only exception being that (8.14c) may not be true for whole sequence $\{\phi_1^e\}$. However the entire sequence $m(\phi_1^e)$ will converge.

In order to deduce (8.55), we apply the results of §8.3 and in particular the definition of the operator $\widetilde{\Lambda}$. From (8.16) and (8.21) we conclude that

$$\widetilde{\Lambda}\Big\{m(\phi_0^*)\Big\} = \Big\{\psi_1^*, -\psi_0^*\Big\} \ .$$

Recall the right side element is obtained as the limit of $\{m(\psi_1^e) - m(\phi_0^e)\}$ (cf. (8.30)). However we always have $\psi^e = y^e$ and hence our hypothesis (8.50) on the initial data permits us to conclude that $\psi_1^* = y_1^*$ and $\psi_0^* = y_0^*$. Thus we get

$$\widetilde{\Lambda}\Big\{m(\phi_0^*), m(\phi_1^*)\Big\} = \Big\{y_1^*, -y_0^*\Big\} \ . \tag{8.56}$$

Since $\widetilde{\Lambda}$ is an isomorphism, this immediately implies (8.55).

For its importance, we single out the convergence (8.19b) which shows that the lateral boundary control $v^e = \frac{\partial \phi^e}{\partial \nu}$ on $\Sigma(z^0)$ converges to the boundary control $\widetilde{v} = \frac{\partial \phi^*}{\partial \nu}$ of the problem (8.32) and (8.33) described in §8.3 with initial conditions given by

$$\widetilde{y}_0 = m(\phi_0^*)\,, \quad \widetilde{y}_1 = m(\phi_1^*)\ , \tag{8.57}$$

On other hand, the convergence (8.19a) shows that the top-bottom boundary controls $w_\pm^e = e(\frac{\partial^2 \phi^e}{\partial t^2} - \phi^e)$ on $\Sigma_\pm$ converges strongly to zero in the space $[H^1(0,T;L^2(\Gamma_\pm))]$.

Let us summarize these results in our concluding theorem:

Theorem 8.5 *Let us consider the exact controllability problem (2.8) in three-dimensions. We assume that the initial conditions $\{y_0^e, y_1^e\}$ satisfy the requirement (8.50). Furthermore, the time of exact controllability T is fixed as in Theorem 7.2.*

i) *Then the exact controls $\{v^e, w_\pm^e\}$ enjoy the following properties:*

$$w_\pm^e = 0(e) \quad in \ \Big[H^1(0,T;L^2(\Gamma_\pm))\Big]'\ , \tag{8.58a}$$

$$v^e \to \widetilde{v} = \frac{\partial \phi^*}{\partial \nu} \quad in \ \ L^2(\Sigma(z^0)) \ weak\ . \tag{8.58b}$$

ii) *The solution y^e of the original problem (2.8) has the following behaviour:*

$$m(y^e) \to y^* \quad in \ \ L^\infty(0,T;L^2(\omega)) \ weak*\ , \tag{8.59}$$

where y^* satisfies

$$\frac{\partial^2 y^*}{\partial t^2} - \left(\frac{\partial^2 y^*}{\partial z_1^2} + \frac{\partial^2 y^*}{\partial z_2^2}\right) = 2\left(\frac{\partial^2 \phi^*}{\partial t^2} - \phi^*\right) \quad in \ \ \omega \times (0,T) \ , \tag{8.60a}$$

$$y^* = \frac{\partial \phi^*}{\partial \nu} \quad on \ \ \gamma(z^0) \times (0,T) \ , \tag{8.60b}$$

$$y^* = 0 \quad on \ \ \gamma_* \times (0,T) \ , \tag{8.60c}$$

$$y^*(0) = y_0^* \ \ and \ \ \frac{\partial y^*}{\partial t}(0) = y_1^* \quad in \ \ \omega \ . \tag{8.60d}$$

Here ϕ^* is the unique solution of (8.16), (8.50) and (8.56).

Remark 8.6. We choose to control by w_+ and w_- on the top and the bottom boundaries because these play the same role geometrically. But it is sufficient to control by w on one of these two boundaries and to set a homogeneous Neumann boundary condition on the other one (see Lions [5]). Then the factor 2 which appears in (8.21c), Theorem 8.3 and in similar relations and which comes from the two control w_+ and w_- has to be replaced by 1 when there is only one control w. The proofs are the same.

Remark 8.7. After the completion of our work, we came to know that the same exact controllability problem was treated by Yan [8]. We now make a few remarks of comparison between his article and our work. First of all, Yan follows an approach different from ours. The exact controls suggested by him are different from the ones suggests by us, naturally hence their limit behaviour of the exact controls: Yan proves strong convergences with the hypothesis that the initial conditions converges strongly. We show weak convergence with the weak convergence property of the initial conditions. Secondly, the nature of the limit problem: in particular, there is an interior control in our case apart from the boundary one whereas Yan has only a boundary control. As a consequence, the minimal time of exact controllability will be bounded away from zero as thickness goes to zero in Yan. In our case, the limit problem is exactly controllable for all $T > 0$ and so the possibility that the minimal time of exact controllability of the three-dimensional problem tends to zero when the thickness tends to zero is not ruled out.

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