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CONVERGENCE IN SPACES OF RAPIDLY INCREASING DISTRIBUTIONS

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Abstract: In this note we show that if (T_j) is a sequence in K'_M, the space of distributions of rapid growth (resp. O'_c the space of its convolution operators), and $(T_j \star \phi)$ converges to 0 in K'_M (resp. in O'_c) for all ϕ in K_m, then (T_j) converges to 0 in K'_M (resp. O'_c). Moreover, if (ψ_j) is in O_c such that $(\psi_j \star \phi)$ converges to 0 in O_c for every ϕ in K_M, then (ψ_j) converges to 0 in O_c. This is no more true if the sequence (ψ_j) is in K_M.

1 – Introduction

When one considers the convolution of elements from K'_M (the space of distributions of rapid growth) with elements from K_M (the space of C^∞ functions which are very rapidly decreasing at infinity), it follows trivially that if (T_j) is any sequence which converges to 0 in K'_M, then the sequence $(T_j \star \phi)$ converges to 0 for every ϕ in K_M. Moreover, if (T_j) is a sequence in O'_c (the space of convolution operators in K'_M), and $T_j \to 0$ in O'_c, then $T_j \star \phi \to 0$ in K_M for every ϕ in K_M. In this note we consider the following questions: given $(T_j) \subset K'_M$ such that $T_j \star \phi \to 0$ in K'_M for every $\phi \in K_M$, does it follow that $T_j \to 0$ in K'_M? Similarly, if $(T_j) \subset O'_C$ and $T_j \star \phi \to 0$ in K_M for every $\phi \in K_M$, does it follow that $T_j \to 0$ in O'_C? In both cases we show that the answer is affirmative. Similar questions have been considered by K. Keller [5] for the space S' of tempered distributions, our methods of proof are different from those of Keller, and they work if we replace K_M by any complete metric space of test functions. Finally we consider these questions of convergence for sequences of functions in O_c and K_M.

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By D, E, D' and E' we denote Schwartz spaces of test functions and distributions, N^n consists of all n-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in N$, and the differential operator D^α, $\alpha \in N^n$, denotes $\left(-i\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(-i\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. Let $M(x)$, $x \geq 0$, be a function which is continuous, increasing and convex with $M(0) = 0$, $M(\infty) = \infty$. For $x < 0$ define $M(x)$ to be $M(-x)$, and for $x = (x_1, x_2, \ldots, x_n)$, we define $M(x) = M(x_1) + M(x_2) + \ldots + M(x_n)$. Examples of such function are $M(x) = \frac{x^p}{p}$, $p > 1$, and $M(x) = e^x$.

For a function M as above, we define the space K_M to be the space of all infinitely differentiable functions ϕ on R^n such that

$$\nu_k(\phi) = \sup_{\substack{x \in R^n \\ |\alpha| \leq k}} e^{M(kx)} |D^\alpha \phi(x)| < \infty \,, \qquad \alpha \in N^n, \quad k = 0, 1, 2, \ldots$$

The space K_M is provided with the topology generated by the semi-norm ν_k, $k = 0, 1, 2,\ldots$ It follows that K_M is a Frechet Montel space. Moreover, it is a normal space of distributions. By K'_M we denote the space of all continuous linear functionals on K_M provided with the strong dual topology. By O'_c we denote the subspace of K'_M consisting of all $S \in K'_M$ such that for every ϕ in K_M the convolution $S \star \phi$ is in K_M, and the map $\phi \to S \star \phi$ from K_M into itself is continuous. O'_c is the space of convolution operators on K'_M, and will be provided with the topology of uniform convergence on bounded subsets of K_M. The space O_c consists of all c^∞-functions such that $D^\alpha f(x) = O(e^{M(kx)})$ for all $\alpha \in N^n$, and some positive integer k independent of α. It turns out that O_c is the strong dual of O'_c, we provide it with the strong dual topology. Another equivalent topology is τ_b of uniform convergence on bounded subset of K_M (see [2] and [3]).

We denote by $V(K_M \star K_M)$ the subspace of K_M generated by the elements of $K_M \star K_M$, and we provide it with the relative topology inherited from K_M. In particular $V(K_M \star K_M)$ is metrizable.

2 – The results

Lemma 1. *$V(K_M \star K_M)$ is dense in K_M.*

Proof: Let ψ be any element of K_M, let $(\phi_\varepsilon;\ \varepsilon > 0)$ be a sequence in D converging to δ in E'. Since the convolution map Λ_ψ from O'_c into K_M which maps S to $S \star \psi$ is continuous, and $\{\phi_\varepsilon \colon \varepsilon > 0\}$ is bounded in E' which is continuously embedded in O'_c it follows that the sequence $(\phi \star \psi;\ \varepsilon > 0)$ converges to ψ in K_M. ∎

Lemma 2. *The space $V(K_M \star K_M)$ is Montel.*

Proof: We show first that every bounded subset of $V(K_M \star K_M)$ is relatively compact. Let U be a bounded subset of $V(K_M \star K_M)$, by $C\ell V(U)$ and $C\ell K_M(U)$ we denote the closures of U in V and K_M respectively. One has $C\ell V(U) = V \cap C\ell K_M(U)$. Let $\{O_j,\, j = 1, 2, \ldots\}$ be an open cover of $C\ell V(U)$ in $V(K_M \star K_M)$, then $O_j = V \cap G_j$, where the G_j's, $j = 1, 2, \ldots$, are open subsets of K_M, and one has

$$V \cap C\ell\, K_M(U) = C\ell\, V(U) \subset \bigcup_{j=1}^{\infty} O_J = \bigcup_{j=1}^{\infty} (V \cap G_J) = V \cap (\bigcup_{j=1}^{\infty} G_j) \ .$$

Since $V(K_M \star K_M)$ is dense in K_M it follows that $C\ell K_M(U)$ is contained in $\cup_{j=1}^{\infty} G_j$. Since K_M is Montel it follows that there exists a finite set of indices $j_1, j_2, \ldots, j_m$ such that $C\ell K_M(U) \subset \cup_{i=1}^{m} G_{j_i}$. Hence

$$C\ell\, V(U) = V \cap C\ell\, K_M(U) \subset V \cap \Big(\bigcup_{i=1}^{m} G_{j_i}\Big) = \bigcup_{i=1}^{m} (V \cap G_{j_i}) \ ,$$

i.e. $C\ell V(U)$ is compact in $V(K_M \star K_M)$.

Finally we show that $V(K_M \star K_M)$ is barreled. Let F be a barrel in $V(K_M \star K_M)$, F is a closed, absorbing, balanced and convex subset of V. We show that F is a neighborhood of 0 in V. Let $F_M = C\ell K_M(F)$. It is clear that $F = V \cap F_M$. We claim that F_M is a barrel in K_M. First we show that it is absorbing. Let $\phi \in K_M$, $\phi \notin F_M$. Since $V(K_M \star K_M)$ is dense in K_M it follows that there exists a sequence $(\phi_j) \subset V$, such that $\phi_j \to \phi$ in K_M. Since F is absorbing subset of V there exists a sequence $(\lambda_j) \subset R$, $\lambda_j > 0$ such that $\lambda_j \phi_j \in F$ for all $j = 1, 2, \ldots$. Without loss of generality we can assume that $0 < \lambda_j \leq 1$. Thus the sequence $(\lambda_j \phi_j)$ is bounded in F_M, hence it has a convergent subsequence, call it also $(\lambda_j \phi_j)$, $\lambda_j \phi_j \to \psi$ in F_M. We can assume also that $\lambda_j \to \lambda$ in R. Let ρ be the metric on K_M. Given any $\varepsilon > 0$, it follows that for j large enough

$$\begin{aligned}
\rho(\lambda_j \phi_j - \lambda \phi) &\leq \rho(\lambda_j \phi_j - \lambda_j \phi) + \rho(\lambda_j \phi - \lambda \phi) \\
&\leq \lambda_j\, \rho(\phi_j - \phi) + |\lambda_j - \lambda|\, \rho(\phi) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .
\end{aligned}$$

Thus $(\lambda_j \phi_j)$ converges to $\lambda \phi$ in K_M. Hence $\lambda \phi = \psi$, and $\lambda \phi \in F_M$, i.e. F_M is absorbing.

Next, we show that F_M is convex. Let ϕ_1, ϕ_2 be in F_M, α real number, $0 \leq \alpha \leq 1$, we show that $\alpha \phi_1 + (1 - \alpha) \phi_2 \in F_M$. We will consider the general case that ϕ_1 ane ϕ_2 are not in F. Since $V(K_M \star K_M)$ is dense in K_M it follows that there exist sequences (ϕ_{j_1}), (ϕ_{j_2}) of functions in V such that $\phi_{j_1} \to \phi_1$ and $\phi_{j_2} \to \phi_2$ in K_M. Since F is absorbing there exist sequences of positive real numbers (k_{j_1}), (k_{j_2}) such that $\{k_{j_1}\phi_{j_1}\}$ and $\{k_{j_2}\phi_{j_2}\}$ are contained in F. Since F is convex it follows that $0 < k_{j_1} \leq 1$ and $0 < k_{j_2} \leq 1$. Let

$$\lambda_{j_1} = \sup\{k_{j_1} \colon\ k_{j_1}\phi_{j_1} \in F\} - \frac{1}{j}\ ,$$

$$\lambda_{j_2} = \sup\{k_{j_2} \colon\ k_{j_2}\phi_{j_2} \in F\} - \frac{1}{j}\ .$$

For each $j = 1, 2, 3, \ldots$, $i = 1, 2$, one has $0 < k_{j_i} - \frac{1}{j} < k_{j_i} \leq 1$, and

$$\left(k_{j_i} - \frac{1}{j}\right)\phi_{j_i} = \left(\frac{k_{j_i} - \frac{1}{j}}{k_{j_i}}\right) \cdot k_{j_i}\,\phi_{j_i} + \left(1 - \frac{k_{j_i} - \frac{1}{j}}{k_{j_i}}\right) \cdot 0$$

is in F. Hence $(\lambda_{j_i}\phi_{j_i}) \subset F$, $i = 1, 2$. Since $C\ell K_M(F) = F_M$, it follows that

$$\lim_{j\to\infty} \sup\{k_{j_i} \colon\ k_{j_i}\phi_{j_i} \in F\} = 1\ ,$$

and $\lambda = \lim_{j\to\infty} \lambda_{j_i} = 1$, $i = 1, 2$. Thus $\lambda_{j_1}\phi_{j_1} \to \phi_1$ and $\lambda_{j_2}\phi_{j_2} \to \phi_2$ in K_M as $j \to \infty$. Hence

$$\alpha\,\phi_1 + (1-\alpha)\,\phi_2 = \lim_{\substack{j_1\to\infty \\ j_2\to\infty}} \left[\alpha\,\lambda_{j_1}\,\phi_{j_1} + (1-\alpha)\,\lambda_{j_2}\,\phi_{j_2}\right] .$$

Since for each j_1, j_2 the term in the bracket is in F (by convexity), it is in F_M. Since F_M is closed it follows that $\alpha\phi_1 + (1-\alpha)\phi_2$ is in F_M, i.e. F_M is convex.

Finally we show that F_M is balanced. Let $\phi \in F_M$, $\alpha \in R$, $|\alpha| \leq 1$. If $\phi \in F$ there is nothing to prove. Otherwise, as in the proof of convexity, there exist sequences (ϕ_j), (λ_j) in V and R respectively, such that for all $j = 1, 2, \ldots$, $\lambda_j\phi_j \in F$, $\phi_j\lambda_j \to \phi$ as $j \to \infty$. Since F is balanced one has $\alpha\lambda_j\phi_J \in F$, and since F_M is closed it follows that $\alpha\phi = \lim_{j\to\infty} \alpha\lambda_j\phi_j$ is in F_M, i.e. F_M is balanced.

Thus F_M is a neighborhood of 0 in K_M because K_M is Montel. Hence $F = V \cap F_M$ is a neighborhood of 0 in $V(K_M \star K_M)$. ∎

From the definition of $V(K_M \star K_M)$ and its topology it follows that K'_M is contained in $(V(K_M \star K_M))'$. Now we give the main result of this paper.

Theorem 1. *Let (T_j) be a sequence in K'_M such that for every ϕ in K_M the sequence $(T_J \star \phi)$ converges to 0 in K'_M, then (T_j) converges to 0 in K'_M.*

Proof: Since K'_M is the strong dual of the Montel space K_M it suffices to show that (T_j) converges to 0 weakly in K'_M. Let $\phi \in K_M$, we show that $\langle T_j, \phi\rangle \to 0$. Let $(\phi_\varepsilon;\ \varepsilon > 0)$ be the sequence as in the proof of Lemma 1,

$\phi_\varepsilon \star \phi \to \phi$ in K_M as $\varepsilon \to 0$. Moreover, the set $\{\phi_\varepsilon \star \phi \colon \varepsilon > 0\}$ is bounded in $V(K_M \star K_M)$. Thus

$$\text{(I)} \qquad \lim_{j\to\infty} \langle T_j, \phi\rangle = \lim_{j\to\infty} \lim_{\varepsilon\to 0} \langle T_j, \phi_\varepsilon \star \phi\rangle \ .$$

Since $T_j \star \check{\phi} \to 0$ in K'_M, and the bilinear map $(T, \psi) \to T \star \psi$ from $K'_M \times K_M \to O_c$ is continous in each variable (see [2]), it follows that $(T_j \star \phi) \star \psi \to 0$ in O_c as $j \to \infty$. Hence

$$\lim_{j\to\infty} \langle T_j, \phi \star \psi\rangle = \lim_{j\to\infty} \langle T_j \star \check{\phi}, \psi\rangle = \lim_{j\to\infty} \Big((T_j \star \check{\phi}) \star \psi\Big)(0) = 0 \ .$$

Thus (T_j) converges weakly to O in $(V(K_M \star K_M))'$. Since $V(K_M \star K_M)$ is Montel by Lemma 2, it follows that (T_j) converges strongly to O in $(V(K_M \star K_M))'$, i.e. it converges uniformly on bounded subsets of $V(K_M \star K_M)$. Since $\{\phi_\varepsilon \star \phi;\ \varepsilon > 0\}$ is bounded in $V(K_M \star K_M)$ it follows that $\lim_{j\to\infty} \langle T_j, \phi \star \phi_\varepsilon\rangle = 0$ uniformly in ε. Thus we can interchange the limits on the right hand side of (I), and one gets,

$$\begin{aligned} \lim_{j\to\infty} \langle T_j, \phi\rangle &= \lim_{j\to\infty} \lim_{\varepsilon\to 0} \langle T_j, \phi \star \phi_\varepsilon\rangle \\ &= \lim_{\varepsilon\to 0} \lim_{j\to\infty} \langle T_j, \phi \star \phi_\varepsilon\rangle = 0 \ . \end{aligned}$$

This completes the proof of the theorem. ∎

Next, we consider the same question of convergence in O'_c. In this direction we have:

Theorem 2. *Let (T_j) be a sequence in O'_c such that, for every ϕ in K_M the sequence $(T_j \star \phi)$ converges to 0 in O'_c, then (T_j) converges to 0 in O'_c.*

Proof: It is clear that $T_j \star \phi \in O'_c$ for every T_j in O'_c and ϕ in K_M. Let T be any element in K'_M, we claim that $T_j \star T \to 0$ in K'_M. For given ϕ in K_M, one has

$$(T_j \star T) \star \phi = (T_j \star \phi) \star T \to 0 \quad \text{ in } \ K'_M \ .$$

From Theorem 1 it follows that $T_j \star T \to 0$ in K'_M. Let B be a bounded subset of K_M, then for any $T \in K'_M$ one has

$$\text{(II)} \qquad \langle T_j \star \phi, T\rangle = \langle \check{T}_j \star T, \phi\rangle \to 0 \quad \text{ uniformly in } \ \phi \in B \ .$$

Since K_M is reflexive and K'_M is Montel (being the strong dual of a Montel space). (II) implies that $T_j \star \phi \to 0$ in K_M, uniformly in $\phi \in B$. This complete the proof of the theorem. ∎

Corollary. *Let (T_j) be a sequence in O'_c such that for any $\phi \in K_M$, $(T_j \star \phi)$ converges to 0 in K_M, then (T_j) converges to 0 in O'_c.*

As in the case of the space K'_1 of distribution of exponential growth, it is possible to extend the definition of Fourier transform of distributions of compact support to the elements of O'_c. It turns out that for $S \in O'_c$, its Fourier transform $\hat{S}$ could be extended to $\mathbb{C}^n$ as an entire function, which satisfies a Paley–Wiener type theorem, see Pahk [6] (the theorem was quoted and used in [1]). In [8], Zielezny proved that the space O'_c $(K'_1 : K'_1)$ is bornologic. A simple modification of the proof of Theorem 9 of [8] shows that O'_c is bornologic. Since O'_c is the projective limit of the Montel spaces $w^{-k}S'$, and the topology of $w^{-k}S'$ is finer than the topology of $w^{-j}S'$ for $k \geq j$, it follows from the Corollary to Proposition 3.9.6 of Horvath [4] that O'_c is semi-Montel. Thus O'_c is Montel. Hence its strong dual O_c is Montel. As in Lemma 2, one can show that K_M as a subspace of O'_c with the relative topology of O'_c is Montel. Following the idea of the proof of Theorem 1, we can prove the following.

Theorem 3. *Let (ψ_j) be a sequence in O_c such that $(\psi_j \star \phi)$ converges to 0 in O_c for every ϕ in K_M, then (ψ_j) converges to 0 in O_c.*

The last result of this note is of negative nature, it simply says that in Theorem 1, one can not replace K'_M by K_M. More precisely we have

Theorem 4. *There exist a sequence (ψ_j) in K_M and ϕ in K_M, where the map $\phi \star \psi \to \psi$ from $\phi \star K_M$ to K_M is well-defined, such that $(\phi \star \psi_j)$ converges to 0 in K_M but (ψ_j) does not converge to 0 in K_M.*

Proof: Assume the contrary, since for given ϕ in K_M the space $\phi \star K_M$ with the relative topology of K_M is metric, it follows that the linear map Λ from $\phi \star K_M$ into K_M which takes $\phi \star \psi$ is continuous. We claim that $\check{\phi} \star K_M = K'_M$. Indeed, given T in K'_M, let S be the Hahn-Banach extension to K_M of $T \circ \Lambda$ from $\phi \star K_M$ into $\mathbb{C}$. S is in K'_M and $\check{\phi} \star S = T$. But on the other hand the equality of $\check{\phi} \star K_M$ and K'_M is impossible, because $\phi \star S$ is infinitely differentiable for all S in K'_M and can never be equal to δ. The contradiction completes the proof of the theorem. ∎

Remarks.

(1) It will be nice to have a concrete example of a sequence (ψ_j) and a function ϕ in K_M which satisfy the conditions of the above result.

(2) Theorem 4 and its proof remain valid if the sequence (ψ_j) is in ε and ϕ is in D.

Added in proof. In a recent article Stevan Pilipovic (Proceedings of the AMS, Vol. 111, N° 4, April 1991) has shown that, if (T_j) is a sequence in S' such that $(T_j \star \phi)$ converges to 0 in S' for any ϕ in D, then (T_j) converges to 0 in S'. In his proof he followed the method of Keller [5].

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