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# A UNIFIED APPROACH TO MIN-MAX CRITICAL POINT THEOREMS

M. Ramos and C. Rebelo

**Abstract:** We present in a unified way some abstract theorems on critical point theory in Banach spaces. The approach is elementary and concentrates on the deformation theorems and on the general min-max principle.

## 1 – Introduction

In the last two decades variational methods have proved to be fruitful and flexible in attacking nonlinear problems. This method consists on trying to find solutions of a given equation by searching for stationary points of a real functional defined in the function space in which the solution is to lie; the given equation is the Euler-Lagrange equation satisfied by a stationary point. This functional is often unbounded so one cannot look for (global) maxima and minima. Instead one seeks saddle-points by a min-max argument.

This paper is intended to give a unified presentation of some results of critical point theory which appeared or have been used under a number of variants in the literature in recent years. We have tried to make it as self-contained as possible. We believe it will prove to be useful both for the user of critical point theorems and for further development of the theory, namely for quick proofs (and in some cases improvement) of the available general multiplicity results (as those in [Li, LL, MMP, Si]), the extensions to equivariant theory or the applications in nonlinear problems.

One of the useful techniques in obtaining critical points is based on deformation arguments. The first part of the paper is devoted to them. It consists of known theorems. However, we think it is worthwhile to present them in a rather general and unified way, so that in applications some technical computations become avoidable. Concerning Theorem 4.5 below for instance, this is a quite useful

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known theorem but we don't know of any complete published proof of the full statement.

On the other hand, in spite of being quite elementary, those theorems have been successively improved in some of its details ; in general this research is motivated by some specific feature on differential equations, let us quote [Ma, RT, BN, Se].

The second part of the paper concerns the general min-max theorem as formulated in [BN, Gh]. Here we slightly modify the new argument introduced by Brézis and Nirenberg [BN] on the deformation lemma (cf. Theorem 5.1 below) in order to unify the main abstract results quoted above. In fact – and this was suggested to us by an interesting paper of Silva [Si] – we formulate the min-max principle under an "homotopical linking" setting and this enables us to recover in the same theorem the recent examples of general critical point theorems, namely those in [Fe, MMP].

As a consequence of this point of view those examples are improved in what concerns the use of inequalities (rather than strict inequalities) in the statements, or in the weak version of the Palais-Smale condition that is assumed. More important than this, it is desirable to have min-max characterizations of the critical points, for example in order to evaluate their Morse indexes (this subject was developed in [RS]).

We prefer to leave further comments to the last section. Let us however remark that some important topics are not focused here, namely the use of Ekeland's principle for Gateaux differentiable mappings [CG, Sz]; the use of the $(PS)^*$ condition in Galerkin schemes [Li, LL]; dual classes and relative category [FLRW, So]; the structure of the critical set [FG]; critical manifolds and problems with symmetry [MW]...

## 2 – The Cauchy problem

Let us settle some notation that will be used throughout. Let $X$ be an open subset of a real Banach space $E$ and $f \in \mathcal{C}^1(X;\mathbb{R})$. We denote by $f'(u)$ the differential of $f$ at the point $u$, $f'(u) = df(u) \in E^*$ and by $\langle\cdot,\cdot\rangle$ the duality mapping between $E^*$ and $E$. Both norms in $E$ and $E^*$ are denoted by $\|\cdot\|$. Also, $d(u,v) := \|u-v\|$ is the distance in $E$.

A *critical point* of $f$ is a point $u \in X$ such that $f'(u) = 0$; the image $f(u)$ is a *critical value*. We denote by $K$ the critical set of $f$, $K := \{u\colon\ f'(u) = 0\}$. For each $c \in \mathbb{R}$, define

$$f^c := \{u\colon\ f(u) \le c\} \quad \text{and} \quad K_c := \{u \in K\colon\ f(u) = c\}\ .$$

The proofs of our first two lemmas are quite elementary.

**Lemma 2.1.** *Let $V: X \to E$ be a locally Lipschitz continuous map and $A, B \subseteq X$ be two disjoint closed nonempty subsets. Then*

**(i)** *the map $\chi$: $E \to [0,1]$ given by $\chi(u) = d(u,A)/(d(u,A) + d(u,B))$ is locally Lipschitz continuous;*

**(ii)** *if $A$ is compact then $V$ is Lipschitz continuous and bounded in a neighbourhood of $A$.*

**Lemma 2.2.** *Let $G$ be a locally Lipschitz continuous map $G\colon X\backslash K \to E$ and $A, B \subseteq X$ be closed disjoint subsets with $K \subseteq A$.*
*Then, for each closed subset $\tilde{A}$ such that $A \subseteq \mathrm{int}(\tilde{A}) \subseteq \tilde{A} \subseteq X\backslash B$, there exist two locally Lipschitz continuous maps $\chi$: $X \to [0,1]$ and $F\colon X \to E$ such that*

**(i)** $\chi\,|_{\tilde{A}} \equiv 0, \quad \chi\,|_B \equiv 1$;

**(ii)** $F(u) = \begin{cases} \chi(u)G(u) & \text{if } u \in X\backslash\tilde{A} \\ 0 & \text{if } u \in \tilde{A}\ . \end{cases}$

**Remark.** It is clear that such a set $\tilde{A}$ exists: take $\tilde{A} := \{u\colon\, d(u,A) \le d(u,B)\}$ for example.

We turn now to the construction of a *pseudo gradient vector field*.

**Lemma 2.3.** *Given positive constants $0 < \alpha < \beta$ there exists a locally Lipschitz continuous map $V: X\backslash K \to E$ such that for every $u \in X\backslash K$*

$$\alpha \le \langle f'(u), V(u)\rangle \le \|f'(u)\|\, \|V(u)\| \le \beta\ .$$

**Proof:** For each $x \in X\backslash K$, since $2\alpha/(\alpha+\beta) < 1$ and $\|f'(x)\| \ne 0$, the definition of the norm in $E^*$ allows us to choose a vector $w_x \in E$ with unit norm such that

$$\langle f'(x), w_x\rangle > \frac{2\alpha}{\alpha+\beta}\, \|f'(x)\|\ .$$

The vector $V_x := \frac{\alpha+\beta}{2}\, \|f'(x)\|^{-1}\, w_x$ satisfies $\alpha < \langle f'(x), V_x\rangle$ and $\|V_x\| < \beta \|f'(x)\|^{-1}$. The usual argument based on the continuity of $f'$ and paracompactness of $X$ yields the result. ∎

**Remark.** If $X$ is an Hilbert space and $V$ is of class $\mathcal{C}^2$ we can take $V(u) := \frac{\alpha+\beta}{2}\, \frac{\nabla f(u)}{\|\nabla f(u)\|^2}$.

Next we shall recall the following version of the Cauchy theorem on ordinary differential equations. Let $F\colon X \to E$ be continuous and $(t_0, u_0) \in \mathbb{R} \times X$, $r > 0$ be such that $B_r(u_0) := \{u \in E\colon \|u - u_0\| < r\} \subseteq X$. Denote

$$M := \sup_{u \in B_r(u_0)} \|F(u)\| \quad \text{and} \quad K := \sup_{u,v \in B_r(u_0)} \frac{\|F(u) - F(v)\|}{\|u - v\|} .$$

It is well-known that whenever $M\ell < r$ and $K < +\infty$ then the Cauchy problem $\dot\sigma(t) = F(\sigma(t))$, $\sigma(t_0) = u_0$ has a unique solution $\sigma(\cdot)$ defined on $I := [t_0 - \ell, t_0 + \ell]$ and taking values in $B_r(u_0)$. From this we derive the following.

**Proposition 2.4.** *If $F\colon X \to E$ is locally Lipschitz continuous then for each $u \in X$ the problem*

$$\dot\sigma(t) = F(\sigma(t)) , \qquad \sigma(0) = u$$

*has a unique solution defined on a maximal interval $]\omega_-(u), \omega_+(u)[$ containing 0. The set $\Omega := \{(t,u)\colon u \in X,\, t \in ]\omega_-(u), \omega_+(u)[\}$ is open and the map $\sigma \equiv \sigma(t,u)\colon \Omega \to X$ is locally Lipschitz continuous.*
*Moreover, if for some $u \in X$ the set $\sigma(\cdot, u)$ lies on a complete subset of $X$, then*

$$\omega_+(u) < +\infty \implies \int_0^{\omega_+(u)} \|F(\sigma(s))\|\, ds = +\infty .$$

**Proof:** From the previous remark, and for each $u \in X$, we have existence and uniqueness of a solution for the problem $\dot\sigma = F(\sigma)$, $\sigma(0) = u$, defined in a closed neighbourhood of 0, $[-\ell(u), \ell(u)]$ with $\ell(u) > 0$. Defining

$$\omega_+(u) := \sup\Big\{t\colon \text{the problem admits solution in } [0,t]\Big\},$$
$$\omega_-(u) := \inf\Big\{t\colon \text{the problem admits solution in } [t,0]\Big\} ,$$

we easily obtain the first assertion of the proposition.

Let us fix now $(t_0, u_0) \in \Omega$ with $t_0 \geq 0$ and $t_1 \in ]t_0, \omega_+(u_0)[$. We will show that if $u$ is sufficiently close to $u_0$ then $t_1 < \omega_+(u)$. A similar argument applies to the interval $]\omega_-(u_0), t_0[$ and this proves in particular that $\Omega$ is open.

Let us consider the compact set $\mathcal{C} := \sigma([0, t_1] \times \{u_0\})$. According to Lemma 2.1.(ii) we can fix positive constants $r$, $K$ with $r < 1$ such that

$$u, v \in \mathcal{U} := \{u\colon d(u, \mathcal{C}) < 2r\} \implies$$
$$\implies \|F(u)\| \leq K \quad \text{and} \quad \|F(u) - F(v)\| \leq K\|u - v\| .$$

Let us fix $\ell < r/(2K)$ such that $t_1/\ell \in \mathbf{N}$. From the remark above it follows that if we have $\|u - \sigma(\alpha, u_0)\| < r$ for some $\alpha \leq t_1$ then the problem

$$\dot\eta(t) = F(\eta(t)) , \qquad \eta(\alpha) = u$$

admits a unique solution, defined in the interval $[\alpha - \ell, \alpha + \ell]$ and with image in $\mathcal{U}$ (notice that $B_r(u) \subset B_{2r}(\sigma(\alpha, u_0)) \subset \mathcal{U}$).

Let $k := t_1/\ell \in \mathbf{N}$ and let us suppose that $\|u - u_0\| \le r/2^k$. According to what we just said, $\sigma(t, u)$ is defined in $[0, \ell]$, has image in $\mathcal{U}$ and for every $t \in [0, \ell]$

$$\begin{aligned}\|\sigma(t,u) - \sigma(t,u_0)\| &= \left\| u - u_0 + \int_0^t (F(\sigma(s,u)) - F(\sigma(s,u_0)))\, ds \right\| \\ &\le \|u - u_0\| + \ell K \sup_s \|\sigma(s,u) - \sigma(s,u_0)\| \ .\end{aligned}$$

Therefore, since we have $\ell K < 1/2$,

$$\|\sigma(\ell,u) - \sigma(\ell,u_0)\| \le \sup_s \|\sigma(s,u) - \sigma(s,u_0)\| \le 2\|u - u_0\| \le 2r/2^k \le r \ .$$

We can thus construct a solution of the problem $\dot\eta = F(\eta)$, $\eta(\ell) = \sigma(\ell, u)$, defined in $[0, 2\ell]$ and with image in $\mathcal{U}$. By uniqueness we have $\eta(t) \equiv \sigma(t,u)$, so that $2\ell < \omega_+(u)$. By iterating the argument it is then possible to construct $\sigma(\cdot, u)$ in $[(p-1)\ell, p\ell]$ with image in $\mathcal{U}$ and satisfying

$$\|\sigma(p\ell,u) - \sigma(p\ell,u_0)\| \le 2^p \|u - u_0\| \le 2^{p-k}\, r \le r \ .$$

When $p = k$ we conclude that $t_1 = k\ell < \omega_+(u)$, and this shows that $\Omega$ is open.

The previous argument has shown in particular that for $u, v \in B_\varepsilon(u_0)$ with $\varepsilon := r/2^k$ we have

$$\|F(\sigma(s,u))\| \le K \quad \text{and} \quad \|F(\sigma(s,u)) - F(\sigma(s,v))\| \le K\|u - v\|$$

for every $s \in [0, t_1]$. Therefore we have for every $t, t' \in [0, t_1]$,

$$\|\sigma(t',v) - \sigma(t,v)\| \le \left| \int_t^{t'} \|\dot\sigma(s,v)\|\, ds \right| = \left| \int_t^{t'} \|F(\sigma(s,v))\|\, ds \right| \le K|t - t'| \ ;$$

on the other hand,

$$\begin{aligned}\|\sigma(t,u) - \sigma(t,v)\| &\le \|u - v\| + \int_0^t \|F(\sigma(s,u)) - F(\sigma(s,v))\|\, ds \\ &\le \|u - v\| + K \int_0^t \|\sigma(s,u) - \sigma(s,v)\|\, ds\end{aligned}$$

and Gronwall inequality implies that

$$\|\sigma(t,u) - \sigma(t,v)\| \le \|u - v\|\, e^{Kt} \le \|u - v\|\, e^{Kt_1} \ .$$

Consequently,

$$\begin{aligned}\|\sigma(t,u) - \sigma(t',v)\| &\le \|\sigma(t,u) - \sigma(t,v)\| + \|\sigma(t,v) - \sigma(t',v)\| \\ &\le e^{Kt_1}\|u - v\| + K|t - t'| \ ,\end{aligned}$$

and this proves that $\sigma$ is locally Lipschitz continuous.

Finally, suppose that $\sigma(t) \equiv \sigma(t,u)$ varies in a complete set and, arguing by contradiction, that $\omega_+(u) < +\infty$ and $\int_0^{\omega_+(u)} \|F(\sigma(s))\|\, ds = \lim_{t\to\omega_+(u)} \int_0^t \|F(\sigma(s))\|\, ds < +\infty$. As

$$\begin{aligned}\|\sigma(t,u)-\sigma(s,u)\| &\le \Big|\int_s^t \|F(\sigma(\tau))\|\, d\tau\Big| \\ &= \Big|\int_0^t \|F(\sigma(\tau))\|\, d\tau - \int_0^s \|F(\sigma(\tau))\|\, d\tau\Big| \longrightarrow 0\end{aligned}$$

when $s,t\to\omega_+(u)$, the limit $\lim_{t\to\omega_+(u)}\sigma(t)$ exists and this clearly contradicts the definition of $\omega_+(u)$. ∎

We shall refer to $\sigma$ as the *flow* associated to the vector field $F$. We conclude the section with two remarks.

**Proposition 2.5.** *If $F\colon X\to E$ is locally Lipschitz continuous and the flow $\sigma$ is defined on $\mathbb{R}\times X$ then*

**(i)** *$\sigma(t,\cdot)$ is an homeomorphism for every $t$;*

**(ii)** *given any compact set $I\subset\mathbb{R}$ and any closed subset $A\subseteq X$, $\sigma(I\times A)$ is closed in $X$.*

**Proof:** The uniqueness of the Cauchy problem implies that $\sigma^{-1}(t,u) = \sigma(-t,u)$ for every $t$,$u$, and this shows that $\sigma(t,\cdot)$ is an homeomorphism.

As for (ii), let us suppose that $\sigma(t_n,u_n)\to v\in X$ for some sequence $(t_n,u_n)\in I\times A$. Passing if necessary to a subsequence, we have $t_n\to t\in I$. Since

$$u_n = \sigma^{-1}(t_n,\sigma(t_n,u_n)) \to \sigma^{-1}(t,v)\ ,$$

we conclude that $\sigma^{-1}(t,v)\in A$ and therefore $v=\sigma(t,\sigma(-t,v))\in\sigma(I\times A)$. ∎

**Proposition 2.6.** *Let $F: X\to E$ be a locally Lipschitz continuous map. Suppose*

$$\|F(u)\| \le A\|u\| + B \quad \forall\, u\in X$$

*for some constants $A,B$ and that the flow $\sigma$ always lies on complete subsets of $X$. Then $\omega(u)=\infty$ for every $u\in X$, $\sigma(t,\cdot)$ is an homeomorphism for every $t$ and $\sigma\colon \mathbb{R}\times X\to X$ is locally Lipschitz continuous and maps bounded sets into bounded sets.*

**Proof:** Given $u\in X$, suppose by contradiction that $\omega_+(u)<+\infty$. In the interval $[0,\omega_+(u)[$ the flow $\sigma(\cdot)\equiv\sigma(\cdot,u)$ satisfies

$$\|\sigma(t)\| \le \|u\| + \int_0^t \|\dot\sigma(s)\|\, ds \le \|u\| + A\int_0^t \|\sigma(s)\|\, ds + B\omega_+(u)\ .$$

By Gronwall inequality we deduce that $\sigma$ has a bounded image. Consequently, by our assumption, $F(\sigma)$ also has a bounded image — and this contradicts Proposition 2.4.

We conclude then that $\omega_+(u) = +\infty$ for every $u \in X$. In the same way we see that $\omega_-(u) = -\infty$. From previous propositions we deduce that $\sigma\colon \mathbb{R} \times X \to X$ is locally Lipschitz continuous, $\sigma(t, \cdot)$ is an homeomorphism, and the previous computations show that for every $s \in [0, t]$,

$$\|\sigma(s,u)\| \le (\|u\| + Bt)\, e^{At} ,$$

and therefore $\sigma$ takes bounded sets into bounded sets. ∎

## 3 – The deformation lemma

A continuous map $h\colon [0,1] \times X \to X$ such that $h_0(u) = u$ for every $u \in X$ is called an *homotopy*. We also write $h_t\colon X \to X$ for $h$. We say that $h$ is an *homotopy of homeomorphisms* if each map $h_t(\cdot)$ is an homeomorphism. Given $f \in \mathcal{C}^1(X; \mathbb{R})$, the homotopy is called *$f$-decreasing* if one has $f(h(t,u)) \le f(h(s,u))$ for every $u \in X$ and $s < t$. We shall always assume without further reference that the following holds

$$f^{-1}([a,b]) \text{ is complete} \qquad \forall\, a < b \in \mathbb{R} .$$

**Theorem 3.1.** *Let $a < b \in \mathbb{R}$, $\delta > 0$ and $S \subseteq X$ be a closed subset. Assume*

$$\|f'(u)\| \ge \frac{2(b-a)}{\delta} \qquad \forall\, u \in S \cap f^{-1}([a,b]) .$$

*Then, for each $\varepsilon > 0$ and for each closed subset $S' \subseteq X$ with $S \cap S' = \emptyset$, there is an $f$-decreasing and locally Lipschitz continuous homotopy of homeomorphisms $h_t\colon X \to X$ such that*

**(i)** *if $u \in f^b$ and $h(t,u) \in S$ for all $t \in [0,1]$ then $h_1(u) \in f^a$.*
*Moreover, if $u \in f^b$ and $h(t,u) \in S \cap \{f \ge a\}$ for all $t \in [0,s]$ then*

$$f(h(s,u)) \le f(u) - (b-a)s .$$

**(ii)** *$h_t(u) = u$ if $u \in A$, where*

$$A = \{f \le a - \varepsilon\} \cup \{f \ge b + \varepsilon\} \cup \{u\colon \|f'(u)\| \le (b-a)/\delta\} \cup S' .$$

**(iii)** *$d(h_t(u), u) \le 2\delta t$ for all $t, u$.*

**Proof:** Consider $A$ defined above and denote $B := f^{-1}([a,b]) \cap S$. Let $V : X\backslash K \to E$ be the vector field given by Lemma 2.3, with $\alpha = 1$ and $\beta = 2$. Let $\sigma$ be the flow obtained from the Cauchy problem

$$\dot{\sigma} = -F(\sigma) , \quad \sigma(0) = u \in X ,$$

where $F = \chi V$ is the vector field associated to $G \equiv V$ given by Lemma 2.2. In view of the definition of $\chi$ we have $\|F(u)\| \leq 2\delta/(b-a)$ in $X$, and Proposition 2.6 shows that $\sigma \colon [0,+\infty[ \times X \to X$ is locally Lipschitz continuous, maps bounded sets into bounded sets and for every $t \geq 0$, $\sigma(t,\cdot)$ is an homeomorphism in $X$.

For every $u \in X$, the map $\sigma(t,u)$ satisfies

$$\frac{d}{dt} f(\sigma(t,u)) = \langle f'(\sigma(t,u)), \dot{\sigma}(t,u)\rangle \leq -\chi(\sigma(t,u)) .$$

Observe that by the uniqueness of the Cauchy problem we have

$$u \in \tilde{A} \iff \exists\, t\colon \sigma(t,u) \in \tilde{A} \iff \forall\, t\colon \sigma(t,u) = u$$

and $f(\sigma(\cdot,u))$ is strictly decreasing for all $u \in X\backslash\tilde{A}$.

By the inequality above, if $\sigma(t,u) \in B$ for all $t \in [0,s]$, we have

$$f(\sigma(s,u)) \leq f(u) - s .$$

Since we also have

$$\begin{aligned} d(\sigma(t,u),u) \leq \int_0^t \|\dot{\sigma}(s,u)\|\, ds &= \int_0^t \|F(\sigma(s,u))\|\, ds \\ &\leq \frac{2\delta}{b-a} \int_0^t \chi(\sigma(s,u))\, ds \leq \frac{2\delta}{b-a} t , \end{aligned}$$

we can take $h(t,u) := \sigma((b-a)t,u)$. ∎

Notice that in the previous theorem it suffices to suppose that $f^{-1}([a-\varepsilon, b+\varepsilon])$ is complete. A similar remark holds for the subsequent results but, for convenience, we shall assume the completeness of the inverse images of every compact interval.

In the same way we will not insist neither in the regularity of the homotopy nor in the condition (iii). Observe that this condition shows in particular that $h$ maps bounded sets into bounded sets.

An interesting choice for $S$ is to take $S := \{u : \|f'(u)\| \geq 2(b-a)/\delta\}$. By specializing $b = c+\varepsilon$, $a = c-\varepsilon$ and $\delta = \sqrt{\varepsilon}$ we obtain

**Corollary 3.2.** *Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Then there is an $f$-decreasing homotopy of homeomorphisms $h_t \colon X \to X$ such that*

**(i)** *if* $u \in f^{c+\varepsilon}$ *and* $\|f'(h(t,u))\| \geq 4\sqrt{\varepsilon}$ *for all* $t \in [0,1]$ *then* $h_1(u) \in f^{c-\varepsilon}$. *Moreover, if* $c - \varepsilon \leq f(h(t,u)) \leq c + \varepsilon$ *and* $\|f'(h(t,u))\| \geq 4\sqrt{\varepsilon}$ *for all* $t \in [0,s]$ *then*

$$f(h(s,u)) \leq f(u) - 2\,\varepsilon s \ .$$

**(ii)** $h_t(u) = u$ *if* $\|f'(u)\| \leq 2\sqrt{\varepsilon}$ *or* $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon])$.

**(iii)** $d(h_t(u), u) \leq 2\sqrt{\varepsilon}\, t$ *for all* $t, u$.

The speed of decrease of the map $f(h(\cdot,u))$, indicated in (i), can be improved if we are less precise in the estimate in (iii):

**Corollary 3.3.** *Let* $c \in \mathbb{R}$ *and* $0 < \varepsilon < 1/2$. *Then there is an* $f$*-decreasing homotopy of homeomorphisms* $h_t \colon X \to X$ *such that*

**(i)** *if* $u \in f^{c+\varepsilon}$ *and* $\|f'(h(t,u))\| \geq 4\sqrt{\varepsilon}$ *for all* $t \in [0,1]$ *then* $h_1(u) \in f^{c-\varepsilon}$. *Moreover, if* $c - \varepsilon \leq f(h(t,u)) \leq c + \varepsilon$ *and* $\|f'(h(t,u))\| \geq 4\sqrt{\varepsilon}$ *for all* $t \in [0,s]$ *then*

$$f(h(s,u)) \leq f(u) - s \ .$$

**(ii)** $h_t(u) = u$ *if* $\|f'(u)\| \leq 2\sqrt{\varepsilon}$ *or* $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon])$.

**(iii)** $d(h_t(u), u) \leq \min\{t/\sqrt{\varepsilon}, 4\sqrt{\varepsilon}\}$ *for all* $t, u$.

**Proof:** Let $\sigma$ be the flow built in the proof of Theorem 3.1, with $b = c + \varepsilon$, $a = c - \varepsilon$ and $\delta = \sqrt{\varepsilon}$. As we showed before, we have $d(\sigma(t,u), u) \leq \frac{2\sqrt{\varepsilon}}{2\varepsilon} t = \frac{t}{\sqrt{\varepsilon}}$. On the other hand, since

$$d(\sigma(t,u),u) \leq \frac{1}{\sqrt{\varepsilon}} \int_0^t \chi(\sigma(s,u))\, ds \leq \frac{1}{\sqrt{\varepsilon}} \Big( f(u) - f(\sigma(t,u)) \Big) \leq \frac{4\varepsilon}{\sqrt{\varepsilon}} = 4\sqrt{\varepsilon} \ ,$$

we can define $h(t,u) := \sigma(t,u)$. ∎

Theorem 3.1 as stated in its generality allows us to locate the homotopy. For each nonempty set $F \subseteq X$ and each $\delta > 0$ denote $F_\delta := \{u \colon d(u,F) \leq \delta\}$.

**Corollary 3.4.** *Given constants* $a < b$, $\delta > 0$ *and two closed subsets* $F, G \subseteq X$ *with* $F_\delta \cap G = \emptyset$, *suppose that*

$$\|f'(u)\| \geq 4(b-a)/\delta \qquad \forall\, u \in F_\delta \cap f^{-1}([a,b]) \ .$$

*Then, for each* $\varepsilon > 0$, *there is an* $f$*-decreasing homotopy of homeomorphisms* $h_t \colon X \to X$ *such that*

**(i)** $h_1(f^b \cap F) \subseteq f^a$;

**(ii)** $h_t(u) = u$ *if* $u \in G$ *or* $u \notin f^{-1}([a-\varepsilon, b+\varepsilon])$;

**(iii)** $d(h_t(u), u) \le \delta t$ *for all* $t, u$.

**Proof:** It suffices to apply Theorem 3.1 with $S := F_\delta$ and $S' := G$. Indeed, if $u \in f^b \cap F$, it follows from (iii) that $h(t,u) \in S$ for all $t \in [0,1]$ so that $h_1(u) \in f^a$. ∎

## 4 – The Palais–Smale condition

Let us now deduce some consequences of the theorems just stated. The condition upon $\|f'\|$ in Theorem 3.1 will be assured by some assumptions on $f$ of Palais-Smale type. We continue to assume that $f^{-1}([a,b])$ is complete for every $a < b$.

Given $c \in \mathbb{R}$ we say that $f$ satisfies the *Palais-Smale condition at level* $c$ (the $(PS)_c$ condition for short) if every sequence $(u_n) \subset X$ such that $f(u_n) \to c$ and $\|f'(u_n)\| \to 0$ has a convergent subsequence in $X$. In particular, $K_c$ is compact.

**Theorem 4.1.** *If $f$ has no critical values in $[a,b]$ and satisfies the Palais-Smale condition at every level $c \in [a,b]$, there exist $\varepsilon > 0$ and an $f$-decreasing homotopy of homeomorphisms $h_t \colon X \to X$ such that*

$$h_1(f^b) \subseteq f^a \quad \text{and} \quad h_t(u) = u \ \ \forall\, u \in X \backslash f^{-1}([a-\varepsilon, b+\varepsilon]) \ .$$

**Proof:** Since the interval $[a,b]$ has no critical values, we can fix $\varepsilon > 0$ sufficiently small such that

$$\|f'(u)\| \ge \frac{2(b-a)}{1/\varepsilon} \qquad \forall\, u \in f^{-1}([a,b]) \ ,$$

and the conclusion follows from Theorem 3.1 with $S := X$. ∎

Another useful version of the theorem is the following.

**Theorem 4.2.** *If $f$ satisfies the $(PS)_c$ condition and $\mathcal{N}$ is an open neighbourhood of $K_c$, there exist $\varepsilon > 0$ and an $f$-decreasing homotopy of homeomorphisms $h_t \colon X \to X$ such that*

$$h_1(f^{c+\varepsilon} \backslash \mathcal{N}) \subseteq f^{c-\varepsilon} \quad \text{and} \quad h_t(u) = u \ \ \forall\, u \in X \backslash f^{-1}([c-2\varepsilon, c+2\varepsilon]) \ .$$

*Moreover, $h$ is locally Lipschitz continuous and satisfies $d(h_t(u), u) \le \sqrt{\varepsilon}\, t$ for all $t, u$.*

**Proof:** Denote $F := X\backslash\mathcal{N}$. From the $(PS)_c$ condition there is a positive constant $\varepsilon$ such that $F_{\sqrt{\varepsilon}} \cap K_c = \emptyset$ and

$$\|f'(u)\| \geq 8\sqrt{\varepsilon} \quad \forall\, u \in F_{\sqrt{\varepsilon}} \cap f^{-1}([c-\varepsilon, c+\varepsilon]) \ .$$

The conclusion follows then from Corollary 3.4 (with $G = \emptyset$). ∎

The following two results can be seen as two typical consequences of the above arguments. Many more of them could of course be selected from the existent literature but we ommit their statement since we do not intend to go here into the particular situations to which they apply.

**Proposition 4.3.** *Given $c \in \mathbb{R}$, let $F, G \subseteq X$ be two closed and disjoint subsets such that $F \cap K_c = \emptyset$ and*

$$\sup_F f \leq c \leq \inf_G f \ .$$

*If $f$ satisfies the $(PS)_c$ condition there exist $\varepsilon > 0$ and an $f$-decreasing homotopy of homeomorphisms $h_t\colon X \to X$ such that*

$$h_1(F) \subseteq f^{c-\varepsilon} \quad \text{and} \quad h_t(u) = u \ \ \forall\, u \in G \cup (X\backslash f^{-1}([c-2\varepsilon, c+2\varepsilon]) \ .$$

**Proof:** Theorem 4.2 implies the existence of $\varepsilon > 0$ and of an $f$-decreasing homotopy of homeomorphisms $h_t\colon X \to X$ such that $h_1(F) \subseteq f^{c-\varepsilon}$ and $h_t(u) = u$ for every point $u \in X\backslash f^{-1}([c-2\varepsilon, c+2\varepsilon])$.

Recall that for each $u$ the map $h(t) \equiv h(t,u)$ is the solution of some Cauchy problem

$$\dot{h}(t) = -W(h(t))\,, \qquad h(0) = u \ ,$$

where $W$ is bounded and satisfies $W(u) = 0 \ \ \forall\, u \in X\backslash f^{-1}([c-2\varepsilon, c+2\varepsilon])$. As we noticed in the proof of Theorem 3.1, the map $f(h(\cdot,u))$ is strictly decreasing for each $u \in F$. Consequently, the set $\tilde{F} := h([0,1] \times F)$ does not intersect $G$.

By Proposition 2.5, $\tilde{F}$ is closed and therefore we can fix a locally Lipschitz map $\chi\colon X \to [0,1]$ such that $\chi\,|_{\tilde{F}} \equiv 1$ and $\chi\,|_G \equiv 0$. Since the map $\chi W$ is still locally Lipschitz continuous and bounded, the Cauchy problem

$$\dot{\sigma}(t) = -\chi(\sigma(t))\, W(\sigma(t))\,, \qquad \sigma(0) = u$$

furnishes an $f$-decreasing homotopy of homeomorphisms $\sigma\colon [0,1] \times X \to X$ such that $\sigma(t,u) = u$ for every $u \in G \cup X\backslash f^{-1}([c-2\varepsilon, c+2\varepsilon])$. On the other hand, by the definition of $\chi$ and by the uniqueness of the Cauchy problem, we have $\sigma(t,u) = h(t,u)$ for each $u \in F$, and thus $\sigma_1(F) = h_1(F) \subseteq f^{c-\varepsilon}$. ∎

**Proposition 4.4.** *Given a Banach space $X$ and constants $a \leq b$, let us suppose that $f$ satisfies the $(PS)_c$ condition for every $c \in [a,b]$.*
*Then, for each $r > 0$ and $\varepsilon > 0$, there exist $R > r$, $c_1 > 0$ and an $f$-decreasing homotopy of homeomorphisms $h_t\colon X \to X$ such that*

**(i)** $h_1(f^b \backslash B_R(0)) \subseteq f^a$;

**(ii)** $h_t(u) = u \quad \forall\, u \in B_r(0) \cup (X \backslash f^{-1}([a-\varepsilon, b+\varepsilon]))$;

**(iii)** $d(h_t(u), u) \leq c_1 t \quad$ *for all* $t, u$.

**Proof:** In view of the $(PS)$ condition there is $R_0 > r$ such that

$$\|f'(u)\| \geq \frac{4}{R_0}(b-a) \qquad \forall\, u \in f^{-1}([a,b]), \quad \|u\| \geq R_0 \ .$$

Let us take $R{:=}3R_0$ and denote $G{:=}B_r(0)$, $F{:=}\{u\colon \|u\| \geq R\}$. As $R_0 < d(F,G)$ and $F_{R_0} \subseteq \{u\colon \|u\| \geq R_0\}$, the conclusion follows from Corollary 3.4. ∎

The next theorem is currently known as the "second deformation theorem". In it we allow $b$ to be $+\infty$ and in this case the set $f^b \backslash K_b$ is the whole open set $X$.

**Theorem 4.5.** *Given constants $a < b$, suppose that $f$ has no critical values in the interval $]a,b[$ and that $f^{-1}(\{a\})$ contains at most a finite number of critical points of $f$. Then, if $f$ satisfies the $(PS)_c$ condition for every $c \in [a,b[$, there exists an $f$-decreasing homotopy $h_t\colon f^b \backslash K_b \to X$ such that*

$$h_1(f^b \backslash K_b) \subseteq f^a \quad \text{and} \quad h_t(u) = u \ \ \forall\, u \in f^a \ .$$

**Proof:** Let us fix a map $V$ given by Lemma 2.3 (associated to $\alpha = 1$, $\beta = 2$). By Proposition 2.4, for each $u \in f^{-1}(]a,b]) \backslash K_b$, the Cauchy problem

$$\dot\sigma(t) = -V(\sigma(t)) \,, \qquad \sigma(0) = u$$

has a unique solution $\sigma(t,u)$ defined in $[0, \omega_+(u)[$. Over this interval we have

$$\frac{d}{dt} f(\sigma(t,u)) \leq -1 \ .$$

Lemma 1. *If $f(\sigma(t(u),u)) = a$ for some $t(u) < \omega_+(u)$ then $t(u)$ is unique and the map $u \mapsto t(u)$ is continuous.*

Indeed, the uniqueness of $t(u)$ is an obvious consequence of the previous inequality, which implies in particular that this point is characterized by the following relations

$$f(\sigma(s,u)) > a > f(\sigma(t,u)) \quad \text{if} \ \ s < t(u) < t < \omega_+(u) \ .$$

Given $\varepsilon > 0$, we have $f(\sigma(t(u)-\varepsilon,u)) > a > f(\sigma(t(u)+\varepsilon,u))$. In view of the continuity of $\sigma$, there is a neighbourhood $\mathcal{U}$ of $u$ such that $f(\sigma(t(u)-\varepsilon,v)) > a > f(\sigma(t(u)+\varepsilon,v))$ for every $v \in \mathcal{U} \cap f^{-1}(]a,b])\backslash K_b$. By the Intermediate Value Theorem we conclude that $|t(u)-t(v)| < \varepsilon$, and this proves the continuity of $t(u)$.

Given $u \in f^{-1}(]a,b])\backslash K_b$, we say that $t(u) = \omega_+(u)$ if $f(\sigma(t,u)) > a$ for every $t < \omega_+(u)$.

Lemma 2. *Let $(u_n)_{n\geq 1} \subset f^{-1}(]a,b])\backslash K_b$ and $v \in f^{-1}(\{a\})$, and suppose that $v = \lim \sigma(s_n,u_n)$ for some sequence $0 \leq s_n < t(u_n)$. Then, for every sequence $(t_n)$ with $s_n \leq t_n < t(u_n)$, we have $v = \lim \sigma(t_n,u_n)$.*

Indeed, fix a small $\varepsilon > 0$ in such a way that $K \cap B_\varepsilon(v) \cap f^{-1}([a,b]) \subseteq \{v\}$, $b_1 := \sup f(B_\varepsilon(v)) < b$ and let us prove that $\sigma(t_n,u_n) \in B_\varepsilon(v)$ for every large $n$. If not, there exists a sequence $(\sigma(t_i,u_i))$ with $d(\sigma(t_i,u_i),v) > \varepsilon$; on the other hand, our assumption implies that $d(\sigma(s_i,u_i),v) < \varepsilon/2$ for every large $i$. We can thus find points $\alpha_i$, $\beta_i$ with $s_i \leq \alpha_i < \beta_i \leq t_i$ such that

$$d(\sigma(\alpha_i,u_i),v) = \varepsilon/2, \;\; d(\sigma(\beta_i,u_i),v) = \varepsilon \;\text{ and }\; \sigma(\cdot,u_i) \in \mathcal{A} \quad \text{over } [\alpha_i,\beta_i] \;,$$

where $\mathcal{A}$ denotes the "ring" $\mathcal{A} := \{u\colon \varepsilon/2 \leq d(u,v) \leq \varepsilon\}$. From the $(PS)$ condition we have

$$\delta := \inf\Big\{ \|f'(u)\|\colon u \in f^{-1}([a,b_1]) \cap \mathcal{A}\Big\} > 0 \;.$$

On the other hand, as

$$\begin{aligned}\varepsilon/2 \leq d(\sigma(\alpha_i,u_i),\sigma(\beta_i,u_i)) &\leq \int_{\alpha_i}^{\beta_i} \|\dot\sigma(s,u_i)\|\, ds \\ &\leq 2\int_{\alpha_i}^{\beta_i} \|f'(\sigma(s,u_i))\|^{-1}\, ds \leq 2(\beta_i-\alpha_i)/\delta \;,\end{aligned}$$

we deduce

$$a \leq f(\sigma(\beta_i,u_i)) \leq f(\sigma(\alpha_i,u_i)) - (\beta_i-\alpha_i) \leq f(\sigma(s_i,u_i)) - \delta\varepsilon/4 \;.$$

Since $f(\sigma(s_i,u_i)) \to f(v) = a$, we obtain a contradiction and this proves the lemma.

Lemma 3. *If $u \in f^{-1}(]a,b])\backslash K_b$ is such that $t(u) = \omega_+(u)$, then there exists the limit $v := \lim_{t\to\omega_+(u)} \sigma(t,u)$ and $v \in K_a$.*

Suppose the lemma is false. Since $K_a$ is compact, Lemma 2 (with $u_n \equiv u$) implies that no sequence $(s_n) \subset [0,\omega_+(u)[$ can be such that $d(\sigma(s_n,u),K_a) \to 0$. Therefore we can fix $\varepsilon > 0$ and $\delta \in ]0,\omega_+(u)[$ such that $d(\sigma(t,u),K_a) > \varepsilon$ for every $t \in [\delta, \omega_+(u)[$. And since $\sigma([0,\delta],u)$ is a compact set disjoint from $K_a$, by choosing if necessary a smaller $\varepsilon$, we deduce

$$\sigma(t,u) \in f^{-1}([a,f(u)]) \cap \{u\colon d(u,K_a) \geq \varepsilon\} \quad \forall\, t \in [0,\omega_+(u)[\ .$$

Since this set is complete and

$$a < f(\sigma(t,u)) \leq f(u) - t \quad \forall\, t \in [0,\omega_+(u)[\ ,$$

we conclude that $\omega_+(u) \leq f(u) - a < +\infty$ and, by Proposition 2.4,

$$2\int_0^{\omega_+(u)} \|f'(\sigma(s,u))\|^{-1}\, ds \geq \int_0^{\omega_+(u)} \|V(\sigma(s,u))\|\, ds = +\infty\ .$$

Therefore there is a sequence $t_n \to \omega_+(u)$ such that $\|f'(\sigma(t_n,u))\| \to 0$. Now, since $(\sigma(t_n,u)) \subset f^{-1}([a,b_1])$ for some $b_1 < b$, we deduce from the $(PS)$ condition that there is a subsequence $(s_n)$ from $(t_n)$ such that $\sigma(s_n,u) \to v$ for some critical value $v \in f^{-1}([a,b_1])$. From the assumption we conclude that $f(v) = a$, therefore $v \in K_a$ and this contradicts the choice of $\varepsilon$.

Taking into account Lemmas 1 and 3, the limit

$$\sigma(t(u),u) := \lim_{t\to t(u)} \sigma(t,u)$$

is well-defined for each $u \in f^{-1}(]a,b])\backslash K_b$.

Lemma 4. *Let $(u_n) \subset f^{-1}(]a,b])\backslash K_b$, $u \in f^{-1}(\{a\})$ and suppose $u = \lim u_n$. Then, for every sequence $(s_n)$ with $0 \leq s_n \leq t(u_n)$, we have $u = \lim \sigma(s_n,u_n)$.*

By Lemma 2, we can assume that $s_n = t(u_n)$. Taking into account the definition of $\sigma(t(u_n),u_n)$, there exist $t_n < t(u_n)$ such that

$$d(\sigma(t_n,u_n), \sigma(t(u_n),u_n)) \leq 1/n\ .$$

As, by Lemma 2, the sequence $(\sigma(t_n,u_n))$ converges to $u$, so does $(\sigma(t(u_n),u_n))$.

Lemma 5. *Let $(u_n)_{n\geq 1} \subset f^{-1}(]a,b])\backslash K_b$, $u \in f^{-1}(]a,b])\backslash K_b$ be such that $t(u) = \omega_+(u)$ and $u = \lim u_n$. Then, for every sequence $(t_n)$ with $0 < t_n < t(u_n)$ and $\liminf t_n \geq \omega_+(u)$, we have $\sigma(t(u),u) = \lim \sigma(t(u_n),u_n) = \lim \sigma(t_n,u_n)$.*

Denote $v := \sigma(t(u),u)$. To prove that $v = \lim \sigma(t_n,u_n)$ we only have to show that any arbitrary subsequence of $(t_n)$ (still denoted by $(t_n)$) has a subsequence $(t_{n_k})$ such that $\sigma(t_{n_k},u_{n_k}) \to v$. Let us fix $s_1 \in ]0,\omega_+(u)[$ such that $\sigma(s_1,u) \in B_{1/2}(v)$. For large $n$, we then have $\sigma(s_1,u_n) \in B_1(v)$ and, since $\liminf t_n \geq \omega_+(u)$, we can choose a sufficiently large order $n_1$ such that $t_{n_1} > s_1$. By iterating this construction, we find points $s_k < t_{n_k}$ such that $\sigma(s_k, u_{n_k}) \in B_{1/k}(v)$. Lemma 2 shows that $\sigma(s_k, u_{n_k}) \to v$, and therefore $(\sigma(t_{n_k}, u_k))$ converges to $v$ as well.

Finally, by the definition of $t(u_n)$ there are points $t_n < t(u_n)$ such that

$$d\Big(\sigma(t(u_n), u_n), \sigma(t_n, u_n)\Big) \to 0 \quad \text{ and } \quad f(\sigma(t_n, u_n)) \to a \ .$$

In view of the continuity of the flow, we cannot have $\liminf t_n < \omega_+(u)$; otherwise there should exist a convergent subsequence $t_{n_k} \to c < \omega_+(u)$, thus $f(\sigma(c, u)) = a$ and this contradicts the assumption $t(u) = \omega_+(u)$. In this way we conclude that $\liminf t_n \geq \omega_+(u)$. Now, from the first part of the proof we deduce $\sigma(t_n, u_n) \to v$, and therefore the same holds for $(\sigma(t(u_n), u_n))$.

For each $u \in f^a$, we will say that $t(u) := 0$.

Consider now the map $\rho\colon [0, +\infty[ \times f^b \backslash K_b \to f^b$ defined as

$$\rho(t, u) = \begin{cases} u & \text{if } \ t(u) = 0 \\ \sigma(t, u) & \text{if } \ 0 \leq t < t(u) \\ \sigma(t(u), u) & \text{if } \ 0 < t(u) \leq t \ . \end{cases}$$

Lemma 6. *The map $\rho$ is continuous.*

Suppose $(t_n, u_n) \to (t, u)$ and let us prove that $\rho(t_n, u_n) \to \rho(t, u)$ (at least for some subsequence). Assume $f(u) \geq a$.

If $t(u) = 0$, since $\rho(t_n, u_n) = \sigma(s_n, u_n)$ with $s_n \leq t(u_n)$, we deduce from Lemma 4 that $\rho(t_n, u_n) \to u = \rho(t, u)$.

Suppose now that $t(u) > 0$. If $t < t(u)$, as $f(\sigma(t, u)) > a$, we also have $f(\sigma(t_n, u_n)) > a$ for large $n$, therefore $t_n < t(u_n)$ and

$$\rho(t_n, u_n) = \sigma(t_n, u_n) \longrightarrow \sigma(t, u) = \rho(t, u) \ .$$

Finally, suppose that $0 < t(u) \leq t$ and let us show that

$$\rho(t_n, u_n) \longrightarrow \sigma(t(u), u) \ .$$

If $t(u) < \omega_+(u)$, Lemma 1 implies that $t(u_n) \to t(u)$ and the conclusion is a consequence of the continuity of the flow. If $t(u) = \omega_+(u)$, Lemma 5 yields the conclusion.

From the definition of $\rho$ and taking into account Lemma 3, the limit $\bar{\rho}(u) := \lim_{t \to +\infty} \rho(t, u)$ is well defined for each $u \in f^b \backslash K_b$. Let $h\colon [0,1] \times f^b \backslash K_b \to f^b$ be the map defined as

$$h(t, u) = \begin{cases} \rho(\frac{t}{1-t}, u) & \text{if } \ 0 \leq t < 1 \\ \bar{\rho}(u) & \text{if } \ t = 1 \ . \end{cases}$$

The map $h$ is $f$-decreasing and satisfies $h_0(u) = u$, $h_1(u) \in f^a$ for every $u$, and $h_t(u) = u$ over $f^a$. It remains to prove that $h$ is continuous. This is a consequence of the previous lemma and of the following remark.

Lemma 7. *If $u_n \to u$ and $t_n \to +\infty$ then $\rho(t_n, u_n) \to \bar{\rho}(u)$.*

In order to see this, consider again several different situations.

The case $t(u) = 0$ is analogous to the corresponding situation in Lemma 6.

If $0 < t(u) < \omega_+(u)$ we have $0 < t(u_n) < 2t(u) < +\infty$ for large $n$, and therefore

$$\rho(t_n, u_n) = \sigma(t(u_n), u_n) \longrightarrow \sigma(t(u), u) = \bar{\rho}(u) \ .$$

Finally, suppose $t(u) = \omega_+(u)$. As $\liminf t_n = +\infty \geq \omega_+(u)$, we deduce from Lemma 5 that $\rho(t_n, u_n) \to \sigma(t(u), u) = \bar{\rho}(u)$, and this completes the proof of the lemma and of the theorem. ∎

The next three results concern different situations where there is a lack of compactness. The Palais-Smale condition is then replaced by some special assumptions. We have chosen those three examples both because they include some interesting ideas and because they proved to be useful in some particular applications in O.D.E.'s.

**Theorem 4.6.** *Given $c \in \mathbb{R}$, suppose there exist $g \in \mathcal{C}^1(X; \mathbb{R})$, $\varepsilon_0 > 0$ and $\beta \in ]0, 1[$ such that*

**(a)** *$f$ satisfies the $(PS)_c$ condition in $\{f \geq g\} := \{u\colon f(u) \geq g(u)\}$;*

**(b)** *$\|g'(u)\| \leq \beta \|f'(u)\|$ for all $u \in f^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \cap \{f = g\}$.*

*Then, for each open neighbourhood $\mathcal{N}$ of $K_c \cap \{f \geq g\}$, there exist $0 < \varepsilon < \varepsilon_0$ and an $f$-decreasing homotopy of homeomorphisms $h_t\colon X \to X$ such that*

**(i)** *$h_1(f^{c+\varepsilon} \backslash \mathcal{N}) \subseteq f^{c-\varepsilon} \cup \{f \leq g\}$;*

**(ii)** *$h_t(u) = u \quad \forall\, u \in X \backslash f^{-1}([c - 2\varepsilon, c + 2\varepsilon])$;*

**(iii)** *$h_t(\{f \leq g\}) \subseteq \{f \leq g\}$.*

**Proof:** The proof follows the same steps as those in Theorem 3.1. Denote $S := X \backslash \mathcal{N}$ and let us fix $\alpha \in ]\beta, 1[$. From the $(PS)_c$ condition we deduce that there is $\varepsilon \in ]0, \varepsilon_0/2[$ such that

$$\|f'(u)\| \geq 2\sqrt{\varepsilon}/\alpha \qquad \forall\, u \in f^{-1}([c - \varepsilon, c + \varepsilon]) \cap S_{2\sqrt{\varepsilon}} \cap \{f \geq g\} \ .$$

Let

$$A := X \backslash f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cup \{u\colon \|f'(u)\| \leq \sqrt{\varepsilon}/\alpha\} \ ,$$

$$B := f^{-1}([c - \varepsilon, c + \varepsilon]) \cap S_{2\sqrt{\varepsilon}} \cap \{f \geq g\} \ .$$

According to Lemma 2.3, fix a vector field $V$ associated to $1 < 1/\alpha$ and consider the flow associated to the Cauchy problem $\dot{\sigma} = -F(\sigma)$, $\sigma(0) = u \in X$, where $F = \chi V$ is given by Lemma 2.2 (with $G \equiv V$). In this way we obtain an $f$-decreasing homotopy of homeomorphisms $h_t(u) := \sigma(2\varepsilon t, u)$ which satisfies condition (ii) of the theorem.

Let us prove that $h_t(\{f \le g\}) \subseteq \{f \le g\}$. Take $u \in X \setminus \tilde{A}$ (cf. Lemma 2.2) such that $f(u) \le g(u)$ and denote $\theta(t) := f(\sigma(t,u)) - g(\sigma(t,u))$. We then have $\theta(0) \le 0$ and it suffices to prove that $\dot{\theta}(t_0) < 0$ whenever $\theta(t_0) = 0$. Indeed, letting $v := \sigma(t_0, u)$, we have

$$\begin{aligned}\dot{\theta}(t_0) &= -\chi(v)\langle f'(v) - g'(v), V(v)\rangle \\ &\le \chi(v)\Big(\|g'(v)\|\,\|V(v)\| - 1\Big) \le \chi(v)\left(\frac{\beta}{\alpha} - 1\right) < 0\ .\end{aligned}$$

Finally, let us prove that property (i) holds. Otherwise, there would exist $u \in f^{c+\varepsilon} \cap S$ such that $f(\sigma(2\varepsilon, u)) > c - \varepsilon$ and $f(\sigma(2\varepsilon, u)) > g(\sigma(2\varepsilon, u))$. Since the set $\{f \le g\}$ is invariant for the flow, $d(\sigma(t,u), u) \le 2\varepsilon \frac{1}{\alpha} \frac{\alpha}{\sqrt{\varepsilon}} = 2\sqrt{\varepsilon}$ in $[0, 2\varepsilon]$ and

$$\frac{d}{dt} f(\sigma(t,u)) \le -\chi(\sigma(t,u)) \le 0\ ,$$

we deduce that $\sigma(t,u) \in B$ for every $t \in [0, 2\varepsilon]$ and

$$c - \varepsilon < f(\sigma(2\varepsilon, u)) \le c + \varepsilon - 2\varepsilon = c - \varepsilon\ .$$

This contradiction proves (i) and ends the proof. ∎

The next theorem partially extends Theorem 4.2 and uses the following definition. Given $c \in \mathbb{R}$ we say that $f$ satisfies the *Palais-Smale-Cerami condition at level c* ($(PSC)_c$ condition for short) if any sequence $(u_n) \subset X$ with $f(u_n) \to c$ and $(1 + \|u_n\|)\,\|f'(u_n)\| \to 0$ has a convergent subsequence in $X$.

It is easy to see that this condition is equivalent to ask for the $(PS)_c$ condition on bounded subsets of $X$ and for the existence of some positive constants $R$, $\alpha$, and $\varepsilon$ in such a way that $\|f'(u)\| \ge \alpha/\|u\|$ for every $u$ satisfying $\|u\| \ge R$ and $|f(u) - c| \le \varepsilon$.

**Theorem 4.7.** *If $f$ satisfies the $(PSC)_c$ condition and $\mathcal{N}$ is an open neighbourhood of $K_c$, there exist $\varepsilon > 0$ and an $f$-decreasing homotopy of homeomorphisms $h_t \colon X \to X$ such that*

$$h_1(f^{c+\varepsilon} \setminus \mathcal{N}) \subseteq f^{c-\varepsilon} \quad \text{and} \quad h_t(u) = u \ \ \forall\, u \in X \setminus f^{-1}([c - 2\varepsilon, c + 2\varepsilon])\ .$$

*Moreover, $h$ is locally Lipschitz continuous and maps bounded sets into bounded sets.*

**Proof:** Since $K_c$ is compact, we can assume without loss of generality that the neighbourhood $\mathcal{N}$ is such that $\mathcal{N} = \{u\colon\ d(u, K_c) < 4\delta\}$ for some $0 < \delta < 4$. Fix positive constants $\alpha$, $R$ and $\varepsilon$ with $\varepsilon < \min\{\delta, \varepsilon_0/2\}$, such that $\mathcal{N} \subseteq B_R(0) := \{u\colon\ \|u\| \le R\}$ and

$$|f(u) - c| \le 2\varepsilon, \quad \|u\| \ge R \quad \Longrightarrow \quad \|f'(u)\| \ge \alpha/\|u\| \ ,$$
$$|f(u) - c| \le 2\varepsilon, \quad \|u\| \le R, \quad d(u, K_c) \ge \delta \quad \Longrightarrow \quad \|f'(u)\| \ge 4\varepsilon/\delta \ge \varepsilon \ .$$

Consider the flow $\sigma$ built in the proof of Theorem 3.1, associated to the closed disjoint sets

$$A := K \cup \{u\colon\ |f(u) - c| \ge 2\varepsilon\} \cup \{u\colon\ d(u, K_c) \le \delta\} \ ,$$
$$B := f^{-1}([c - \varepsilon, c + \varepsilon]) \cap \{u\colon\ d(u, K_c) \ge 2\delta\} \ .$$

As $\|F(u)\| = \|\chi(u)\, V(u)\| \le 2/\|f'(u)\|$ for every $u \in X \backslash A$, we conclude that

$$\|F(u)\| \le 2\varepsilon^{-1} + 2\alpha^{-1}\|u\|$$

in $X$ (consider the cases $\|u\| \le R$ or $\|u\| \ge R$).

Taking into account Proposition 2.6, the homotopy $h(t, u) := \sigma(2\varepsilon t, u)$ is well defined and it remains to prove that $h_1(f^{c+\varepsilon} \backslash \mathcal{N}) \subseteq f^{c-\varepsilon}$. Assume on the contrary that there is $u \in X \backslash A$ such that $d(u, K_c) \ge 4\delta$ and $c - \varepsilon < f(\sigma(t)) \le c + \varepsilon$ for every $t \in [0, 2\varepsilon]$, where we have written $\sigma(t) \equiv \sigma(t, u)$. We cannot have $\sigma(t) \in B$ for every $t$, otherwise

$$c - \varepsilon < f(\sigma(2\varepsilon)) \le f(u) - \int_0^{2\varepsilon} \langle f'(\sigma(s)), V(s)\rangle\, ds \le c + \varepsilon - 2\varepsilon = c - \varepsilon \ ,$$

a contradiction. So we deduce that there are $0 \le t_1 < t_2 \le 2\varepsilon$ such that

$$d(\sigma(t_1), K_c) = 4\delta \ge d(\sigma(t), K_c) \ge 2\delta = d(\sigma(t_2), K_c)$$

for every $t \in [t_1, t_2]$. In particular $\sigma([t_1, t_2]) \subset B \cap B_R(0)$ and

$$\begin{aligned} 2\delta \le d(\sigma(t_1), \sigma(t_2)) &\le \int_{t_1}^{t_2} \|V(\sigma(s))\|\, ds \\ &\le 2 \int_{t_1}^{t_2} \frac{1}{\|f'(\sigma(s))\|}\, ds \le 2\,|t_2 - t_1|\, \frac{\delta}{4\varepsilon} \le \delta \ , \end{aligned}$$

and from this contradiction we may conclude. ∎

We end the section with one more example. Given $c \in \mathbb{R}$ we say that $f$ satisfies the *Palais–Smale–Séré condition at level* $c$ ($(PSS)_c$ condition for short) if every sequence $(u_n) \subset X$ such that $f(u_n) \to c$, $\|f'(u_n)\| \to 0$ and $\|u_n - u_{n+1}\| \to 0$ has a convergent subsequence in $X$.

**Theorem 4.8.** *If $f$ satisfies the $(PSS)$ condition at every point of $[a-\varepsilon, b+\varepsilon]$ for some $\varepsilon > 0$ and this interval does not contain any critical values of $f$, then there exists an $f$-decreasing homotopy of homeomorphisms $h_t : X \to X$ such that*

$$h_1(f^b) \subseteq f^a \quad \text{and} \quad h_t(u) = u \ \ \forall\, u \in X \backslash f^{-1}([a-\varepsilon, b+\varepsilon]) \ .$$

**Proof:** The proof of the theorem makes use of the following elementary result:

Lemma. *Let $\omega > 0$ and $\theta \in \mathcal{C}([0,\omega[; \mathbb{R})$, $\theta > 0$, be such that $\int_0^\omega \theta(s)\, ds = +\infty$. Then there is an increasing sequence $(t_n) \subset [0,\omega[$, convergent to $\omega$ and such that*

$$\theta(t_n) \to +\infty \quad \text{and} \quad \int_{t_n}^{t_{n+1}} \theta(s)\, ds \to 0 \ .$$

Indeed, define by recurrence a strictly increasing sequence $(s_n) \subset [0,\omega[$ by taking $s_0 = 0$ and $\int_{s_n}^{s_{n+1}} \theta(s)\, ds = \sqrt{\omega - s_n}$. Let $L := \lim s_n$. If $L < \omega$ we would have $\int_0^L \theta(s)\, ds = \sum_{n \geq 0} \sqrt{\omega - s_n}$ and this is impossible because the integral is finite while the series diverges.

Therefore, $\omega = \lim s_n$. From the definition of $s_n$ we have

$$\max_{[s_n, s_{n+1}]} \theta \geq \frac{\sqrt{\omega - s_n}}{s_{n+1} - s_n} \geq \frac{1}{\sqrt{\omega - s_n}}$$

and this implies the existence of an increasing sequence $(t_n) \subset [s_n, s_{n+1}]$ convergent to $\omega$, with $\theta(t_n) \to +\infty$. Since

$$\int_{t_n}^{t_{n+1}} \theta(s)\, ds \leq \int_{s_n}^{s_{n+2}} \theta(s)\, ds = \sqrt{\omega - s_n} + \sqrt{\omega - s_{n+1}} \longrightarrow 0 \ ,$$

$(t_n)$ is the required sequence.

Now, let $A$, $B$ be the closed disjoint sets:

$$A := \Big(X \backslash f^{-1}([a-\varepsilon, b+\varepsilon])\Big) \ , \qquad B := f^{-1}([a,b]) \ ,$$

(notice that $K \subseteq A$) and consider the flow $\sigma$ associated to $A$ and $B$, built as in the proof of Theorem 4.7.

For each $u$, we have $\omega_+(u) = +\infty$. Otherwise, in view of Proposition 2.4, the map $\theta(t) := \|F(\sigma(t))\|$ (where we have written $\sigma(t) \equiv \sigma(t,u)$) with $u \in X \backslash A$

would verify the assumptions of the previous lemma, and this would imply the existence of a sequence $(t_n) \subset [0, w_+(u)[$ satisfying $a - \varepsilon \le f(\sigma(t_n)) \le b + \varepsilon$,

$$\|f'(\sigma(t_n))\| \le 2/\theta(t_n) \to 0 \quad \text{and} \quad d(\sigma(t_{n+1}), \sigma(t_n)) \le \int_{t_n}^{t_{n+1}} \theta(s)\, ds \to 0 \ .$$

Using condition $(PSS)$ we contradict the assumption made in the theorem. Analogously we can prove that $\omega_-(u) = -\infty$ for every $u$, therefore $\sigma \equiv \sigma(t,u)$ is a locally Lipschitz continuous map defined in $\mathbb{R} \times X$ and $\sigma(t,\cdot)$ is an homeomorphism.

Finally, since $\frac{d}{dt} f(\sigma(t,u)) \le -1$ if $\sigma(t,u)$ varies in $B$, we let $h(t,u) := \sigma((b-a)t, u)$. ∎

## 5 – Homotopical linking

In this section we prove a general theorem of min-max type by combining an argument in [BN] with a notion of linking similar to the ones in [BR, Si]. The subset $T$ introduced in Theorem 5.1 below is suggested by the results in [Gh] on the location of the critical points.

As before we take $f \in \mathcal{C}^1(X;\mathbb{R})$ and assume $f^{-1}([a,b])$ is complete for every constants $a < b$.

Consider three subsets $\partial Q$, $Q$ and $A$ of $X$ where $\partial Q \subseteq Q$ are both compact and $Q \cap A = \emptyset$ (the sets $\partial Q$ and $A$, but not $Q$, may be empty). We define a class $\Gamma$ of homotopies

$$\Gamma := \Big\{ \gamma_t \colon Q \to X \backslash A \colon \gamma_t\,|_{\partial Q} \equiv Id \ \ \forall\, t \Big\}$$

and the number

$$c := \inf_{\gamma_t \in \Gamma} \sup_{u \in Q} f(\gamma_1(u)) \ .$$

Here $Id$ denotes (the restriction of) the identity mapping. Note that $\Gamma$ is nonempty since $Id \in \Gamma$. From the definition of $c$ we also see that

$$\sup_{\partial Q} f \le c \le \sup_{Q} f \ .$$

We shall also assume that

$$(H) \qquad \sup_{Q} f \le \inf_{A} f \ .$$

By definition, a *minimizing sequence* for $c$ is a sequence of homotopies $(\gamma_t^n)_{n \ge 1} \subset \Gamma$ satisfying

$$\sup_{u \in Q} f(\gamma_1^n(u)) \to c \quad \text{and} \quad \sup_{u \in Q} f(\gamma_1^n(u)) \le \inf_{A} f \ .$$

Such a sequence always exists. This is clear in case $c < \inf_A f$; and if $c = \inf_A f$ it follows from $(H)$ that we can choose $\gamma^n \equiv Id$ as a minimizing sequence.

**Theorem 5.1.** *Assume $c \in \mathbb{R}$ and that condition $(H)$ holds. Suppose there exists $T \subseteq X$ such that*

$$(H') \qquad \forall\, \gamma_t \in \Gamma \;\; \exists u \in Q \backslash \partial Q\colon \quad f(\gamma_1(u)) \geq c \;\; \text{and} \;\; \gamma_1(u) \in T\ .$$

*Let $(\gamma_t^n)_{n\geq 1}$ be a minimizing sequence for $c$. Then, up to a subsequence, there exists $(u_n) \subset X$ such that*

$$f(u_n) \to c, \;\; \|f'(u_n)\| \to 0, \;\; d(u_n, T) \to 0 \;\; \text{and} \;\; d(u_n, \gamma_1^n(Q)) \to 0\ .$$

**Proof:** For each $\varepsilon \in ]0, 1/2[$ and $n_0 \in \mathbb{N}$ let us fix $n \geq n_0$ sufficiently large such that the homotopy $\gamma_t := \gamma_t^n$ satisfies

$$\sup_{\gamma_1(Q)} f \leq c + \varepsilon\ .$$

Consider the homotopy $h_t\colon X \to X$ given by Corollary 3.3 and let us prove that there exist $t \in [0, 1]$ and $x \in Q$ such that $v := \gamma_1(x)$ satisfies

$$c \leq f(h(t, v)), \;\; \|f'(h(t, v))\| \leq 4\sqrt{\varepsilon} \;\; \text{and} \;\; d(h(t, v), T) \leq 4\sqrt{\varepsilon}\ .$$

By the arbitrariness of $\varepsilon$ and $n_0$, the theorem is then proved by choosing $u_n = h(t, v)$ (observe that $d(u_n, T) \leq d(u_n, v) + d(v, T) \leq 8\sqrt{\varepsilon}$).

In order to prove the claim we argue by contradiction and suppose that the previous condition does not hold. In particular, and by property (iii) of Corollary 3.3, for every point $t_1 \in [0, 1]$ and $v = \gamma_1(x)$,

$$c \leq f(h(t_1, v)) \;\; \text{and} \;\; h(t_1, v) \in T \implies \|f'(h(t, v))\| \geq 4\sqrt{\varepsilon} \quad \forall\, 0 \leq t \leq t_1\ .$$

Also, property (i) of that corollary implies $t_1 < 1$ and

$$(*) \qquad f(h(t_1, v)) \leq f(v) - t_1\ .$$

On the other hand, since $f$ is locally Lipschitz continuous and $\partial Q$ is a compact set, $f$ is Lipschitz continuous in a neighbourhood of $\partial Q$ and we can fix positive constants $a$ and $C$ such that

$$(**) \qquad d(u, \partial Q) \leq a \implies f(u) \leq \sup_{\partial Q} f + C d(u, \partial Q)\ .$$

Let us fix a constant $M > \max\{C, 1/a\}$ and define a continuous map $\varphi\colon Q \to X$,

$$\varphi(u) := \min\Big\{1, \max\{d(u, \partial Q), M d(\gamma_1(u), \partial Q)\}\Big\}\ .$$

Consider the homotopy

$$\alpha(t,u) := \begin{cases} \gamma(2t,u) & \text{if } \ 0 \le t \le 1/2 \\ h((2t-1)\varphi(u), \gamma_1(u)) & \text{if } \ 1/2 \le t \le 1 \ . \end{cases}$$

It is clear that $\alpha$ is continuous and $\alpha_t|_{\partial Q} \equiv Id$. On the other hand, as it was explicitly observed in the proof of Theorem 3.1, the map $f(h(\cdot,u))$ is strictly decreasing for each $u \in X$, unless $h(t,u) = u \ \forall t \in [0,1]$. Since $\gamma([0,1]) \cap A = \emptyset$ and $\sup_{\gamma_1(Q)} f \le \inf_A f$, we conclude that $\alpha([0,1]) \cap A = \emptyset$ and thus $\alpha \in \Gamma$.

According to assumption $(H')$, we can fix $x \in Q \backslash \partial Q$ such that $v = \alpha_1(x)$ satisfies

$$c \le f(h(\varphi(x), v)) \quad \text{ and } \quad h(\varphi(x), v) \in T \ .$$

By the previous remark we must have $\varphi(x) < 1$, and then

$$1 \ge \varphi(x) \ge M d(v, \partial Q) \ .$$

Consequently, $d(v, \partial Q) \le a$. On the other hand, since we have $\sup_{\partial Q} f \le c$, both $(*)$ and $(**)$ imply

$$\begin{aligned} c \le f(h(\varphi(x), v)) \le f(v) - \varphi(x) &\le c + C d(v, \partial Q) - \varphi(x) \\ &\le c + (C - M)\, d(v, \partial Q) \ , \end{aligned}$$

therefore $d(v, \partial Q) = \varphi(x) = 0$. This last equality shows that $x \in \partial Q$ and this contradicts the choice of $x$ and proves the theorem. ∎

Condition $(H')$ of Theorem 5.1 can be checked by means of the following notion.

**Definition 5.2.** Given a closed subset $S \subseteq X$ we say that $Q$ and $S$ *link homotopically throught* $\partial Q$ (in $X \backslash A$) if $S \cap \partial Q = \emptyset$ and $\gamma_1(Q) \cap S \ne \emptyset$ for every $\gamma_t \in \Gamma$.

In the context of Theorem 5.1, given a minimizing sequence $(\gamma_t^n) \subset \Gamma$ for $c$ we say that $f$ satisfies the $(PS)_c$ *near* $(\gamma_t^n)$ if every sequence $(u_n) \subset X$ such that $f(u_n) \to c$, $\|f'(u_n)\| \to 0$ and $\liminf d(u_n, \gamma_1^n(Q)) = 0$ possesses a convergent subsequence in $X$.

**Theorem 5.3.** *Assume $S$ and $\partial Q$ link homotopically throught $\partial Q$ in $X \backslash A$ and that*

$$\sup_{\partial Q} f \le \inf_S f \quad \textit{and} \quad \sup_Q f \le \inf_A f \ .$$

*If $c \in \mathbb{R}$ and $(\gamma_t^n) \subset \Gamma$ is a given minimizing sequence for $c$ then, up to a subsequence, there exist $u_n \in X$ such that*

$$f(u_n) \to c\,, \qquad f'(u_n) \to 0 \qquad \text{and} \qquad d(u_n, \gamma_1^n(Q)) \to 0\ .$$

*In particular, if $f$ satisfies the $(PS)_c$ condition near $(\gamma_t^n)$ then*

- **(i)** $K_c \cap (X \backslash \partial Q) \neq \emptyset$;
- **(ii)** *if* $c = \inf_S f$ *then* $K_c \cap S \neq \emptyset$;
- **(iii)** *if* $c = \sup_Q f$ *and $f$ satisfies the $(PS)_c$ condition (or the $(PS)$ condition on the bounded subsets of $X$) then* $K_c \cap Q \neq \emptyset$.

**Proof:** From the definition of $c$ we have

$$\sup_{\partial Q} f \le \inf_S f \le c \le \sup_Q f \le \inf_A f\ .$$

We can easily verify that condition $(H')$ of Theorem 1.1 is satisfied if we choose for $T$ the set $T = \{u:\ f(u) \ge c\}$ or $T = S$ according to whether $\inf_S f < c$ or $\inf_S f = c$ respectively (for the first case, take into account that $Q$ is, by assumption, compact). The conclusion follows then from the previous theorem.

As for (iii), observe that we can choose $\gamma_t^n \equiv Id$ as a minimizing sequence, whenever $c = \sup_Q f$. ∎

**Remark 5.4.** It readily follows from the proof of the theorems that they remain true (with simpler proofs) if we replace the homotopies in $\Gamma$ by continuous maps $\gamma\colon Q \to X \backslash A$. In particular if we choose $A = \emptyset$, the class $\Gamma$ thus obtained is the usual min-max class considered in the literature.

It is clear that the above results still hold true in a slightly more general setting. One could start with a compact metric space $Q$, a closed subset $\partial Q$ of $Q$ and a continuous map $p\colon Q \to X$. Given $A \subseteq X$ disjoint from the image set $p(Q)$, the class $\Gamma$ is now defined by

$$\Gamma := \Big\{\gamma \in \mathcal{C}([0,1] \times Q; X \backslash A):\ \ \gamma_0 \equiv p,\ \gamma_t\,|_{\partial Q} \equiv p\,|_{\partial Q}\ \ \forall\, t\Big\}\ .$$

## 6 – Examples

We conclude the paper by showing three specializations of Theorem 5.3. We start with the Saddle-Point theorem of Rabinowitz [Ra]. It is well-known that this theorem can be deduced from Theorem 5.3 (taking into account Remark 5.4);

we shall combine it with a recent theorem of Feireisl [Fe] to obtain a multiplicity result.

Unless otherwise stated, we let $f \in \mathcal{C}^1(X;\mathbb{R})$ be defined on a Banach space $X$. We assume $X = X_1 \oplus X_2$ (topological direct sum) where $X_1$ is a finite dimensional subespace. For each $R > 0$ we denote by $B_R$ the closed ball of radius $R$ centered at the origin and by $\partial B_R$ its sphere. It will also be convenient to introduce the following convention: we say that $f$ satisfies the $(PS)$ condition on a given interval $I \subseteq \mathbb{R}$ whenever $f$ satisfies the $(PS)_c$ condition for every point $c \in I$.

**Theorem 6.1.** *Assume that $X = X_1 \oplus X_2$ where $\dim X_1 < \infty$. Moreover, $X_1 = V_1 \oplus \mathbb{R}^+ e$ with $e \neq 0$. Suppose there exists $R > 0$ such that*

$$\sup_{\partial B_R \cap X_1} f \leq \inf_{X_2} f \quad \text{and} \quad -\infty < a := \inf_{\mathbb{R}^+ e \oplus X_2} f \ .$$

*Denote $b := \sup_{B_R \cap X_1} f$. If $f$ satisfies the $(PS)$ condition on $[a,b]$ then $f$ admits two distinct critical points $u_0$ and $u_1$ such that*

$$a \leq f(u_0) \leq \sup_{\partial B_R \cap X_1} f \leq \inf_{X_2} f \leq f(u_1) \leq b \ .$$

**Proof:** The critical point $u_1$ is given by the Saddle–Point theorem (that is by a specialization of Theorem 5.3 with $Q := B_R \cap X_1$, $\partial Q := \partial B_R \cap X_1$, $S := X_2$ and $A := \emptyset$).

On the other hand, choose now $Q := \partial B_R \cap X_1$, $\partial Q := \emptyset$, $A := X_2$ and $S := \mathbb{R}^+ e \oplus X_2$. It is proved in [Fe] that $Q$ and $S$ link homotopically throught $\partial Q$ in $X \backslash A$ and thus Theorem 5.3 provides the second critical point $u_0$. The theorem is then proved in case $f(u_0) < f(u_1)$ holds.

Suppose now that $f(u_0) = f(u_1)$. Then we have $f(u_0) = \sup_{\partial B_R \cap X_1} f = \inf_{X_2} f = f(u_1)$ and the statements in (ii), (iii) of Theorem 5.3 show that $u_0 \in \partial B_R \cap X_1$ and $u_1 \in X_2$. In particular $u_0 \neq u_1$ and this completes the proof. ∎

We turn now to the "local linking" theorem in [MMP]. Given positive constants $R_1$ and $R_2$ we let $B_1 := B_{R_1} \cap X_1$ and $B_2 := B_{R_2} \cap X_2$. The corresponding spheres in $X_1$ and in $X_2$ are denoted $\partial B_1$ and $\partial B_2$ respectively.

**Theorem 6.2.** *Assume that $X = X_1 \oplus X_2$ where $\dim X_1 < \infty$ and that there exist positive constants $R_1$ and $R_2$ such that*

$$\sup_{\partial B_1} f \leq \inf_{B_2} f \leq \sup_{B_1} f \leq \inf_{\partial B_2} f \ .$$

*If $f$ satisfies the $(PS)$ condition on $[\inf_{B_2} f, \sup_{B_1} f]$ then this interval contains a critical value of $f$.*

**Proof:** Let $Q := B_1$, $\partial Q := \partial B_1$, $A := \partial B_2$ and $S := B_2$. It is proved in [MMP] that the required linking condition is verified and then the conclusion follows readily from Theorem 5.3. ∎

We end up with the Linking theorem of Rabinowitz [Ra, page 28]. We assume $X = X_1 \oplus \mathbb{R}e \oplus X_2$ where $\|e\| = 1$ and $X_1$ is finite dimensional. For given constants $R_1, R > 0$, define $Q_0 := B_1 \oplus [0, R]e$ where $B_1$ denotes the ball $B_{R_1} \cap X_1$. Also, for given $\rho > 0$, denote $B_2 := \{u \in \mathbb{R}e \oplus X_2 \colon \|u\| \le \rho\}$ and let $\partial Q_0$, $\partial B_2$ be the corresponding boundaries in the spaces $X_1 \oplus \mathbb{R}e$ and $\mathbb{R}e \oplus X_2$ respectively. Now we may state the following version of a multiplicity result in [MMP, Theorem 3.6].

**Theorem 6.3.** *Assume that $X_1$ is finite dimensional and that, for some $0 < \rho < R$ and $R_1 > 0$,*

$$\sup_{\partial Q_0} f \le \inf_{\partial B_2} f \quad \text{and} \quad -\infty < \inf_{B_2} f \ .$$

*If $f$ satisfies the $(PS)$ condition on $[\inf_{B_2} f, \sup_{Q_0} f]$ then there exist two distinct critical points $u_0$ and $u_1$ of $f$ such that*

$$\inf_{B_2} f \le f(u_0) \le \sup_{\partial Q_0} f \le \inf_{\partial B_2} f \le f(u_1) \le \sup_{Q_0} f \ .$$

**Proof:** The critical point $u_1$ is given by Rabinowitz's theorem (this is a specialization of Theorem 5.3 by letting $Q := Q_0$, $\partial Q := \partial Q_0$, $S := \partial B_2$ and $A := \emptyset$). The existence of $u_0$ can again be deduced from our theorem by taking $Q := \partial Q_0$, $\partial Q := \emptyset$, $S := B_2$ and $A := \partial B_2$; indeed, it is proved in [MMP] that the linking condition is fullfilled.

To end the proof observe that if $f(u_0) = f(u_1)$ then again both (ii) and (iii) in Theorem 5.3 imply that $u_0 \in \partial Q_0$ and $u_1 \in \partial B_2$ so that $u_0 \ne u_1$. ∎

## 7 – Historical note

Earlier works as those in [Br], [D] or [Sc] for example already use deformations of gradient type. The pseudo-gradient vector fields were introduced by Palais [Pa]. The condition $(C)$ of Palais and Smale, here called $(PS)$ condition, appears in [PS], and goes back to Krasnosel'skii book [Kr].

Both Theorem 3.1 and Corollary 3.2, 3.4 are quantitative versions of a theorem of Clark [Cl], due to Willem [Wi]. Our exposition is slightly different and was inspired by the one in Brézis and Nirenberg [BN], to whom we owe Corollary 3.3.

The first two theorems in section 4 were taken from [Wi] (those are classic results that already appeared in the literature), Theorem 4.3 from Yihong Du [Du] and Theorem 4.4 from Silva [Si]. Theorem 4.5 was proved by Marino and Prodi [MP] in the case of a $\mathcal{C}^2$ functional, see also [Ch]. Theorem 4.6 is a version of a result of Majer [Ma] while Theorems 4.7 and 4.8 were taken from corresponding results of Bartolo, Benci and Fortunato [BBF] (where the authors make use of a variant of the $(PS)$ condition due to Cerami [Ce]), and of Séré [Se] (see also [CES]).

For some textbooks with many developments and applications, we mention [Co], [MW], [Ra], [So], [Wi] and [Ze].

The idea of an "homotopical index" already appears with Benci and Rabinowitz [BR]. A different definition was proposed by Silva [Si], see also [Te]. A somehow dual notion is the *relative category*, an extension of the Lusternik-Schnirelmann category introduced by Fournier and Willem [FW], see also [FLRW].

Theorem 5.1 is an improved version of Theorem 1 in [BN] while Theorem 5.3 is a version of Theorem 1 in [Gh]. The general formulation of the latter theorem allows to prove in a unified way the examples in section 6; Theorem 6.1 is due to Feireisl [Fe] while the other two were proved by Micheletti, Marino and Pistoia [MMP] (Theorem 6.2 was proved by Castro [Ca] in the case $X_1$ has dimension one; the geometrical setting of the theorem – the so called "local linking" – was first studied by Li and Liu [Li], [LL]).

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M. Ramos and C. Rebelo,
Faculdade de Ciências da Universidade de Lisboa, Departamento de Matemática,
R. Ernesto de Vasconcelos, Bloco C1, 1700 Lisboa – PORTUGAL