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NON LOCAL SOLUTIONS OF A NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

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Abstract: In this work we prove that the mixed problem for a temporally nonlinear Kirchhoff-Carrier model, for vibrations of a nonhomogeneous stretched string, has unique nonlocal solution for small data. The solution is obtained in S.L. Sobolev spaces.

Introduction

The nonlinear model of Kirchhoff-Carrier, cf. Carrier [5], for vibrations of an elastic string, of lenght L, is given by:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_o}{\rho.h} + \frac{E}{2L\rho}\int_0^L \left|\frac{\partial u}{\partial s}(s,t)\right|^2 ds\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

where $0 \le x \le L$ and $t > 0$ represent the string in repose, $u(x,t)$ is the vertical displacement of the point x at the instant t, ρ is the mass density, h is the area of the cross section of the string, L is the lenght of the string, P_o the initial tension on the string and E the Young's modulus of the material.

The natural generalization of the model (1) is given by the following nonlinear mixed problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 dx\right)\Delta u = f \text{ on } \mathbf{Q} = \Omega\times(0,T) \\ u = 0 \text{ on } \Sigma = \Gamma\times(0,T) \\ u(x,0) = \phi_o(x) \text{ on } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_1(x) \text{ on } \Omega \end{cases} \tag{2}$$

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where Ω is a bounded open set of $\mathbf{R}^n$ with smooth boundary Γ, $M\colon[0,\infty)\to\mathbf{R}$ is a positive real function and $\Delta=\sum_{i=1}^{n}\dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

Remark 1. In the Kirchhoff-Carrier model (1), $M\colon[0,\infty)\to\mathbf{R}$ is $M(\lambda)=\dfrac{P_o}{\rho.h}+\dfrac{E}{2L\rho}\lambda$.

Several authors have investigated the nonlinear problem (2). When $n=1$ and $\Omega=(0,L)$, it was studied by Dickey [8] and Bernstein [3] whom considered ϕ_o and ϕ_1 analytic functions with some growth conditions. Assuming Ω bounded open set of $\mathbf{R}^n$, ϕ_o and ϕ_1 analytic functions, Pohozaev [18] obtained existence and uniqueness of global solutions for the mixed problem (2). In Lions [12] he formulated the Pohozaev's results in an abstract context obtaining better results and presenting a collection of problems. One of the problems proposed by Lions [12] was the study of the problem (2) with $M\colon\Omega\times[0,\infty)\to\mathbf{R}$, i.e., the problem

$$(3)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}-M\Big(x,\displaystyle\sum_{i=1}^{n}\int_\Omega\Big|\frac{\partial u}{\partial x_i}\Big|^2dx\Big)\Delta u=f \text{ on } \mathbf{Q}\\ u=0 \text{ on } \Sigma\\ u(x,0)=\phi_o(x) \text{ on } \Omega\\ \dfrac{\partial u}{\partial t}(x,0)=\phi_1(x) \text{ on } \Omega \end{cases}$$

that is, for nonhomogeneous materials. This case has it's origin in the model (1) when the physic elements ρ, h and E are not constants, but depends on the point x in the string. In Rivera Rodrigues [20] the author proved the existence and uniqueness of local solutions for the problem (3).

In a more general context it is correct to consider ρ, h and E changing not only with the point x in the string but with the instant t too, i.e., $\rho=\rho(x,t)$, $h=h(x,t)$ and $E=E(x,t)$. In this case, we have the problem

$$(4)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}-M\Big(x,t,\displaystyle\sum_{i=1}^{n}\int_\Omega\Big|\frac{\partial u}{\partial x_i}\Big|^2dx\Big)\Delta u=f \text{ on } \mathbf{Q}\\ u=0 \text{ on } \Sigma\\ u(x,0)=\phi_o(x) \text{ on } \Omega\\ \dfrac{\partial u}{\partial t}(x,0)=\phi_1(x) \text{ on } \Omega \end{cases}$$

where $M\colon\Omega\times[0,T]\times[0,\infty)\to\mathbf{R}$.

In this work we study the problem (4) and making use of the same technique used by Rivera Rodrigues [20], we prove that if ϕ_o, ϕ_1, f and $\dfrac{\partial M}{\partial t}$ are small in some sense, then exist one, and only one, nonlocal solution for the problem (4). It's important to observe that it's a good assumption to consider $\dfrac{\partial M}{\partial t}$ small, because in normal conditions ρ, h and E have a small variation with the time.

For the study of problem (2) with dissipative terms we have, for instance, Brito [4] and Medeiros-Milla Miranda [14]. The problem (2) in the degenerate case can be find in Arosio-Spagnolo [1], Ebihara-Medeiros-Milla Miranda [9], Arosio-Garavaldi [2], Crippa [6], Yamada [21], Nishihara-Yamada [17] and Nishihara [16].

The plan of this paper is the following:

1) Notations and preliminary results;

2) Assumptions and statement of the principal result;

3) Galerkin's approximation and a priori estimates;

4) Proof of the theorem;

5) Uniqueness.

1 – Notation and preliminary results

Let Ω be a bounded open set of $\mathbf{R}^n$ with smooth boundary Γ. By $L^2(\Omega)$ we represent the usual space of Lebesgue square integrable functions on Ω whose inner product and norm will be denoted by $(\cdot\,,\cdot)$ and $|\cdot|$ respectively. In the Sobolev space $H_o^1(\Omega)$ we consider the norm

$$||u||^2 = \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^2 dx \tag{5}$$

and inner product

$$((u,v)) = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx~. \tag{6}$$

Let $(-\Delta)$ be the operator defined by $\{H_o^1(\Omega), L^2(\Omega), ((\cdot\,,\cdot))\}$. Then as we well known $(-\Delta)$ is an unbounded selfadjoint operator in $L^2(\Omega)$ with domain

$$D(-\Delta) = \Big\{u \in H_o^1(\Omega); \Delta u \in L^2(\Omega)\Big\} = H_o^1(\Omega) \cap H^2(\Omega) \tag{7}$$

and it has the following properties:

(a) There exist $m_o > 0$ such that

(8) $$(-\Delta u, u) \geq m_o\, |u|^2 \,,\ \forall\, u \in D(-\Delta)\ ;$$

(b)

(9) $$(-\Delta u, u) = ||u||^2 \,,\ \forall\, u \in D(-\Delta)\ ;$$

(c) There exist a sequence $(\lambda_j)_{j\in \mathbf{N}}$ of real numbers and $(w_j)_{j\in \mathbf{N}}$ a sequence of $L^2(\Omega)$ vectors such that

(10) $$m_o \leq \lambda_1 \leq \lambda_2 \leq \dots$$

(11) $$-\Delta w_j = \lambda_j w_j \,,\ \forall\, j \in \mathbf{N}$$

(12) $$\lim_{j\to\infty} \lambda_j = \infty$$

(13) $\{w_j\}$ is a orthonormal complete set in $L^2(\Omega)$ and orthogonal complete set in $H^1_o(\Omega)$ and in $H^1_o(\Omega) \cap H^2(\Omega)$.

Remark 2. We introduce the equivalent norm

(14) $$||u||_{H^1_o(\Omega)\cap H^2(\Omega)} = |-\Delta u|, \qquad \forall\, u \in H^1_o(\Omega) \cap H^2(\Omega)$$

for smooth boundary Γ.

In order to complete this section we introduce a compactness result. It is a version of Arzela's theorem and it's proof follows the same argument as the usual proof of scalar Arzela's theorem.

Lemma 1. *Let E and F be Banach spaces, $E \hookrightarrow F$ with compact injection. Let $(\sigma_m)_{m\in\mathbf{N}}$ be a sequence of functions from the interval $[a,b] \subset \mathbf{R}$ into E. If $(\sigma_m)_{m\in\mathbf{N}}$ is uniformly bounded in $[a,b]$ with respect to the norm of E and equicontinuous with respect to the norm of F, then there exist a subsequence $(\sigma_{m_\nu})_{\nu\in\mathbf{N}}$ of $(\sigma_m)_{m\in\mathbf{N}}$ and a continuous function $\sigma\colon [a,b] \to F$ such that*

(15) $$\lim_{\nu\to\infty} \sigma_{m_\nu}(t) = \sigma(t) \text{ in } F \text{ uniformly for } t \in [a,b]\ .$$

Moreover, if E is a reflexive Banach space then we find that $\sigma \in L^\infty(a,b;E)$.

2 – Assumptions and principal result

Let Ω be as in section 1, $T > 0$ a real number. We consider a real function

$$\begin{aligned} M\colon\ \Omega\times[0,T]\times[0,\infty) &\longrightarrow \mathbf{R} \\ (x,t,\lambda) &\longmapsto M(x,t,\lambda) \end{aligned}$$

such that the following assumptions are satisfied:

(H.1) $M \in L^\infty_{\rm loc}([0,\infty); W^{1,\infty}(\Omega\times(0,T)))$, i.e., for each $k > 0$ we have $M \in L^\infty(\Omega\times(0,T)\times(0,k))$, $\dfrac{\partial M}{\partial t} \in L^\infty(\Omega\times(0,T)\times(0,k))$ and $\dfrac{\partial M}{\partial x_i} \in L^\infty(\Omega\times(0,T)\times(0,k))$ for $i = 1,\ldots,n$.

(H.2) For each $L > 0$ we have $\dfrac{\partial M}{\partial \lambda} \in L^\infty(\Omega\times(0,T)\times(0,L))$.

(H.3) There exist a real number $m_1 > 0$ such that $m_1 \le M(x,t,\lambda)$, $\forall x \in \Omega$, $t \in [0,T]$ and $\lambda \ge 0$.

Now we define

$$\begin{aligned} & k_o = 4(m_o m_1^3)^{-1/2}, \qquad k_1 = \frac{1}{m_1} \\ & \theta_o = \operatorname*{ess\,sup}_{\substack{\vec{x}\in\Omega \\ 0<t<T}} \left|\frac{\partial M}{\partial t}(x,t,0)\right| \\ & k_2 = \frac{1}{2}\Big[1 + ||M||_{L^\infty(\Omega\times(0,T)\times(0,1))}\Big] \\ & k_3 = \frac{4}{m_o m_1}\Big[\Big(k_2 + \frac{T}{2}\Big)\Big(1 + e^{(1+k_1\theta_o)T}\Big)\Big] \\ & k_4 = \Big\|\frac{\partial M}{\partial \lambda}\Big\|_{L^\infty(\Omega\times(0,T)\times(0,k_3))} \end{aligned} \tag{16}$$

$$\delta = \min\left\{1; m_o^{1/2}; \frac{\ln 2}{3T[1 + Tk_o k_4 + Tk_o k_4\, e^{(1+k_1\theta_o)T}]}; \left[\frac{\ln\, 2}{6Tk_o k_2 k_4(1 + e^{(1+k_1\theta_o)T})}\right]^{1/2}\right\} \tag{17}$$

$$k_\delta = k_2\delta^2 + \frac{T}{2}\,\delta\ . \tag{18}$$

Theorem. *Let $M\colon\Omega\times[0,T]\times[0,\infty) \to \mathbf{R}$ be a real function satisfying (H.1)–(H.3), $\phi_o \in H^1_o(\Omega)\cap H^2(\Omega)$, $\phi_1 \in H^1_o(\Omega)$ and $f\colon[0,T] \to H^1_o(\Omega)$ a continuous* *function. If*

$$|\Delta\phi_o|^2 + ||\phi_1||^2 + 0 \le t \le T \to \text{Máx}\,||f(t)||^2 \le \delta^2 \tag{19}$$

and

$$\left\| \frac{\partial M}{\partial t} \right\|_{L^\infty(\Omega\times(0,T)\times(0,k_3))} \le \frac{\ln 2}{3Tk_1} \ . \tag{20}$$

Then there exist one, and only one, function $u\colon [0,T] \to H_o^1(\Omega)$ *such that*

$$u \in C([0,T]; H_o^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap C^2([0,T]l H^{-1}(\Omega)) \ , \tag{21}$$

$$\begin{cases} u \in L^\infty(0,T; H_o^1(\Omega) \cap H^2(\Omega)) \\ u' \in L^\infty(0,T; H_o^1(\Omega)) \\ u'' \in L^\infty(0,T; L^2(\Omega)) \ , \end{cases} \tag{22}$$

$$\begin{cases} u''(t) - M(t, ||u(t)||^2)\, \Delta u(t) = f(t) \ \textit{in}\ L^2(\Omega), \ \ 0 \le t \le T \\ u(0) = \phi_o \\ u'(0) = \phi_1 \ . \end{cases} \tag{23}$$

Remark 3. In $(23)_1$ we are making use of the following notation: if $\psi\colon \Omega \times (0,T) \to \mathbf{R}$ is a function then $\psi(t)\colon \Omega \to \mathbf{R}$ is defined by $\psi(t)(x) = \psi(x,t)$.

3 – Galerkin's approximation and a priori estimates

We consider $V_o = \{0\}$ and $V_m = [w_1, \ldots, w_m]$ for $m = 1, 2, \ldots$ i.e., V_m is the vector space spanned by $w_1, \ldots, w_m$; where $(w_m)_{m\in\mathbf{N}}$ is as in the section 1. The sequence of Galerkin's approximation is defined by induction as follows: we put

$$\begin{array}{rccc} u_o\colon & [0,T] & \longrightarrow & V_o \\ & t & \longmapsto & u_o(t) = 0 \end{array}$$

and for $m = 1, 2, \ldots$, we consider

$$\begin{array}{rccc} u_m\colon & [0,T_m] & \longrightarrow & V_m \\ & t & \longmapsto & u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j \end{array}$$

the unique solution of the initial value problem, with the coefficient of $-\Delta u_m(t)$ depends on the time t:

$$\begin{cases} u_m''(t) - M(t, ||u_{m-1}(t)||^2)\, \Delta u_m(t) = f_m(t) \ \text{in}\ V_m, \ \ \forall\, t \in [0, T_m] \\ u_m(0) = \varphi_{om} \\ u_m'(0) = \varphi_{1m} \end{cases} \tag{24}$$

where

$$T_m = \sup\Big\{\tau; 0 < \tau \le T_{m-1} \text{ and } u_m\colon [0,\tau] \to V_m \text{ is solution of (24)}\Big\}\ , \tag{25}$$

$$f_m(t) = \sum_{j=1}^{m} (f(t), w_j)\, w_j, \qquad 0 \le t \le T\ , \tag{26}$$

$$\varphi_{om} = \sum_{j=1}^{m} (\phi_o, w_j)\, w_j\ , \tag{27}$$

$$\varphi_{1m} = \sum_{j=1}^{m} (\phi_1, w_j)\, w_j\ . \tag{28}$$

Remark 4. The Galerkin's approximation is well defined. It's sufficient we note that the initial value problem (24) is equivalent to the following system of ordinary differential equations:

$$\begin{cases} g''_{jm}(t) + \displaystyle\sum_{k=1}^{m} \lambda_k g_{km}(t) \Big(M(t, ||u_{m-1}(t)||^2) w_k, w_j\Big) = (f(t), w_j) \\ \qquad\qquad\qquad\qquad 0 \le t \le T_m;\ j = 1, \ldots, m \\ g_{jm}(0) = (\phi_o, w_j) \\ g'_{jm}(0) = (\phi_1, w_j)\ . \end{cases} \tag{29}$$

Estimate (i) From $(24)_1$ we have the approximate equation

$$(u''_m(t), v) - \Big(M(t, ||u_{m-1}(t)||^2\Big)\Delta u_m(t), v\Big) = (f_m(t), v), \quad \forall\, v \in V_m\ . \tag{30}$$

Take $v = -\Delta u'_m(t)$ in (30) we get

$$\frac{1}{2}\frac{d}{dt}||u'_m(t)||^2 + \int_\Omega M(x, t, ||u_{m-1}||^2)\, \Delta u_m(x,t).\Delta u'_m(x,t)\, dx = ((f_m(t), u'_m(t))),$$

since

$$\int_\Omega M\Big(x, t, ||u_{m-1}(t)||^2\Big)\, \Delta u_m(x,t)\, \Delta u'_m(x,t)\, dx =$$

$$= \frac{1}{2}\frac{d}{dt}\Big(M(t, ||u_{m-1}(t)||^2)\Delta u_m(t), \Delta u_m(t)\Big)$$

$$- \frac{1}{2}\int_\Omega \frac{\partial M}{\partial t}(x, t, ||u_{m-1}(t)||^2)\, (\Delta u_m(x,t))^2\, dx$$

$$- ((u_{m-1}(t), u'_{m-1}(t))) \int_\Omega \frac{\partial M}{\partial \lambda}(x, t, ||u_{m-1}(t)||^2)\, (\Delta u_m(x,t))^2\, dx$$

we have

$$(31)\quad \frac{d}{dt}\left\{\frac{1}{2}\Big[||u'_m(t)||^2 + \Big(M(t,||u_{m-1}(t)||^2)\Delta u_m(t), \Delta u_m(t)\Big)\Big]\right\} =$$

$$= ((f_m(t), u'_m(t))) + \frac{1}{2}\int_\Omega \frac{\partial M}{\partial t}(x,t,||u_{m-1}(t)||^2)\,(\Delta u_m(x,t))^2\,dx$$

$$+ ((u_{m-1}(t), u'_{m-1}(t)))\int_\Omega \frac{\partial M}{\partial \lambda}(x,t,||u_{m-1}(t)||^2)\,(\Delta u_m(x,t))^2\,dx,$$

$$\forall\, t\in[0,T_m],\ m=1,2,\ldots$$

Lemma 2. *Let be*

$$(32)\qquad \begin{cases} Z_o(t)=0\\ Z_m(t) = \dfrac{1}{2}\Big[||u_m(t)||^2 + \Big(M(t,||u_{m-1}(t)||^2)\Delta u_m(t), \Delta u_m(t)\Big)\Big]\\ \qquad\qquad 0\le t\le T_m,\ m=1,2,\ldots\ , \end{cases}$$

$$\alpha = \sup_{0\le t\le T_m} Z_m(t)\,,\ \alpha'_m = \frac{2}{m_o m_1}\,\alpha_m\ ,$$

$$\theta_m = \left\|\frac{\partial M}{\partial t}\right\|_{L^\infty(\Omega\times(0,T)\times(0,\alpha'_m))}, \qquad \beta_m = \left\|\frac{\partial M}{\partial \lambda}\right\|_{L^\infty(\Omega\times(0,T)\times(0,\alpha'_m))}.$$

Then, $T_m = T$, α_m is finite $\forall\, m\in\mathbf{N}$ and

$$(33)\qquad Z_m(t) \le \left[Z_m(0) + \frac{1}{2\delta}\int_0^t ||f_m(s)||^2\,ds\right] e^{(\delta+k_1\theta_{m-1}+k_o\alpha_{m-1}\beta_{m-1})t}.$$

Proof: The proof will be done by induction on m. Clearly the solution of the problem

$$\begin{cases} g''_{11}(t) + \lambda_1(M(t,0)w_1,w_1)\,g_{11}(t) = (f(t),w_1)\\ g_{11}(0) = (\phi_o,w_1)\\ g'_{11}(0) = (\phi_1,w_1) \end{cases}$$

is defined in all $[0,T]$. This show us that $T_1 = T$. Moreover if we consider the assumption (H.3) on M we have

$$(34)\qquad |\Delta u_1(t)|^2 \le \frac{2}{m_1}\,Z_1(t), \qquad \forall\, t\in[0,T]\ .$$

From (31) and (34) we get

$$Z'_1(t) - (\delta + k_1\theta_o)\,Z_1(t) \le \frac{1}{2\delta}\,||f_1(t)||^2\ ,$$

where δ is given by (17). By the last inequality we obtain

$$Z_1(t) \le \left[Z_1(0) + \frac{1}{2\delta}\int_0^t ||f_1(s)||^2\, ds\right] e^{(\delta+k_1\theta_o)t}$$

and it proves that α_1 is finite and (33) is true when $m = 1$. Now we make the induction assumption, i.e., we assume that for $m \ge 1$ we have $T_m = T$, α_m finite and (34) true for this m. Then (31) for $m+1$ implies

$$\begin{aligned} Z'_{m+1}(t) \le{}& \frac{1}{2\delta}||f_{m+1}(t)||^2 + \delta Z_{m+1}(t) \\ &+ \frac{1}{2}\int_\Omega \left|\frac{\partial M}{\partial t}(x,t,||u_m(t)||^2)\right| (\Delta u_{m+1}(x,t))^2\, dx \\ &+ ||u_m(t)||\; ||u'_m(t)|| \int_\Omega \left|\frac{\partial M}{\partial \lambda}(x,t,||u_m(t)||^2)\right| (\Delta u_{m+1}(x,t))^2\, dx\ . \end{aligned}$$

By the other hand, we note that

$$\begin{aligned} ||u_m(t)||^2 \le \frac{1}{m_o}|\Delta u_m(t)|^2 &\le \frac{2}{m_o m_1}\, Z_m(t) \\ &\le \frac{2}{m_o m_1}\,\alpha_m = \alpha'_m, \quad 0 \le t \le T. \end{aligned} \tag{35}$$

It follows that:

$$Z'_{m+1}(t) - (\delta + k_1\theta_m + k_o\alpha_m\beta_m)\, Z_{m+1}(t) \le \frac{1}{2\delta}\, ||f_{m+1}(t)||^2\ .$$

The above inequality shows that (33) is true for $(m+1)$, α_{m+1} is finite and $T_{m+1} = T$, i.e., the proof of Lemma 2 is complete. ∎

We denote,

$$\tau_m = Z_m(0) + \frac{1}{2\delta}\int_0^T ||f_m(t)||^2\, dt\,,\ m = 1, 2, \dots\ , \tag{36}$$

and then the sequence $(\tau_m)_{m\in\mathbf{N}}$ is bounded. In fact, by (26), (27) and (28) we have that

$$\begin{cases} \Delta\varphi_{om} \to \Delta\phi_o \text{ strong in } L^2(\Omega) \\ \varphi_{1m} \to \phi_1 \text{ strong in } H_o^1(\Omega) \\ f_m(t) \to f(t) \text{ strong in } H_o^1(\Omega),\ \text{uniformly on } [0,T] \end{cases} \tag{37}$$

and from the hypothesis of small data (17) we obtain

$$|\Delta\varphi_{om}|^2 + ||\varphi_{1m}||^2 + 0 \le t \le T \to \text{Máx}\, ||f_m(t)||^2 \le \delta^2, \quad \forall m \in \mathbf{N}\ . \tag{38}$$

Therefore,

$$||\varphi_{om}||^2 \leq \frac{1}{m_o} |\Delta \varphi_{om}|^2 \leq \frac{1}{m_o} \delta^2 \leq 1, \qquad \forall\, m \in \mathbf{N} \ ,$$

and then,

$$\begin{aligned} \tau_m = \frac{1}{2}\Big[||\varphi_{1m}||^2 + \int_\Omega M(x,0,||\varphi_{o(m-1)}||^2)\,(\Delta\varphi_{om}(x))^2\,dx\Big] \\ + \frac{1}{2\delta}\int_0^T ||f_m(t)||^2\,dt \leq k_2\delta^2 + \frac{T}{2}\,\delta = k_\delta \ . \end{aligned}$$

We conclude that:

$$0 \leq \tau_m \leq k_\delta \,,\ \forall\, m \in \mathbf{N} \ , \tag{39}$$

and

$$Z_m(t) \leq \tau_m\, e^{(\delta + k_1\theta_{m-1} + k_o\alpha_{m-1}\beta_{m-1})t}, \quad \forall\, t \in [0,T], m \in \mathbf{N} \ . \tag{40}$$

Lemma 3. *Exists a constant c_o (independent of $m \in \mathbf{N}$ and $t \in [0,T]$) such that*

$$Z_m(t) \leq 2c_o \,,\ \forall\, t \in [0,T],\ \forall\, m \in \mathbf{N} \ . \tag{41}$$

Proof: We consider $c_o = k_\delta[1 + e^{(1+k_1\theta_o)T}]$. Then, we have by (39):

$$\tau_m \leq c_o \,,\ \forall\, m \in \mathbf{N} \ , \tag{42}$$

and by (40)

$$Z_1(t) \leq \tau_1\, e^{(\delta + k_1\theta_o)t} \leq k_\delta\, e^{(1+k_1\theta_o)T} \leq c_o \leq 2c_o \ ,$$

it shows that (41) is true for $m = 1$. Now, we do the follows induction assumption: given $m \geq 1$ we assume that (41) is true for this m. In order to prove that (41) is true for $(m+1)$ we have

$$\alpha_m = \sup_{0 \leq t \leq T} Z_m(t) \leq 2c_o$$

and

$$\begin{aligned} \alpha'_m = \frac{2\alpha_m}{m_o m_1} \leq \frac{4c_o}{m_o m_1} = \frac{4}{m_o m_1}\Big\{k_\delta[1 + e^{(1+k_1\theta_o)T}]\Big\} = \\ = \frac{4}{m_o m_1}\Big\{\Big(k_2\delta^2 + \frac{T}{2}\,\delta\Big)\Big(1 + e^{(1+k_1\theta_o)T}\Big)\Big\} \leq k_3 \ . \end{aligned}$$

Therefore, we can see that

$$\beta_m \leq \left\| \frac{\partial M}{\partial \lambda} \right\|_{L^\infty(\Omega\times(0,T)\times(0,k_3))} = k_4 \tag{43}$$

and

$$\theta_m \leq \left\| \frac{\partial M}{\partial t} \right\|_{L^\infty(\Omega\times(0,T)\times(0,k_3))} \leq \frac{\ln 2}{3Tk_1} \; . \tag{44}$$

By (40), (42), (43) and (44) we get

$$Z_{m+1}(t) \leq \tau_{m+1}\, e^{(\delta + k_1\theta_m + k_o\alpha_m\beta_m)t} \leq c_o\, e^{(\delta + \frac{\ln 2}{3T} + 2k_ok_4c_o)t} \; .$$

We note that, from our choice we have

$$\begin{aligned}\left(\delta + \frac{\ln 2}{3T} + 2k_ok_4c_o\right) &= \left[1 + Tk_ok_4 + Tk_ok_4\, e^{(1+k_1\theta_o)T}\right]\delta + \\ &\quad + 2k_ok_2k_4[1 + e^{(1+k_1\theta_o)T}]\,\delta^2 + \frac{\ln 2}{3T} \leq \frac{\ln 2}{3T} + \frac{\ln 2}{3T} + \frac{\ln 2}{3T} = \frac{\ln 2}{T} \; .\end{aligned}$$

Therefore,

$$\left(\delta + \frac{\ln 2}{3T} + 2k_ok_4c_o\right)t \leq \ln\, 2\, ,\ \forall\, t \in [0,T] \; , \tag{45}$$

and then

$$Z_{m+1}(t) \leq 2c_o\, ,\ \forall\, t \in [0,T] \; .$$

The above relation complete the proof of lemma 3. ∎

We obtain from (41) the first estimate: There exists a constant c_1 such that

$$||u_m(t)||^2 + ||u_m'(t)||^2 + |\Delta u_m(t)|^2 \leq c_1\, ,\ \forall\, t \in [0,T],\ \forall\, m \in \mathbf{N} \tag{46}$$

Estimate (ii) We start observing that

$$\begin{aligned}\left|M(t, ||u_{m-1}(t)||^2)\, \Delta u_m(t)\right|^2 &= \int_\Omega \left|M(x,t,||u_{m-1}(t)||^2)\right|^2 |\Delta u_m(x,t)|^2\, dx \\ &\leq ||M||_{L^\infty(\Omega\times(0,T)\times(0,c_1))} \cdot c_1\end{aligned}$$

and

$$|f_m(t)|^2 = \sum_{j=1}^{m} |(f(t),w_j)|^2 \leq |f(t)|^2 \leq \frac{1}{m_o}||f(t)||^2 \leq \frac{\delta^2}{m_o} \leq 1 \; .$$

Thus, using $(24)_1$ we obtain the existence of a constant c_2 such that

$$|u''_m(t)|^2 \leq c_2\,, \ \forall\, t \in [0,T],\ \forall\, m \in \mathbf{N}\ . \tag{47}$$

By (46), (47) and the fundamental theorem of calculus we choose $t, s \in [0,T]$ and we have that

$$||u_m(t) - u_m(s)|| \leq \sqrt{c_1}\,|t-s|\ , \tag{48}$$

$$|u'_m(t) - u'_m(s)| \leq c_2\,|t-s|\ . \tag{49}$$

In order to obtain an estimate for (u''_m) analogous to (48) and (49) we choose $t, s \in [0,T]$ and by $(24)_1$ we get

$$\begin{aligned} u''_m(t) - u''_m(s) &= M(t, ||u_{m-1}(t)||^2)\,\Delta(u_m(t) - u_m(s)) + \\ &\quad + \Big[M(t, ||u_{m-1}(t)||^2) - M(s, ||u_{m-1}(s)||^2)\Big]\,\Delta u_m(s) + (f_m(t) - f_m(s))\ . \end{aligned}$$

On the other hand, for $v \in H^1_o(\Omega)$ we note that

$$\Big\|M(t, ||u_{m-1}(t)||^2).v\Big\|^2 =$$

$$\begin{aligned} &= \sum_{i=1}^{m} \int_\Omega \left|\frac{\partial M}{\partial x_i}(x,t,||u_{m-1}(t)||^2).v(x) + M(x,t,||u_{m-1}(t)||^2)\frac{\partial v}{\partial x_i}(x)\right|^2 dx \\ &\leq 2|v|^2 \sum_{i=1}^{n} \left\|\frac{\partial M}{\partial x_i}\right\|^2_{L^\infty(\Omega\times(0,T)\times(0,c_1))} + 2||M||^2_{L^\infty(\Omega\times(0,T)\times(0,c_1))} \cdot \sum_{i=1}^{n} \left|\frac{\partial v}{\partial x_i}\right|^2 \\ &\leq 2\left[||M||_{L^\infty(\Omega\times(0,T)\times(0,c_1))} + \sum_{i=1}^{n} \left\|\frac{\partial M}{\partial x_i}\right\|_{L^\infty(\Omega\times(0,T)\times(0,c_1))}\right]\left[|v|^2 + \sum_{i=1}^{n}\left|\frac{\partial v}{\partial x_i}\right|^2\right]. \end{aligned}$$

Whence, there exists a constant c_3 such that

$$\Big\|M(t, ||u_{m-1}(t)||^2).v\Big\|^2 \leq c_3\,||v||^2\,, \ \forall\, t \in [0,T],\ \forall\, m \in \mathbf{N}\ . \tag{50}$$

By the above estimate we have

$$\begin{aligned} (M(t, ||u_{m-1}(t)||^2)\,\Delta(u_m(t) - u_m(s)), v) &= \\ &= \Big(\Delta(u_m(t) - u_m(s)),\ M(t, ||u_{m-1}(t)||^2).v\Big) \\ &= \Big((u_m(s) - u_m(t), M(t, ||u_{m-1}(t)||^2)v)\Big) \\ &\leq \sqrt{c_3}\,||v||\,||u_m(s) - u_m(t)|| \end{aligned}$$

and using (48) we get

$$\Big|\Big(M(t,||u_{m-1}(t)||^2)\,\Delta(u_m(t)-u_m(s)),v\Big)\Big| \le \sqrt{c_1c_3}\,||v||\,|t-s| \ . \tag{51}$$

Now, if we consider $g(x,t)=(x,t,||u_{m-1}(t)||^2)$ then we have

$$\begin{aligned} M(x,t,||u_{m-1}(t)||^2) - M(x,s,||u_{m-1}(s)||^2) &= \\ &= \int_s^t \frac{\partial}{\partial\xi}(M\circ g)(x,\xi)\,d\xi \\ &= \int_s^t \frac{\partial M}{\partial\xi}(x,\xi,||u_{m-1}(\xi)||^2)\,d\xi \\ &\quad + 2\int_s^t \frac{\partial M}{\partial\lambda}(x,\xi,||u_{m-1}(\xi)||^2)\,((u_{m-1}(\xi),u'_{m-1}(\xi)))\,d\xi \ . \end{aligned}$$

Then we can see that there exists a constant c_4 such that

$$\Big|M(x,t,||u_{m-1}(t)||^2) - M(x,s,||u_{m-1}(s)||^2)\Big| \le c_4\,|t-s|$$

and this estimate shows that there exists a constant c_5 such that

$$\Big|\Big(\Big[M(t,||u_{m-1}(t)||^2) - M(s,||u_{m-1}(s)||^2)\Big]\Delta u_m(s),v\Big)\Big| \le c_5\,||v||\,|t-s| \ . \tag{52}$$

Finally, we note that

$$|(f_m(t)-f_m(s),v)| \le \frac{1}{m_o}\,||f(t)-f(s)||\,||v|| \ . \tag{53}$$

From (51), (52) and (53) we obtain that there exists a constant c_6 such that

$$||u''_m(t)-u''_m(s)||_{H^{-1}(\Omega)} \le c_6\Big(|t-s| + ||f(t)-f(s)||\Big) \ . \tag{54}$$

The estimate (ii) is the relations (47), (48), (49) and (54).

4 – Proof of the theorem

By estimates (i) and (ii) we have:

$(u_m)_{m\in\mathbf{N}}$ uniformly bounded in $[0,T]$ with respect to the norm of $H^1_o(\Omega)\cap H^2(\Omega)$ and equicontinuous with respect to the norm of $H^1_o(\Omega)$.

$(u'_m)_{m\in\mathbf{N}}$ uniformly bounded in $[0,T]$ with respect to the norm of $H^1_o(\Omega)$ and equicontinuous with respect to the norm of $L^2(\Omega)$.

$(u''_m)_{m\in\mathbf{N}}$ uniformly bounded in $[0,T]$ with respect to the norm of $L^2(\Omega)$ and equicontinuous with respect to the norm of $H^{-1}(\Omega)$.

Then, by lemma 1, there exists a function $u\colon \Omega\times[0,T]\to\mathbf{R}$ and a subsequence $(u_{m_\nu})_{\nu\in\mathbf{N}}$ extracted from $(u_m)_{m\in\mathbf{N}}$, such that

(55) $$u\in C([0,T];H_o^1(\Omega))\cap C^1([0,T];L^2(\Omega))\cap C^2([0,T];H^{-1}(\Omega))\ ,$$

(56) $$\begin{cases} u_{m_\nu}(t)\to u(t) \text{ strongly in } H_o^1(\Omega), \text{ uniformly in } [0,T]\\ u'_{m_\nu}(t)\to u'(t) \quad \text{strongly in } L^2(\Omega), \text{ uniformly in } [0,T]\\ u''_{m_\nu}(t)\to u''(t) \quad \text{strongly in } H^{-1}(\Omega), \text{ uniformly in } [0,T]\ . \end{cases}$$

Moreover, since $H_o^1(\Omega)\cap H^2(\Omega)$, $H_o^1(\Omega)$ and $L^2(\Omega)$ are reflexive Banach spaces, we still have

(57) $$\begin{cases} u\in L^\infty(0,T;H_o^1(\Omega)\cap H^2(\Omega)\\ u'\in L^\infty(0,T;H_o^1(\Omega))\\ u''\in L^\infty(0,T;L^2(\Omega))\ . \end{cases}$$

The convergences don't allow us to pass to the limit in the approximate equation. Indeed, the sequence $(u_{m_\nu})_{\nu\in\mathbf{N}}$ have the properties, but we can't say the same for $(u_{m_\nu-1})_{\nu\in\mathbf{N}}$. In order to solve this problem we will prove the following lemma.

Lemma 4. $\lim\limits_{m\to\infty}||u_{m+1}(t)-u_m(t)||^2=0$ *uniformly on* $[0,T]$.

Proof: For each $m\in\mathbf{N}$ we define $w_m=u_{m+1}-u_m$. Then

$$||u_{m+1}(t)-u_m(t)||^2=\sum_{i=1}^n\int_\Omega\left(\frac{\partial w_m}{\partial x_i}(x,t)\right)^2dx$$

and making use of the assumption (H.3) we can see that there exists a constant c_7 such that

(58) $$||u_{m+1}(t)-u_m(t)||^2\le \\ \le c_7\left\{\frac{1}{2}\left[|w'_m(t)|^2+\sum_{i=1}^n\left(M(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t),\frac{\partial w_m}{\partial x_i}(t)\right)\right]\right\}.$$

Hence, we are motivated to put

(59) $$\psi_m(t)=\frac{1}{2}\left[|w'_m(t)|^2+\sum_{i=1}^n\left(M(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t),\frac{\partial w_m}{\partial x_i}(t)\right)\right]$$

and then, we will conclude with the proof of lemma showing that $\psi_m(t) \to 0$ uniformly in $[0, T]$.

Differentiating $\psi_m(t)$, we have

$$\begin{aligned}
\psi'_m(t) &= \frac{1}{2}\frac{d}{dt}|w'_m(t)|^2 + \\
&+ \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial M}{\partial t}(t, ||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), \frac{\partial w_m}{\partial x_i}(t)\right) + \\
&+ ((u_m(t), u'_m(t)))\sum_{i=1}^{n}\left(\frac{\partial M}{\partial \lambda}(t, ||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}, \frac{\partial w_m}{\partial x_i}(t)\right) + \\
&+ \sum_{i=1}^{n}\left(M(t, ||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), \frac{\partial w'_m}{\partial x_i}\right).
\end{aligned} \tag{60}$$

From the approximation equation we find

$$\begin{aligned}
w''_m(t) + \Big[M(t, ||u_{m-1}(t)||^2) - M(t, ||u_{m-1}(t)||^2)\Big]\,\Delta u_m(t) - \\
- M(t, ||u_m(t)||^2)\,\Delta w_m(t) = f_{m+1}(t) - f_m(t)
\end{aligned}$$

and then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|w'_m(t)|^2 &= \Big(M(t, ||u_m(t)||^2)\Delta w_m, w'_m(t)\Big) \\
&+ \Big(\Big[M(t, ||u_m(t)||^2) - M(t, ||u_{m-1}(t)||^2)\Big]\Delta u_m(t), w'_m(t)\Big) \\
&+ \Big(f_{m+1}(t) - f_m(t), w'_m(t)\Big)\ .
\end{aligned}$$

From the above relation and (60) we obtain

$$\psi'_m(t) = A_m(t) + B_m(t) + C_m(t) + D_m(t) + E_m(t) \tag{61}$$

where

$$\begin{cases}
A_m(t) = -\displaystyle\sum_{i=1}^{n}\left(\frac{\partial M}{\partial x_i}(t, ||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), w'_m(t)\right) \\
B_m(t) = \Big(\Big[M(t, ||u_m(t)||^2) - M(t, ||u_{m-1}(t)||^2)\Big]\Delta u_m(t), w'_m(t)\Big) \\
C_m(t) = ((u_m(t), u'_m(t)))\displaystyle\sum_{i=1}^{n}\left(\frac{\partial M}{\partial \lambda}(t, ||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), \frac{\partial w_m}{\partial x_i}(t)\right) \\
D_m(t) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\left(\frac{\partial M}{\partial t}(t, ||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), \frac{\partial w_m}{\partial x_i}(t)\right) \\
E_m(t) = (f_{m+1}(t) - f_m(t), w'_m(t))\ .
\end{cases} \tag{62}$$

By (59) and the estimates we find constants c_8, c_9, c_{10} and c_{11} such that

$$\begin{aligned} &A_m(t) \le c_8\, \psi_m(t), && B_m(t) \le c_9\, [\psi_{m-1}(t) - \psi_m(t)] \\ &C_m(t) \le c_{10}\, \psi_m(t), && D_m(t) \le c_{11}\, \psi_m(t) \end{aligned}$$

and $E_m(t) \le \dfrac{1}{2}|f_{m+1}(t) - f_m(t)|^2 + \psi_m(t)$.

Then we prove that there exists a constant c_{12}, independent of m and $t \in [0,T]$, such that

$$\psi_m'(t) - c_{12}\, \psi_m(t) \le \frac{1}{2}\, |f_{m+1}(t) - f_m(t)|^2 + c_{12}\, \psi_{m-1}(t)$$

and then,

$$\begin{aligned} \psi_m(t) \le e^{c_{12}T} &\left[\psi_m(0) + \frac{1}{2}\int_0^T |f_{m+1}(t) - f_m(t)|^2\, dt\right] \\ &+ c_{12}\, e^{c_{12}T} \int_0^t \psi_{m-1}(s)\, ds\ . \end{aligned}$$

Now we denote by

$$\gamma_m = \psi_m(0) + \frac{1}{2}\int_0^T |f_{m+1}(t) - f_m(t)|^2\, dt\ ,$$

and choose

$$c_{13} = \text{Máx}\Big\{e^{c_{12}T}, c_{12}\, e^{c_{12}T}, 0 \le t \le T \to \text{Máx}\, \psi_1(t)\Big\}\ .$$

Then, we can see that

$$\text{(63)} \qquad \begin{cases} \psi_1(t) \le c_{13} \\ \psi_m(t) \le c_{13}\, \gamma_m + c_{13} \displaystyle\int_0^t \psi_{m-1}(s)\, ds\ . \end{cases}$$

By induction we find

$$\text{(64)} \qquad \psi_m(t) \le c_{13} \sum_{j=0}^{m-1} \frac{(c_{13} + t)^j}{j!}\, \gamma_{m-j}\ ,\ \forall\, t \in [0,T],\ m = 2, 3, \dots$$

If we consider (37) we get

$$\text{(65)} \qquad \lim_{m \to \infty} \gamma_m = 0$$

and, as we well know,

$$\text{(66)} \qquad \sum_{j=1}^{\infty} \frac{(c_{13}T)^j}{j!} = e^{c_{13}T}\ .$$

Therefore, from (64), (65) and (66) we conclude that $\psi_m(t) \to 0$ uniformly in $[0,T]$ and the proof of lemma 4 is complete. ∎

The result of lemma 4 implies that

$$\lim_{\nu\to\infty} ||u_{m_\nu-1}(t)||^2 = ||u(t)||^2 \text{ uniformly in } [0,T] \ . \tag{67}$$

Then, we have the following convergences:

$$M(t,||u_{m_\nu-1}(t)||^2).v \to M(t,||u(t)||^2).v \quad \text{strongly in } L^2(\Omega), \text{ uniformly in } [0,T], \ \forall v\in L^2(\Omega) \ , \tag{68}$$

$$\Delta u_{m_\nu}(t) \to \Delta u(t) \text{ weakly in } L^2(\Omega), \ 0\le t\le T \ . \tag{69}$$

The convergences (68) and (69) imply

$$M(t,||u_{m_\nu-1}(t)||^2)\Delta u_{m_\nu}(t) \to M(t,||u(t)||^2)\Delta u(t) \quad \text{weakly in } L^2(\Omega), \ 0\le t\le T \ . \tag{70}$$

We have then by passage to the limit in ν that

$$u''(t) - M(t,||u(t)||^2)\,\Delta u(t) = f(t) \text{ in } L^2(\Omega), \ 0\le t\le T \ .$$

Clearly we also have $u(0)=\phi_o$ and $u'(0)=\phi_2$.

5 – Uniqueness

Let u and v be satisfying (21), (22) and (23). Then, if we define $w=u-v$ we get

$$\begin{cases} w''(t) + M(t,||v(t)||^2)\,\Delta v(t) - M(t,||u(t)||^2)\,\Delta u(t) = 0 \\ w(0) = w'(0) = 0 \ . \end{cases} \tag{71}$$

Now we put

$$\psi(t) = \frac{1}{2}\Big[|w'(t)|^2 + \sum_{i=1}^{n}\Big(M(t,||u(t)||^2)\frac{\partial w}{\partial x_i}(t), \frac{\partial w}{\partial x_i}(t)\Big)\Big]. \tag{72}$$

Therefore, using again the same analysis used in the proof of lemma 4, we obtain a constant c_{14} such that

$$\psi'(t) - c_{14}\,\psi(t) \le 0$$

and this imply

$$\psi(t) \le c^{c_{14}t}\,\psi(0)\,,\ \forall\, t\in[0,T]\ . \tag{73}$$

But, from (72) there exists a constant c_{15} such that

$$0 \le \psi(t) \le c_{15}\Big[|w'(t)|^2 + ||w(t)||^2\Big]\,,\ 0\le t\le T\ .$$

By $(71)_2$, if we take $t=0$ in the above relation, we have $\psi(0)=0$. This fact with (73) shows that $\psi(t)=0$, $0\le t\le T$; and then we have uniqueness.

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