PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 4 – 1994

WHEN DOES FINITE HOLOMORPHY IMPLY HOLOMORPHY?

Leopoldo Nachbin

Abstract: A study of the existence of discontinuous polynomials and its role in holomorphy.

1 – Introduction

Given a Hausdorff complex locally convex space E, any finitely holomorphic map $f : U \to F$ is holomorphic for every nonvoid open subset U of E and complex locally convex space F iff E is countably dimensional and carries its finest locally convex topology (Proposition 3). The proof of this result is based on the following one. Given a complex vector space E endowed with its finest locally convex topology, there is a discontinuous complex valued polynomial on E, equivalently a discontinuous complex valued homogeneous polynomial of degree two on E, iff the algebraic dimension of E is uncountable (Proposition 2). The point of this article is to prove these two results without using the continuum hypothesis, as they are known otherwise. We discuss these questions to the best of our knowledge. References [2] and [3] are standard about holomorphy.

2 – Existence of discontinuous polynomials

In Example 3, [5], we stated without proof that, if E is a complex vector space, whose algebraic dimension is at least equal to continuum power, endowed with its largest locally convex topology, there is a discontinuous homogeneous complex valued polynomial of degree two on E. This fact was used in connection with Lemma 1, [5], about holomorphy on countable inductive limits. That statement in

Received: April 9, 1993.

Example 3, [5], was given two proofs in Lemma 19, [1]. It got used in Example 20, [1], in connection with Lemma 11, [1], about holomorphy on countable inductive limits. The argument used in proving Lemma 19, [1], is quoted and used in [3], page 37.

Lemma 1. *Given a set B, for every function $r : B \times B \to \mathbb{R}_+$ there is a function $t : B \to \mathbb{R}_+$ such that*

$$r(b_1, b_2) \leq t(b_1)\, t(b_2) \tag{1}$$

for all $b_1, b_2 \in B$ iff B is countable.

Proof: Sufficiency is clear. We shall produce three proofs of necessity, that is, if B is uncountable there is a function $r : B \times B \to \mathbb{R}_+$ for which there is no function $t : B \to \mathbb{R}_+$ satisfying (1) for all $b_1, b_2 \in B$. The first proof is due to M.E. Rudin, [7].

Without loss of generality, B may be taken to be an uncountable subset of $\mathbb{R}$. Let

$$r(b_1, b_2) = \frac{1}{|b_2 - b_1|} \in \mathbb{R}_+$$

for $b_1, b_2 \in B$, $b_1 \neq b_2$. Arbitrarily define $r(b, b) \in \mathbb{R}_+$ for $b \in B$. Suppose a function t as described exists. For every integer $n \geq 0$, let

$$B_n = \Big\{ b \in B;\ t(b) \leq n \Big\} .$$

Fix n so that B_n is uncountable. There must be $b_1, b_2 \in B_n$, $b_1 \neq b_2$, such that

$$|b_2 - b_1| < \frac{1}{n^2}, \quad \text{hence} \quad r(b_1, b_2) > n^2 .$$

On the other hand, $r(b_1, b_2) \leq n^2$ by (1). So we have a contradiction. The second proof is due to K. Alster, [6]. Without loss of generality, B may be assumed to have as its power the first uncountable cardinal, and to be identified with the set of all countable ordinal numbers. We define $r : B \times B \to \mathbb{N}$ as follows. We put $r(b_1, b_2) = r_{b_1}(b_2)$ for $b_1, b_2 \in B$, where the functions $r_{b_1} : B \to \mathbb{N}$ for $b_1 \in B$ are defined as follows. If $b_1 \in B$ is finite, then r_{b_1} is an arbitrary function. If $b_1 \in B$ is infinite, then r_{b_1} restricted to the set $\{b \in B;\ b \leq b_1\}$ is defined to be any injective map from this set into $\mathbb{N}$, and restricted to the set $\{b \in B;\ b > b_1\}$ is defined arbitrarily. Suppose that there exists a function t as indicated. There is $n \in \mathbb{N}$ such that $t^{-1}(n)$ is an uncountable set. Thus there is an infinite $b_1 \in t^{-1}(n)$ such that $C = \{b \in t^{-1}(n);\ b \leq b_1\}$ is infinite. Hence $r_{b_1}(C)$ is an infinite subset of $\mathbb{N}$. Since $t(b) = n$ for all $b \in C$, we get a contradiction because

$$r_{b_1}(b) = r(b_1, b) \leq t(b_1)\, t(b) = t(b_1)\, n \quad \text{if} \ \ b \in C ,$$

hence $r_{b_1}(C)$ is a bounded subset of $\mathbb{N}$, but $r_{b_1}(C)$ is an infinite subset of $\mathbb{N}$. The third proof is due to T.J. Jech, [4]. More generally, if B is an uncountable set, there is a function $r\colon B \times B \to \mathbb{N}$ with the property that, for every function $t: B \to \mathbb{N}$, there are $b_1, b_2 \in B$ such that $r(b_1, b_2) > t(b_1), t(b_2)$. It suffices to prove this assertion for a set B of cardinality equal to the first uncountable cardinal. Let us assume that B is the set of all countable ordinal numbers. If $b_1 \in B$, let $W_{b_1} = \{b \in B;\, b \leq b_1\}$. Each W_{b_1} is countable. For each $b \in B$, let r_b be some injective map of W_a into $\mathbb{N}$. If $b_1, b_2 \in B$, we define $r(b_1, b_2) = r_{b_1}(b_2)$ whenever $b_1 > b_2$, and arbitrarily otherwise. We shall show that the function $r: B \times B \to \mathbb{N}$ satisfies the statement that we claimed. Let $t: B \to \mathbb{N}$ be arbitrary. Since B is uncountable, there is an uncountable subset C of B such that t is constant on C, with value n. Let $b_1 \in C$ be such that $W_1 \cap C$ is infinite. Since the values of $r(b_1, b_2)$ for $b_2 \in W_1 \cap C$ are all distinct, there is one such b_2 with $r(b_1, b_2) > n = t(b_1) = t(b_2)$. Once such a function $r\colon B \times B \to \mathbb{N}$ is found, replace r by r^2 to get a function $r^2\colon B \times B \to \mathbb{R}_+$ for which there is no function $t\colon B \to \mathbb{R}_+$ such that (1) holds for all $b_1, b_2 \in B$. If such a function t existed, we would find $b_1, b_2 \in B$ such that $r(b_1, b_2) > t(b_1), t(b_2)$, hence

$$t(b_1)\, t(b_2) < r(b_1, b_2) \leq t(b_1)\, t(b_2) \;,$$

a contradiction. ∎

Proposition 2. *Given a complex vector space E endowed with its finest locally convex topology, there is a discontinuous complex valued polynomial on E. Equivalently a discontinuous complex valued homogeneous polynomial of degree two exists on E, iff the algebraic dimension of E is uncountable.*

Proof: If the algebraic dimension of E is countable, every polynomial $p\colon E \to \mathbb{C}$ is continuous. In fact, a subset U of E is open iff the intersection of U with any finite dimensional vector subspace S of E is open in S. See [2], [3]. If the algebraic dimension of E is uncountable, let B be a basis for the vector space E. Hence B is uncountable. Find a function $r: B \times B \to \mathbb{R}_+$ for which there is no function $t\colon B \to \mathbb{R}_+$ satisfying (1) for all $b_1, b_2 \in B$. Then proceed as in the first proof of Lemma 19, [1]. ∎

3 – Finite holomorphy implies holomorphy

If E and F are complex locally convex spaces, and $U \subset E$ is open, then $f\colon U \to F$ is said to be finitely holomorphic when the restriction of f to $U \cap S$ is holomorphic for every finite dimensional vector subspace S of E.

Proposition 3. *Given a Hausdorff complex locally convex space E, any finitely holomorphic map $f\colon U \to F$ is holomorphic for every open subset U of E and complex locally convex space F iff E is countably dimensional and carries its finest locally convex topology.*

Proof: Necessity is seen as follows. Take $U = E$ and $F = E$ with its largest locally convex topology. The identity map of E is finitely holomorphic, hence holomorphic. It follows that the given topology on E has to be its finest locally convex topology. The dimension of E has to be countable. Otherwise, we could use Proposition 2 and find a discontinuous complex valued homogeneous polynomial of degree two on E. It would be a complex valued finitely holomorphic function on E failing to be holomorphic. Sufficiency is seen as follows. A subset U of E is open iff the intersection of U with any finite dimensional vector subspace S of E is open in S. See [2], [3]. If $f\colon U \to S$ is finitely holomorphic, then f is continuous on every such $U \cap S$. It follows that f is continuous on U. Hence f is holomorphic. ∎

BIBLIOGRAPHY

[1] Barroso, J.A., Matos, M.C. and Nachbin, L. – *On holomorphy versus linearity in classifying locally convex spaces*, in "Infinite Dimensional Holomorphy and Applications" (M.C. Matos, Ed.), North-Holland 1977, 31–74.

[2] Dineen, S. – *Complex Analysis in Locally Convex Spaces*, North-Holland, 1981.

[3] Hervé, M. – *Analyticity in Infinite Dimensional Spaces*, de Gruyter, 1989.

[4] Jech, T.J. – On a problem of L. Nachbin, *Proceedings of the American Mathematical Society*, 79 (1980), 341–342.

[5] Nachbin, L. – *Some holomorphically significant properties of locally convex spaces*, in "Functional Analysis" (D.G. de Figueiredo, Ed.), Marcel Dekker, 1976, 251–275.

[6] Pelczynski, A. – *Letter to L. Nachbin of February 7, 1979.*

[7] Rudin, M.E. – *Letter to L. Nachbin of January 6, 1979.*

Leopoldo Nachbin,
Department of Mathematics, University of Rochester,
Rochester NY 14627 – USA
and
Centro Brasileiro de Pesquisas Físicas,
Rua Xavier Sigaud, 150, 22290-160 Rio de Janeiro RJ – BRASIL