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EXACT CONTROLLABILITY FOR TEMPORALLY WAVE EQUATION

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In memoriam P.H. Rivera (1941–1983)

Presented by J.L. Lions

Summary: Let us consider the wave equation for an operator with coefficients $a_{ij}(x,t)$ dependent of $x \in \Omega$ and $t \in [0,T]$. We study the problem of exact controllability with control fixed on the boundary. Under certain restrictions on $a_{ij}(x,t)$, we prove that the Method HUM (Hilbert Uniqueness Method) can be applied to obtain the stabilization at a large time T.

1 – Introduction

In the present work we study the exact controllability of the system:

$$(*) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\,\frac{\partial u}{\partial x_j}\right) = 0 & \text{on } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{on } \Omega\,. \end{cases}$$

By Ω we denote on open bounded set of $\mathbb{R}^n$ with smooth boundary Γ; $Q = \Omega \times [0,T[$ the cylinder, which lateral boundary we represent by Σ. The coefficients $a_{ij}(x,t)$ satisfy certain conditions of regularity fixed in Section 1, (1) and (2).

The exact controllability for $(*)$ is formulated as follows:

Given $T > 0$, find a Hilbert space H, such that for every set of initial data $\{u_0, u_1\} \in H$, there exists a corresponding control $v \in L^2(\Sigma)$ such that the

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solution $u = u(x,t)$ of $(*)$ satisfies the stabilization condition:

$$(**) \qquad u(x,T) = 0, \quad \frac{\partial u}{\partial t}(x,T) = 0 \quad \text{on } \Omega .$$

We solve the problem, using the method HUM (Hilbert Uniqueness Method) idealized by J.L. Lions in 1986, [4]. He studied, initially, the system $(*)$ for the case $a_{ij}(x,t) = 1$ if $i = j$ and zero if $i \neq j$, that is, the case of $-\Delta$. In [4] he studied the case with a coefficient $a(t)$, that is, the operator is $-a(t)\,\Delta$, with $a(t), a'(t) \in L^\infty(\mathbb{R})$ and $a(t) \geq a_0 > 0$, $a'(t) \geq 0$ for $t > 0$. In this case Lions [4] proved that we have exact controllability. In Rivera [7] he obtained a weakest condition, that is, it is sufficient that $a(t)$ is monotonous on some interval $T_0 \leq t \leq T_1$ such that

$$T - T_0 > \frac{R\sqrt{\|a\|_\infty}}{a_0}, \qquad R \text{ constant} .$$

In the general case $a_{ij}(x)$, depending only on $x \in \Omega$, Komornik [2] obtained exact controllability of $(*)$ with certain regularity on $a_{ij}(x)$, plus the following technical condition:

Exists $0 < \delta < 1$ such that

$$(1-\delta)\, a_{ij}(x)\, \xi_i\, \xi_j - \frac{1}{2}\, \frac{\partial}{\partial x_k}(a_{ij}(x))\, m_k\, \xi_i\, \xi_j \geq 0$$

for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$. For m_k look Section 1 in the present work.

Our objective in this work, is to solve the problem of exact controllability for $(*)$, in the general temporally case, that is, $a_{ij}(x,t)$, $x \in \Omega$, $t \in [0,T]$.

We divide the work in four sections. In Section 1 we fix the notation and do the assumptions. In Section 2 we prove that under convenient hypothesis on $a_{ij}(x,t)$, the method HUM works very well for $(*)$. The Sections 3, 4 contains the proofs of the results on the existence and regularity used in the Section 2.

1 – Notations and terminology

By $\mathbb{R}^n$ we represent the real Euclidean space of dimension n. Fix x^0, any point of $\mathbb{R}^n$, and consider the vector

$$m(x) = x - x^0 = (x_k - x_k^0) = (m_k)_{1 \leq k \leq n} .$$

Let be

$$R(x^0) = \|x - x^0\|_{L^\infty(\Omega)}$$

the radius of the smallest ball, with center in x^0, containing Ω. Represent by $\nu(x)$ the unit normal vector of Γ directed towards the exterior of Ω. We denote by:

$$\Gamma(x^0) = \left\{ x \in \Gamma \mid m(x) \cdot \nu(x) > 0 \right\} ,$$

$$\Gamma_*(x^0) = \left\{ x \in \Gamma \mid m(x) \cdot \nu(x) \leq 0 \right\} .$$

Note that $m(x) \cdot \nu(x)$ represent the inner product in $\mathbb{R}^n$ of the vectors $m(x), \nu(x)$. We also represent:

$$\Sigma(x^0) = \Gamma(x^0) \times]0, T[,$$

$$\Sigma_*(x^0) = \Gamma_*(x^0) \times]0, T[.$$

Throughout this work, we use the summation convention for repeated indexs. Let $A(t)$ be the linear operator defined by:

$$A(t)\, \phi = -\frac{\partial}{\partial x_j} \left(a_{ij}(x,t)\, \frac{\partial \phi}{\partial x_i} \right) .$$

Note that $A(t)$ is a temporally second order operator. We suppose that

$$\text{(1)} \qquad \begin{cases} a_{ij} \in L^\infty(0,T; W^{1,\infty}(\Omega)), \\ a_{ij}(x,t) = a_{ji}(x,t), \quad \text{for all } (x,t) \in Q, \ 1 \leq i,j \leq n. \\ \text{There exists a constant } \alpha > 0 \text{ such that} \\ \qquad a_{ij}(x,t)\, \xi_i\, \xi_j \geq \alpha \|\xi\|^2, \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } (x,t) \in Q . \end{cases}$$

With respect to the variable $t \in [0,T]$ assume:

$$\text{(2)} \qquad \begin{cases} a'_{ij} = \dfrac{\partial}{\partial t} a_{ij} \in L^1(0,+\infty; L^\infty(\Omega)), \\ a_{ij}(\cdot,t) \in C^1(\overline{\Omega}) \quad \text{a.e. in } [0,T], \\ a''_{ij} \in L^\infty(Q), \quad \dfrac{\partial}{\partial x_k}(a'_{ij}) \in L^1(0,+\infty; L^\infty(\Omega)) . \end{cases}$$

For technical reasons we also suppose, as Komornik [2], the existence of $0 < \delta < 1$ such that

$$\text{(3)} \qquad (1-\delta)\, a_{ij}(x,t)\, \xi_i\, \xi_j - \frac{1}{2}\, \frac{\partial}{\partial x_k} a_{ij}(x,t)\, m_k\, \xi_i\, \xi_j \geq 0$$

for all $\xi \in \mathbb{R}^n$, $(x,t) \in Q$.
Now we define $a'(t,\phi,\psi)$ as the bilinear form:

$$\text{(4)} \qquad a'(t,\phi,\psi) = \int_\Omega a'_{ij}\, \frac{\partial \phi}{\partial x_i}\, \frac{\partial \psi}{\partial x_j}\, dx \quad \text{for } \phi, \psi \in H_0^1(\Omega) .$$

Denote by

$$
(5) \qquad \begin{cases} \beta(t) = \|a'_{ij}(x,t)\|_{L^\infty(\Omega)} \in L^1(0,+\infty), \\ P(t) = \dfrac{1}{2} \displaystyle\int_\Omega a_{ij} \, \frac{\partial \phi}{\partial x_i} \, \frac{\partial \phi}{\partial x_j} \, dx \,, \qquad \phi \in H^1_0(\Omega), \\ T_0 = \dfrac{2}{\delta} \, R(x^0) \, C_\alpha \, C_1^2 \,; \qquad C_\alpha = \max\{1, 1/\alpha\} \;. \end{cases}
$$

Note that the constant C_1 will be fixed in Section 3, (23).

2 – The main result and application of HUM

The answer to the question of exact controllability for $(*)$ is given by Theorem 2.1 in below. The proof will be given by method HUM.

Theorem 2.1. *For T_0 given by $(5)_3$. Let be $T > T_0$. Then, for each $\{y_0, y_1\} \in L^2(\Omega) \times H^{-1}(\Omega)$, exists control $v \in L^2(\Sigma)$, such that the solution $y = y(x,t)$ of the system:*

$$
(6) \qquad \begin{cases} y'' + A(t)\, y = 0 & \textit{on } Q, \\ y = v & \textit{on } \Sigma, \\ y(0) = y_0, \quad y'(0) = y_1 & \textit{on } \Omega \;, \end{cases}
$$

satisfies

$$
(7) \qquad y(x,T) = 0, \quad y'(x,T) = 0 \quad \textit{ in } \;\Omega \;.
$$

Proof: It will be given by steps. We use certain results of existence and regularity of ultraweak solutions proved later in the Section 3.

Step 1. We consider a regular problem. In fact, let be $\varphi_0, \varphi_1 \in \mathcal{D}(\Omega)$ and solve the homogeneous system:

$$
(8) \qquad \begin{cases} \varphi'' + A(t)\, \varphi = 0 & \text{on } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(0) = \varphi_0, \quad \varphi'(0) = \varphi_1 & \text{on } \Omega \;. \end{cases}
$$

The unique solution $\varphi = \varphi(x,t)$ of (8) satisfies:

$$
(9) \qquad \frac{\partial \varphi}{\partial \nu} \in L^2(\Sigma) \,, \qquad \text{cf. Section 3.}
$$

Step 2. Using the solution φ of (8) we formulate the following backward problem:

$$\begin{cases} \psi'' + A(t)\,\psi = 0 & \text{on } \; Q, \\ \psi = \begin{cases} a_{ij}\,\nu_i\,\nu_j\,\dfrac{\partial\varphi}{\partial\nu} & \text{on } \Sigma(x^0), \\ 0 & \text{on } \Sigma_*(x^0), \end{cases} \\ \psi(T) = 0\,, \quad \psi'(T) = 0\;. \end{cases} \tag{10}$$

The system (10) has a unique ultraweak solution $\psi = \psi(x,t)$ defined by transposition. The Theorem 4.3 Section 4 gives the following regularity:

$$\psi \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))\;. \tag{11}$$

The operator Λ

Given φ_0, φ_1 in $\mathcal{D}(\Omega)$ we solve (8), obtaining a solution $\varphi = \varphi(x,t)$ satisfying (9). Then we solve the backward problem (10), obtaining $y = y(x,t)$ with regularity (11). Therefore is well defined the map

$$\Lambda\colon\, \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \;\longrightarrow\; H^{-1}(\Omega) \times L^2(\Omega)\;,$$

given by

$$\Lambda\{\varphi_0, \varphi_1\} = \{\psi'(0), -\psi(0)\}\;. \tag{12}$$

Step 3. Multiplying the equation $(8)_1$ by ψ, solution of (10), and integrating on Q, we get:

$$\langle \psi'(0), \varphi_0\rangle - (\psi(0), \varphi_1) = \int_{\Sigma(x^0)} a_{ij}\,\nu_i\,\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 d\Sigma\;. \tag{13}$$

From (12) and (13) we obtain:

$$\langle \Lambda\{\varphi_0, \varphi_1\}, \{\varphi_0, \varphi_1\}\rangle = \int_{\Sigma(x^0)} a_{ij}\,\nu_i\,\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 d\Sigma\;. \tag{14}$$

Let consider in $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ the quadratic from:

$$\|\{\varphi_0, \varphi_1\}\|_F^2 = \int_{\Sigma(x^0)} a_{ij}\,\nu_i\,\nu_j \left(\frac{\partial\varphi}{\partial\nu}\right)^2 d\Sigma\;. \tag{15}$$

This is a seminorm on $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$. In this Section 3 we will prove the following inequality:

$$
\begin{aligned}
r_1 \, \|\{\varphi_0, \varphi_1\}\|^2_{H^1_0(\Omega)\times L^2(\Omega)} &\le \int_{\Sigma(x^0)} a_{ij} \, \nu_i \, \nu_j \left(\frac{\partial \varphi}{\partial \nu}\right)^2 d\Sigma \\
&\le r_2 \, \|\{\varphi_0, \varphi_1\}\|^2_{H^1_0(\Omega)\times L^2(\Omega)}
\end{aligned} \tag{16}
$$

for all $\{\varphi_0, \varphi_1\} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$. The first inequality implies that $\|\{\varphi_0, \varphi_1\}\|_F$ is in fact a norm on $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ and the both inequality (16) imply that the norm $\|\{\varphi_0, \varphi_1\}\|_F$ is equivalent to the norm in $H^1_0(\Omega) \times L^2(\Omega)$, given by:

$$
\|\{\varphi_0, \varphi_1\}\|^2_{H^1_0(\Omega)\times L^2(\Omega)} = \int_\Omega |\nabla \varphi_0(x)|^2 \, dx + \int_\Omega |\varphi_1(x)|^2 \, dx \ . \tag{17}
$$

To prove the first part of inequality (16) we need to fix $T > T_0$, that is, for large T.

Let F the closure of $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ with respect to $\| \ \|_F$. Then, for $T > T_0$, the inequality (16) shows that

$$
F = H^1_0(\Omega) \times L^2(\Omega) \tag{18}
$$

which dual F' is $H^{-1}(\Omega) \times L^2(\Omega)$.

The operator Λ is continuous with respect to $\| \ \|_F$. Then is has a unique continuous extentions to the closure of $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$, which is F given by (18). We have

$$
\Lambda \colon \, F \to F' \tag{19}
$$

is coercive, then it is an isomorphism between F and its dual F'. It follows that given $\{y_1, -y_0\} \in F' = H^{-1}(\Omega) \times L^2(\Omega)$ exists a unique $\{\varphi_0, \varphi_1\} \in F = H^1_0(\Omega) \times L^2(\Omega)$ such that:

$$
\Lambda\{\varphi_0, \varphi_1\} = \{y_1, -y_0\} \ . \tag{20}
$$

Then (12) and (20) says that the solution $\psi = \psi(x,t)$ of the backward system (20) satisfies:

$$
\psi(0) = y_0 \, , \qquad \psi'(0) = y_1 \ .
$$

Then, the unique solution ψ of (10), with control $v = a_{ij} \, \nu_i \, \nu_j \, \frac{\partial \varphi}{\partial \nu}$, is equal to y solution of (6), then y satisfies the stabilization condition (7). ∎

3 – Inequalities

To prove the inequality (16), we need the following identity, which proof was given by J.L. Lions [4].

Lemma 3.1. *For the weak solution $\phi = \phi(x,t)$ of (8), it is true the identity:*

$$\frac{1}{2}\int_\Sigma a_{ij}\,\nu_i\,\nu_j\left(\frac{\partial\phi}{\partial\nu}\right)^2 h_k\,\nu_k\,d\Sigma = \left(\phi', \frac{\partial\phi}{\partial x_k}h_k\right)\Big|_0^T + \frac{1}{2}\int_Q |\phi'|^2\,\frac{\partial h_k}{\partial x_k} - \tag{21}$$
$$-\frac{1}{2}\int_Q a_{ij}\,\frac{\partial\phi}{\partial x_i}\,\frac{\partial\phi}{\partial x_j}\,\frac{\partial h_k}{\partial x_k} + \int_Q a_{ij}\,\frac{\partial\phi}{\partial x_j}\,\frac{\partial\phi}{\partial x_k}\,\frac{\partial h_k}{\partial x_i} - \frac{1}{2}\int_Q \frac{\partial}{\partial x_k}(a_{ij})\,\frac{\partial\phi}{\partial x_i}\,\frac{\partial\phi}{\partial x_j}\,h_k\;,$$

where (h_k) is a vector field in $C^1(\overline{\Omega})$.

Lemma 3.2 *Let $\phi = \phi(x,t)$ be weak solution of (8), then we have*

$$\int_{\Sigma(x^0)} a_{ij}\,\nu_i\,\nu_j\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma \le C\,\|\{\varphi_0,\varphi_1\}\|^2_{H_0^1(\Omega)\times L^2(\Omega)}\;.$$

Proof: We define the energy associated to the system (8) as the quadratic form:

$$E(t) = \frac{1}{2}\int_\Omega\left(|\phi'(t)|^2 + a_{ij}(x,t)\,\frac{\partial\phi}{\partial x_i}\,\frac{\partial\phi}{\partial x_j}\right)dx\;. \tag{22}$$

We used the equality (for the proof cf. J.L. Lions–E. Magenes [5])

$$2E(t) = 2E_0 + \int_0^t a'(s,\phi(s),\phi(s))\,ds\;.$$

By the coerciveness hypothesis of $[a_{ij}]$, we now find the basic estimate:

$$\sum_{i,j=1}^n |a'_{ij}|\left|\frac{\partial\phi}{\partial x_i}\right|\left|\frac{\partial\phi}{\partial x_j}\right| \le \frac{\beta(t)}{\alpha}\,a_{ij}\,\frac{\partial\phi}{\partial x_i}\,\frac{\partial\phi}{\partial x_j}\;.$$

Integrating in Ω, we get:

$$\int_\Omega |a'_{ij}|\left|\frac{\partial\phi}{\partial x_i}\right|\left|\frac{\partial\phi}{\partial x_j}\right|dx \le \frac{2\beta(t)}{\alpha}\,P(t) \le \frac{2\beta(t)}{\alpha}\,E(t)\;,$$

whence,

$$E(t) \le E_0 + \int_0^t \frac{\beta(t)}{\alpha}\,E(s)\,ds\;.$$

From the Gronwall's Lemma, we obtain

$$E(t) \le C_1\, E_0\,, \qquad \forall\, t \in [0,T], \quad E_0 = E(0)\ , \tag{23}$$

i.e.,

$$E(t) \le C\, \|\{\phi_0, \phi_1\}\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\,, \qquad \forall\, t\in[0,T]\ .$$

In the identity (21), we consider a vector field (h_k) such as $h_k\,\nu_k = 1$. We estimate each term in the right side member of (21). From the definition of $\Sigma(x^0)$ and (23), we obtain

$$\int_{\Sigma(x^0)} a_{ij}\,\nu_i\,\nu_j \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma \le C\,E(t) \le C\,\|\{\phi_0,\phi_1\}\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\ . \ \blacksquare$$

Remark 1. By a similar argument used in (23), we prove:

$$C_0\,E_0 \le E(t)\,, \qquad \forall\, t\in[0,T], \quad C_0 = C_1^{-1}\ .$$

Lemma 3.3 (Inverse inequality). *Let $\phi = \phi(x,t)$ be weak solution of the homogeneous problem (8) and $T > T_0$. Then,*

$$(T-T_0)\,E_0 \le C \int_{\Sigma(x^0)} a_{ij}\,\nu_i\,\nu_j \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Sigma\ ,$$

where

$$C = \frac{R(x^0)\,C_1}{\delta}\ .$$

Proof: We consider the vector field $h_k = m_k \in C^1(\overline{\Omega})$, and we observe that

$$\frac{\partial m_k}{\partial x_j} = \frac{\partial}{\partial x_j}(x_k - x_k^0) = \delta_{jk}\ . \tag{24}$$

We write

$$X = \left(\phi', \frac{\partial\phi}{\partial x_k}\,m_k\right)\Big|_0^T\ , \tag{25}$$

$$Y = \int_Q \left(|\phi'|^2 - a_{ij}\,\frac{\partial\phi}{\partial x_i}\,\frac{\partial\phi}{\partial x_j}\right)\ , \tag{26}$$

$$I = \frac{1}{2}\int_\Sigma a_{ij}\,\nu_i\,\nu_j \left(\frac{\partial\phi}{\partial\nu}\right)^2 m_k\,\nu_k\,d\Sigma\ . \tag{27}$$

Substituting (24), (25), (26) and (27) in the identity (21), we obtain the equality

$$X + \frac{n}{2} Y + \int_Q a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} - \frac{1}{2} \int_Q \frac{\partial}{\partial x_k}(a_{ij})\, m_k \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} = I \ .$$

We apply the technical hypothesis (3), of Komornik, and obtain:

$$X + \frac{n}{2} Y + \delta \int_Q a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \leq I \ .$$

Using the equation $(8)_1$, we have:

$$Y = (\phi', \phi)\,|_0^T \ .$$

From the above inequality and definition of $E(t)$ we get:

$$X + \left(\frac{n-\delta}{2}\right) Y + \delta \int_0^T E(t)\, dt \leq I \ .$$

From (25) and (26), we deduce the inequality (cf. [4])

$$\left| X + \left(\frac{n-\delta}{2}\right) Y \right| \leq \frac{R(x^0)}{2} \left(|\phi'|^2 + |\nabla \phi|^2\right) \ .$$

Applying the coerciveness of $[a_{ij}]$ and (23), we obtain:

$$\left| X + \left(\frac{n-\delta}{2}\right) Y \right| \leq C_\alpha\, R(x^0)\, C_1\, E_0 \ . \tag{28}$$

From (28) and Remark 1, it follows:

$$\delta\, C_0\, T\, E_0 - 2R(x^0)\, C_\alpha\, C_1\, E_0 \leq I \ . \tag{29}$$

As an immediate consequence of the definition of $\Sigma(x^0)$ and $R(x^0)$, from (29) it follows that

$$\delta\, C_0\, T\, E_0 - 2R(x^0)\, C_\alpha\, C_1\, E_0 \leq \frac{R(x^0)}{2} \int_{\Sigma(x^0)} a_{ij}\, \nu_i\, \nu_j \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Sigma \ .$$

Finally, we obtain

$$(T - T_0)\, E_0 \leq \frac{R(x^0)\, C_1}{2\delta} \int_{\Sigma(x^0)} a_{ij}\, \nu_i\, \nu_j \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Sigma \ . \ \blacksquare$$

4 – Concept of ultraweak solutions

In this section we study the concept of ultraweak solution by the transposition method, J.L. Lions [4] and J.L. Lions–E. Magenes [5]. First of all we proceed heuristically in order to obtain the natural definition. In fact, let us consider the nonhomogeneous problem

$$\begin{cases} z'' + A(t)\, z = 0 & \text{on } Q, \\ z = v & \text{on } \Sigma, \\ z(0) = z_0, \quad z'(0) = z_1 & \text{in } \Omega\ , \end{cases} \tag{30}$$

for

$$v \in L^2(\Sigma)\,, \qquad z_0 \in L^2(\Omega)\,, \qquad z_1 \in H^{-1}(\Omega)\ . \tag{31}$$

Suppose $f \in L^1(0,T;L^2(\Omega))$ and consider the homogeneous backward problem:

$$\begin{cases} \theta'' + A(t)\, \theta = f & \text{on } Q, \\ \theta = 0 & \text{on } \Sigma, \\ \theta(T) = 0, \quad \theta'(T) = 0 & \text{on } \Omega\ . \end{cases} \tag{32}$$

Multiply both sides of (32) by z, solution of (30), assuming that exists and integrate on Q. We obtain, formally:

$$\int_Q f\, z\, dx\, dt = \int_\Omega \theta(0)\, z_1\, dx - \int_\Omega \theta'(0)\, z_0\, dx - \int_\Sigma a_{ij}\, \nu_i\, \nu_j\, \frac{\partial\theta}{\partial\nu}\, v\, d\Sigma\ . \tag{33}$$

The solution $\theta = \theta(x,t)$ of (32) has the regularity

$$\theta \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))\ . \tag{34}$$

Then, (33), obtained formally, can be written:

$$\int_Q f\, z\, dx\, dt = \langle z_1, \theta(0)\rangle - (z_0, \theta'(0)) - \int_\Sigma a_{ij}\, \nu_i\, \nu_j\, \frac{\partial\theta}{\partial\nu}\, v\, d\Sigma\ . \tag{35}$$

Given $f \in L^1(0,T;L^2(\Omega))$ we obtain $\theta = \theta(x,t)$ solution of the backward problem (32), with regularity (34), and then we obtain the right-hand side of (35). Therefore, we have well defined the mapping S by:

$$\begin{aligned} &S\colon\ L^1(0,T;L^2(\Omega)) \to \mathbb{R} \\ &\langle S, f\rangle = \langle z_1, \theta(0)\rangle - (z_0, \theta'(0)) - \int_\Sigma a_{ij}\, \nu_i\, \nu_j\, \frac{\partial\theta}{\partial\nu}\, v\, d\Sigma\ , \end{aligned} \tag{36}$$

whence

$$|\langle S, f\rangle| \le C\Big(\|z_1\|_{H^{-1}(\Omega)} + |z_0|_{L^2(\Omega)} + \|v\|_{L^2(\Sigma)}\Big)\, \|f\|_{L^1(0,T;L^2(\Omega))} \ . \tag{37}$$

Then S is a linear continuous form on $L^1(0,T;L^2(\Omega))$, that is, $S \in L^\infty(0,T;L^2(\Omega))$, the topological dual of $L^1(0,T;L^2(\Omega))$. By Riesz's representation theorem, exists unique $z \in L^\infty(0,T;L^2(\Omega))$ such that

$$\langle S, f\rangle = \int_Q f\, z\, dx\, dt \ . \tag{38}$$

Whence by (38) we obtain a unique z, solution of (35) for each $f \in L^1(0,T;L^2(\Omega))$. This is called transposition method.

Definition 1. We call ultraweak solution of (30), with boundary and initial data given by (31), a function $z \in L^\infty(0,T;L^2(\Omega))$ satisfying:

$$\int_Q f\, z\, dx\, dt = \langle z_1, \theta(0)\rangle - (z_0, \theta'(0)) - \int_\Sigma a_{ij}\, \nu_i\, \nu_j\, \frac{\partial\theta}{\partial\nu}\, v\, d\Sigma \tag{39}$$

for all $f \in L^1(0,T;L^2(\Omega))$.

Lemma 4.1. *The system (30) has only one ultraweak solution z, verifying:*

$$\|z\|_{L^\infty(0,T;L^2(\Omega))} \le C\Big(\|z_1\|_{H^{-1}(\Omega)} + |z_0|_{L^2(\Omega)} + \|v\|_{L^2(\Sigma)}\Big) \ . \tag{40}$$

Proof: It follows from (36), (37), (38). The uniqueness comes from Du Bois Raymond's Lemma. ∎

In the following we obtain regularity of ultraweak solutions. The method consists in obtaining regularity of ultraweak solution with regular initial and boundary conditions. By density we obtain the regularity for the non regular case.

Lemma 4.2 *Given $\{z_0, z_1, v\} \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(0,T;H^{3/2}(\Gamma))$, exists a ultraweak solution z of the system (30), with the regularity:*

$$z \in C([0,T];H^1(\Omega)) \cap C^1([0,T];L^2(\Omega)) \ . \tag{41}$$

Proof: Let $\overline{v} \in H_0^2(0,T;H^2(\Omega))$ be such that $\gamma_0\, \overline{v} = v$, where γ_0 is the trace operator. Represent by u the solution of the system:

$$\begin{cases} u'' + A(t)\, u = -(\overline{v}'' + A(t)\, \overline{v}) \ \in \ L^2(Q) & \text{on } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = z_0, \quad u'(0) = z_1 & \text{on } \Omega \ . \end{cases} \tag{42}$$

We know that u has the regularity:

$$u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \ .$$

Then,

$$z = u + \overline{v} \ \in \ C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \ . \ \blacksquare$$

Theorem 4.3. *The system (30) has ultraweak solution z for all $\{z_0, z_1, v\} \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma)$, such that:*

$$z \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)) \tag{43}$$

and

$$\|z\|_{L^\infty(0,T;L^2(\Omega))} + \|z'\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq \\ \leq C\Big(|z_0|_{L^2(\Omega)} + \|z_1\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Sigma)}\Big) \ . \tag{44}$$

Proof: We will prove by density. In fact, let us consider $\{z_{0\mu}, z_{1\mu}, v_\mu\} \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(0,T; H^{3/2}(\Gamma))$ such that

$$\begin{cases} z_{0\mu} \to z_0 & \text{in } L^2(\Omega), \\ z_{1\mu} \to z_1 & \text{in } H^{-1}(\Omega), \\ v_\mu \to v & \text{in } L^2(\Sigma) \ . \end{cases} \tag{45}$$

Denote by $\overline{v}_\mu \in H_0^2(0,T; H^2(\Omega))$ the function that $v_\mu = \gamma_0 \, \overline{v}_\mu$. We have the problem

$$\begin{cases} z''_\mu + A(t)\, z_\mu = 0 & \text{on } Q, \\ z_\mu = v_\mu & \text{on } \Sigma, \\ z_\mu(0) = z_{0\mu}, \quad z'_\mu(0) = z_{1\mu} & \text{on } \Omega \ . \end{cases} \tag{46}$$

We have, from (41) the regularity for z_μ ultraweak solution of (46):

$$z_\mu \in C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \ .$$

From the linearity of the system (30), it follows that $z_\mu - z$ is ultraweak solution of (30), for the initial condition $z_{0\mu} - z_0$, $z_{1\mu} - z_1$, and boundary conditions $v_\mu - v$. Applying the estimate (40) $z_\mu - z$ and let $\mu \to \infty$, we obtain

$$z_\mu \to z \quad \text{in } \ L^\infty(0,T; L^2(\Omega)) \ .$$

We obtain $z \in C([0,T]; L^2(\Omega))$ because $z_\mu \in C([0,T]; L^2(\Omega))$. $\blacksquare$

Let us now prove that $z' \in C([0,T]; H^{-1}(\Omega))$. In this step of the proof we have some difficulty motivated by the dependence of the time t. We know that:

$$\langle z', f\rangle = -\int_Q z\, f'\, dx\, dt\,, \qquad f \in \mathcal{D}(Q)\ . \tag{47}$$

By hypothesis, z is ultraweak solution of (30), defined by (39). Then z' satisfies:

$$\langle z', f\rangle = (z_0, \theta'(0)) - \langle z_1, \theta(0)\rangle + \int_\Sigma a_{ij}\,\nu_i\,\nu_j\,\frac{\partial\theta}{\partial\nu}\,v\,d\Sigma\ , \tag{48}$$

where θ is solution of the system:

$$\begin{cases} \theta'' + A(t)\,\theta = f' & \text{on } Q,\\ \theta = 0 & \text{on } \Sigma,\\ \theta(T) = 0, \quad \theta'(T) = 0 & \text{on } \Omega\ . \end{cases} \tag{49}$$

If we prove the inequality

$$|\theta'(0)|_{L^2(\Omega)} + \|\theta(0)\|_{H_0^1(\Omega)} + \left|\frac{\partial\theta}{\partial\nu}\right|_{L^2(\Sigma)} \le C\,\|f\|_{L^1(0,T;H_0^1(\Omega))}\ ,$$

where θ is solution of (49) and $f \in L^1(0,T;H_0^1(\Omega))$, we obtain:

$$|\langle z', f\rangle| \le C\Big(|z_0|_{L^2(\Omega)} + \|z_1\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Sigma)}\Big)\,\|f\|_{L^1(0,T;H_0^1(\Omega))}$$

that is, $z' \in L^\infty(0,T;H^{-1}(\Omega))$ and

$$\|z'\|_{L^\infty(0,T;H^{-1}(\Omega))} \le C\Big(|z_0|_{L^2(\Omega)} + \|z_1\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Sigma)}\Big)\ . \tag{50}$$

From this inequality we use the same argument used to prove the regularity $z \in C([0,T];L^2(\Omega))$ in order to obtain $z' \in C([0,T];H^{-1}(\Omega))$.

We consider first, $f \in \mathcal{D}(Q)$ and by density we obtain the case $f \in L^1(0,T;H_0^1(\Omega))$.

We consider the system:

$$\begin{cases} y'' + A(t)\,y - \displaystyle\int_t^T A'(s)\,y(s)\,ds = f & \text{on } Q,\\ y = 0 & \text{on } \Sigma,\\ y(T) = y'(T) = 0 & \text{on } \Omega\ . \end{cases} \tag{51}$$

It follows that (51) has strong solution, i.e., almost everywhere in Q. The derivative of the solution is equal to the solution of (49), then $y'(t) \in H_0^1(\Omega)\cap H^2(\Omega)$. Whence $y \in C([0,T];H_0^1(\Omega)\cap H^2(\Omega))$. Multiply (51), by $A(t)\,y'$ and integrate on Q. We obtain:

$$\|y'(0)\| + |y''(0)| \le C\|f\|_{L^1(0,T;H_0^1(\Omega))}\ . \tag{52}$$

By the identity (21) for the solution of (49) with appropriate estimates and the hypothesis on a_{ij}, we obtain

$$\left\| \frac{\partial \theta}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C \, \|f\|_{L^1(0,T;H_0^1(\Omega))} \,. \tag{53}$$

From (52) and (53) we obtain the proof of $z' \in C([0,T]; H^{-1}(\Omega))$. ∎

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