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# WEIGHTED FRECHET AND LB-SPACES OF MOSCATELLI TYPE

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**Abstract:** The structure of the weighted Fréchet and LB-spaces of Moscatelli type appears when one combines both the structure of the Köthe sequence spaces [3] and the structure of Fréchet and LB-spaces of Moscatelli type, introduced by Moscatelli in 1980 [11] and developed by Bonet and Dierolf in [4, 5]. The theory of this new structure includes both theories.

The main motivation for our research on these spaces are the questions which remain open in the theory of LB-spaces. The most important one is the question posed by Grothendieck [8] asking whether every regular LB-space is complete. This question is answered positively in our present frame here.

This paper is divided into three sections. In the first section we introduce the weighted LB-spaces of Moscatelli type and study strictness, regularity and bounded retractivity. We also prove that these inductive limits are regular if and only if they are complete (under mild additional assumptions). In the second section we define the weighted Fréchet spaces of Moscatelli type and investigate when they are Montel, Schwartz and when they satisfy property $(\Omega_\varphi)$ or property $(DN_\varphi)$ of Vogt. In our third and last section we establish a certain duality between the weighted Fréchet and LB-spaces of Moscatelli type.

## 1 – Weighted LB-spaces of Moscatelli type

### 1.1 Definition and preliminaries

In what follows, $(L, \|\ \|)$ will denote a normal Banach sequence space, i.e., a Banach sequence space which satisfies:

$(\alpha)$ $\varphi \subset L \subset \omega$ algebraically and the inclusion $(L, \|\ \|) \to \omega$ is continuous, where $\omega = \prod_{k\in\mathbb{N}} \mathbb{K}$ and $\varphi = \bigoplus_{k\in\mathbb{N}} \mathbb{K}$.

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($\beta$) $\forall\, a = (a_k)_{k\in\mathbb{N}} \in L$, $\forall\, b = (b_k)_{k\in\mathbb{N}} \in \omega$ such that $|b_k| \le |a_k|$, $\forall\, k \in \mathbb{N}$, we have $b \in L$ and $\|b\| \le \|a\|$.

Clearly every projection onto the first $n$ coordinates $p_n\colon \omega \to \omega$, $(a_k)_{k\in\mathbb{N}} \to ((a_k)_{k\le n}, (0)_{k>n})$ induces a norm-decreasing endomorphism on $L$.

We shall also consider on $(L, \|\ \|)$ the following properties:

($\gamma$) $\|a\| = \lim_n \|p_n(a)\| \quad \forall\, a \in L$.

($\delta$) If $a \in \omega$, $\sup_n \|p_n(a)\| < \infty$, then $a \in L$ and $\|a\| = \lim_n \|p_n(a)\|$.

($\varepsilon$) $\lim_n \|a - p_n(a)\| = 0 \quad \forall\, a \in L$.

Unexplained notation as in [9, 12].

Following the classical notations (see [3]), given a strictly positive Köthe matrix $A = (a_n)_{n\in\mathbb{N}}$ on $\mathbb{N}$, that is $0 < a_n(k) \le a_{n+1}(k)$ $(n, k \in \mathbb{N})$, we shall denote by $V = (v_n)_{n\in\mathbb{N}}$ the associated decreasing sequence of strictly positive weights with $v_n = \frac{1}{a_n}$ $(n \in \mathbb{N})$ and by $\overline{V}$ the family

$$\overline{V} := \left\{ \overline{v} = (\overline{v}(k)) \in \omega\colon \sup_{k\in\mathbb{N}} \frac{\overline{v}(k)}{v_n(k)} < \infty, \ \ \forall\, n \in \mathbb{N} \right\} .$$

We shall always assume without loss of generality that for every $\overline{v} \in \overline{V}$, we have $\overline{v}(k) > 0$ $(k \in \mathbb{N})$.

Let $(L, \|\ \|)$ be a normal Banach sequence space, $(Y_k, s_k)_{k\in\mathbb{N}}$ a sequence of Banach spaces and $V = (v_n)_{n\in\mathbb{N}}$ a decreasing sequence of strictly positive weights. For each $n \in \mathbb{N}$ we put

$$L(v_n, (Y_k)_{k\in\mathbb{N}}) := \Big\{ (x_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} Y_k\colon \ (v_n(k)\, s_k(x_k))_{k\in\mathbb{N}} \in L \Big\}$$

endowed with the norm $\|(x_k)_{k\in\mathbb{N}}\| := \|(v_n(k)\, s_k(x_k))_{k\in\mathbb{N}}\|$ and $k(V, L, (Y_k)_{k\in\mathbb{N}}\} := \operatorname{ind}_n L(v_n, (Y_k)_{k\in\mathbb{N}})$.

For every $\overline{v} \in \overline{V}$, we define

$$L(\overline{v}, (Y_k)_{k\in\mathbb{N}}) = \Big\{ (x_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} Y_k\colon \ (\overline{v}(k)\, s_k(x_k))_{k\in\mathbb{N}} \in L \Big\}$$

endowed with the norm $\|(x_k)_{k\in\mathbb{N}}\| := \|(\overline{v}(k)\, s_k(x_k))_{k\in\mathbb{N}}\|$.

The inclusion

$$L(v_n, (Y_k)_{k\in\mathbb{N}}) \subset L(\overline{v}, (Y_k)_{k\in\mathbb{N}})$$

is continuous for arbitrary $\overline{v} \in \overline{V}$ and $n \in \mathbb{N}$, so that, if

$$K(\overline{V}, L, (Y_k)_{k\in\mathbb{N}}) := \operatorname{proj}_{\overline{v}\in\overline{V}} L(\overline{v}, (Y_k)) \ ,$$

then $k(V,L,(Y_k)_{k\in\mathbb{N}})$ is continously injected in $K(\overline{V},L,(Y_k)_{k\in\mathbb{N}})$. For the case $Y_k = \mathbb{K}$ $(k\in\mathbb{N})$ we shall omit $(Y_k)_{k\in\mathbb{N}}$ and write $L(v_n)$, $k(V,L)$, $L(\overline{v})$ and $K(\overline{V},L)$, following the classical notations.

Clearly, for the case $Y_k = \mathbb{K}$ $(k\in\mathbb{N})$ and $L = 1_p$, the space $k(V,L,(Y_k)_{k\in\mathbb{N}})$ is the corresponding scalar Köthe co-echelon space and $K(\overline{V},L,(Y_k)_{k\in\mathbb{N}})$ coincides with its well-known projective hull (see [3]). In fact the idea for our inclusion is taken from the one in [3] for this particular case.

Moreover it is easy to see that $k(V,L,(Y_k)_{k\in\mathbb{N}})$ and $K(\overline{V},L,(Y_k)_{k\in\mathbb{N}})$ induce on $\bigoplus_{k\in\mathbb{N}} Y_k$ the same topology. In particular, if $L$ satisfies property $(\varepsilon)$, then $k(V,L,(Y_k)_{k\in\mathbb{N}})$ is a topological subspace of $K(\overline{V},L,(Y_k)_{k\in\mathbb{N}})$ (compare with [3] and 2.4 in [5]).

In order to define the weighted LB-spaces of Moscatelli type, we would like to make the following three conventions for this first section:

- $(L,\|\ \|)$ will denote a normal Banach sequence space with property $(\gamma)$.
- $V = (v_n)_{n\in\mathbb{N}}$ will stand for a decreasing sequence of strictly positive weights.
- $(X_k,r_k)_{k\in\mathbb{N}}$ and $(Y_k,s_k)_{k\in\mathbb{N}}$ will represent two sequences of Banach spaces such that for each $k\in\mathbb{N}$, $Y_k$ is a subspace of $X_k$ and $s_k \geq r_k \mid Y_k$ (in consequence, $B_k := \{y\in Y_k:\ s_k(y)\leq 1\} \subset \{x\in X_k:\ r_k(x)\leq 1\} =: A_k$).

For every $n\in\mathbb{N}$, the space $L(v_n,(X_k,r_k)_{k<n},(Y_k,s_k)_{k\geq n})$ is a Banach space, the inclusion

$$L\Big(v_n,(X_k)_{k<n},(Y_k)_{k\geq n}\Big) \to L\Big(v_{n+1},(X_k)_{k<n+1},(Y_k)_{k\geq n+1}\Big)$$

is continuous and the unit ball of the first space – which we shall always denote by $\mathcal{B}_n$ – is contained in the unit ball of the second one, $\mathcal{B}_{n+1}$. Now the inductive limit

$$k\Big(V,L,(X_k),(Y_k)\Big) := \mathrm{ind}_n\, L\Big(v_n,(X_k)_{k<n},(Y_k)_{k\geq n}\Big)$$

is the *LB-space of Moscatelli type* w.r.t. $(L,\|\ \|)$, $V=(v_n)_{n\in\mathbb{N}}$, $(X_k,r_k)_{k\in\mathbb{N}}$ and $(Y_k,s_k)_{k\in\mathbb{N}}$.

Recall that if $X_k = Y_k = \mathbb{K}$ and $L = 1_p$ we obtain the Köthe co-echelon spaces [3] and for the case $v_n(k) = 1$ $(k,n\in\mathbb{N})$ we get the LB-spaces of Moscatelli type [4].

Clearly $k(V,L,(X_k),(Y_k))$ is a quotient of $\bigoplus_{k\in\mathbb{N}} X_k \times k(V,L,(Y_k)_{k\in\mathbb{N}})$. Therefore there is a basis of 0-neighbourhoods of the form $\bigoplus_{k\in\mathbb{N}} \varepsilon_k A_k + U$, where $(\varepsilon_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers and $U$ is a 0-neighbourhood in $k(V,L,(X_k),(Y_k))$, is given by $\bigoplus_{k\in\mathbb{N}} \varepsilon_k A_k + \mathcal{B}_{\overline{v}} \cap k(V,L,(Y_k)_{k\in\mathbb{N}})$ with $\overline{v}\in\overline{V}$, where $\mathcal{B}_{\overline{v}}$ stands for the unit ball in $L(\overline{v},(Y_k)_{k\in\mathbb{N}})$.

We introduce the following auxiliary spaces.

For each $k \in \mathbb{N}$ let $C_k$ denote the closure of $B_k$ in $(X_k, r_k)$, $Z_k$ its linear span and $t_k$ the Minkowski functional of $C_k$ (we will keep these notations through all section 1). The space $(Z_k, t_k)$ is a Banach space. Obviously $s_k \geq t_{k|Y_k} \geq r_{k|Y_k}$, hence $k(V, L, (X_k), (Y_k))$ is continuously injected in $k(V, L, (X_k), (Z_k))$.

Now given $(\varepsilon_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, $\varepsilon_k > 0$ $(k \in \mathbb{N})$ and $\overline{v} \in \overline{V}$, $p_{\varepsilon_k,\overline{v}}$ will denote the Minkowski functional of $\varepsilon_k A_k + \frac{1}{\overline{v}(k)} B_k$. Then $p_{\varepsilon_k,\overline{v}}$ is a norm on $X_k$ which is equivalent to $r_k$. Therefore $(X_k, p_{\varepsilon_k,\overline{v}})$ is a Banach space. Since $\sup_k \frac{\overline{v}(k)}{v_n(k)} < \infty$ $(n \in \mathbb{N})$, the spaces $L(v_n, (X_k)_{k\geq n}, (Y_k)_{k\geq n})$ and $L(v_n, (X_k)_{k<n}, (Z_k)_{k\geq n})$ are both continuously injected in $L((X_k, p_{\varepsilon_k,\overline{v}})_{k\in\mathbb{N}})$, for arbitrary $(\varepsilon_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, $\varepsilon_k > 0$ $(k \in \mathbb{N})$, $\overline{v} \in \overline{V}$ and $n \in \mathbb{N}$. Consequently, the inclusions:

$$\begin{aligned} k\Big(V, L, (X_k), (Y_k)\Big) &\subset k\Big(V, L, (X_k), (Z_k)\Big) \\ &\subset \operatorname{proj} L(X_k, p_{\varepsilon_k,\overline{v}})_{k\in\mathbb{N}} =: K\Big(\overline{V}, L, (X_k), (Y_k)\Big) \end{aligned}$$

are continuous. For the case $(Y_k, s_k) = (X_k, r_k)$ $(k \in \mathbb{N})$, the space $K(\overline{V}, L, (X_k), (Y_k))$ coincides with $K(\overline{V}, L, (Y_k)_{k\in\mathbb{N}})$ algebraically and topologically.

Furthermore, $K(\overline{V}, L, (X_k), (Y_k))$ has a basis of 0-neighbourhoods formed by the sets

$$\Big(\prod_{k\in\mathbb{N}} \varepsilon_k A_k + \delta \mathcal{B}_{\overline{v}}\Big) \cap K\Big(\overline{V}, L, (X_k), (Y_k)\Big), \qquad (\varepsilon_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}, \quad \varepsilon_k > 0 \quad \delta > 0\ ,$$

and $\overline{v} \in \overline{V}$ where $\mathcal{B}_{\overline{v}}$ stands for the unit ball in $L(\overline{v}, (Y_k)_{k\in\mathbb{N}})$. Moreover $k(V, L, (X_k), (Y_k))$ and $K(\overline{V}, L, (X_k), (Y_k))$ induce the same topology on $\bigoplus_{k\in\mathbb{N}} X_k$.

**1.2 Proposition.** *Let $(Z_k, t_k)_{k\in\mathbb{N}}$ be as in 1.1 and $k(V, L, (X_k), (Y_k))$, $k(V, L, (X_k), (Z_k))$ the corresponding weighted LB-spaces of Moscatelli type. The following assertions hold:*

- **i)** *$k(V, L, (X_k), (Y_k)) = k(V, L, (X_k), (Z_k))$ if and only if for every $n \in \mathbb{N}$ there are $m \geq n$, $M_n > 0$ with $\frac{1}{v_n(k)} C_k \subset \frac{M_n}{v_m(k)} B_k$ $(k \geq m)$.*
- **ii)** *If $k(V, L, (X_k), (Y_k))$ is regular, then $k(V, L, (X_k), (Y_k)) = k(V, L, (X_k), (Z_k))$ and $\operatorname{ind}_n L(v_n)$ is regular.*

**1.3 Remarks.**

**i)** The converse of 1.2 ii) holds whenever any of the following conditions is satisfied:

- **a)** each subset in $k(V, L) = \operatorname{ind}_n L(v_n)$ which is bounded for the relative $K(\overline{V}, L)$-topology is also bounded in $k(V, L)$ (see [3]);
- **b)** there is $m \in \mathbb{N}$ such that $Y_k$ is a topological subspace of $X_k$ $(k \geq m)$.

**ii**) Condition 2.4 ii) in [5] implies our condition 1.2 i) and they are different (of course they coincide if $v_n(k) = 1$, $(k \in \mathbb{N})$). Indeed, from 2.6 in [5], given an infinite dimensional Banach space $(X, r)$, for each $m \in \mathbb{N}$, there is a bounded Banach disc in $(X, r)$, $D_m$, whose closure is contained in $MD_m$, for some $M \geq 1$ but not contained in $mD_m$. According to this, we can obtain a sequence of bounded Banach discs, $(B_k)_{k\in\mathbb{N}}$, and a strictly increasing sequence of natural numbers, $(n_k)_{k\in\mathbb{N}}$, such that $\overline{B}_k \subset n_k B_k$ and $\overline{B}_{k+1} \not\subset n_k B_{k+1}$. Put $Y_k := \mathrm{LIN}(B_k)$ (linear span of $B_k$), $s_k$ the Minkowski functional of $B_k$, and take $(X, r)$, $(Y_k, s_k)_{k\in\mathbb{N}}$ and $V = (v_m)_{m\in\mathbb{N}}$ with $v_m(k) = \left(\frac{1}{n_k}\right)^m$, $m, k \in \mathbb{N}$. For each $k \geq m+1$,

$$\frac{1}{v_m(k)}\,\overline{B_k}^x = (n_k)^m\,\overline{B_k}^x \subset (n_k)^{m+1} B_k = \frac{1}{v_{m+1}}\,B_k\ .$$

It follows from 1.2 i) that

$$k\Big(V, L, (X, r), (Y_k)\Big) = k\Big(V, L, (X, r), (Z_k)\Big)\ .$$

However, there is no $\rho \geq 1$ satisfying $\overline{B_k}^x \subset \rho\, B_k$, for all $k \in \mathbb{N}$.

**1.4 Proposition.** *$k(V, L, (X_k), (Y_k))$ is boundedly retractive if and only if the following two conditions hold:*

**1**) *There exists $m \in \mathbb{N}$ such that $Y_k$ is a topological subspace of $X_k$ $(k \geq m)$;*

**2**) $\mathrm{ind}_n\, L(v_n)$ *is boundedly retractive.*

**Proof:** Assume $k(V, L, (X_k), (Y_k))$ is boundedly retractive. Then for $\mathcal{B}_1$ there is $m \in \mathbb{N}$ such that $L(v_r, (X_k)_{k<r}, (Y_k)_{k\geq r})$ and $L(v_m, (X_k)_{k<m}, (Y_k)_{k\geq m})$ induce the same topology on $\mathcal{B}_1$, for all $r \geq m$. It follows easily that $Y_k$ and $X_k$ induce the same topology on $B_k$, $k \geq m$, thus $Y_k$ is a topological subspace of $X_k$ for all $k \geq m$.

Since $k(V, L, (X_k), (Y_k))$ is regular, $\mathrm{ind}\, L(v_n)$ has to be regular. If it is not boundedly retractive, without loss of generality we may assume that for all $n \in \mathbb{N}$, $L(v_n)$ and $L(v_{n+1})$ dot not induce the same topology on the unit ball of $L(v_1)$, namely $B_{v_1,1}$. Therefore, for each $n \in \mathbb{N}$ we can find $(\lambda_p^n)_{p\in\mathbb{N}} \subset B_{v_1,1}$ which is $L(v_{n+1})$-null and not $L(v_n)$-null. We can also assume that $\lambda_{p,j}^n = 0$, $1 \leq j < n+1$. Take $x_j \in Y_j$ with $s_j(x_j) = 1$ $(j \in \mathbb{N})$, and put $x_p^n = (\lambda_{p,k}^n x_k)_{k\in\mathbb{N}}$ $(p, n \in \mathbb{N})$. Then $\{x_p^n : n, p \in \mathbb{N}\} \subset \mathcal{B}_1$, the sequence $(x_p^n)_{p\in\mathbb{N}}$ is $L(v_{n+1}, (X_k)_{k<n+1}, (Y_k)_{k\geq n+1})$-null but it is not $L(v_n, (X_k)_{k<n}, (Y_k)_{k\geq n})$-null.

For the converse assume conditions 1) and 2). It follows from 1) that for all $k \geq m$ there is $\lambda_k > 0$ such that $\lambda_k\, s_k \leq r_k \leq s_k$. Since $k(V, L, (X_k), (Y_k))$ is regular it suffices to prove that the topologies coincide on $\mathcal{B}_n$ residually, $n \geq m$. For every $n \geq m$, put $D_n := \{((r_k(x_k)_{k<n}, (s_k(x_k))_{k\geq n}) : x \in \mathcal{B}_n\}$; then $D_n \subset$

$B_{v_n,1}$, hence there is $l \in \mathbb{N}$ such that $L(v_l)$ and $L(v_r)$ coincide on $D_n$, for all $r \geq l$, i.e. for every $\varepsilon > 0$, there is $\delta > 0$ such that $\delta B_{v_r,1} \cap D_n \subset \varepsilon B_{v_l,1}$. Take $\delta' = \lambda_n \cdots \lambda_{r-1}\delta$ and let $x \in \delta' \mathcal{B}_r \cap \mathcal{B}_n$ be given. Then,

$$\Big\| \Big( (v_r(k)\, r_k(x_k))_{k<n},\ (v_r(k)\, s_k(x_k))_{k\geq n} \Big) \Big\| \leq$$
$$\leq \frac{1}{\lambda_n\, \lambda_{n+1} \cdots \lambda_{r-1}} \Big\| \Big( (v_r(k) r_k(x_k))_{k<r},\ (v_r(k) s_k(x_k))_{k\geq r} \Big) \Big\| < \delta ,$$

thus we obtain

$$\Big( (r_k(x_k))_{k<n}, (s_k(x_k))_{k\geq n} \Big) \in \delta B_{v_r,1} \cap B_{v_n,1} \subset \varepsilon B_{v_l,1} ;$$

that is, $\|((v_l(k)\, r_k(x_k))_{k<n}, (v_l(k)\, s_k(x_k))_{k\geq n})\| \leq \varepsilon$ and therefore $x \in \mathcal{B}_l$. ∎

**1.5 Proposition.** *The following statements are equivalent:*

**i)** $k(V, L, (X_k), (Y_k))$ *is strict;*

**ii)** *For every* $k \in \mathbb{N}$, $Y_k$ *is a topological subspace of* $X_k$ *and for every* $n \in \mathbb{N}$, *there is* $M_n > 0$ *with* $\frac{1}{v_n(k)} \leq \frac{M_n}{v_1(k)}$ $(k \in \mathbb{N})$.

Regarding Grothendieck's question [8] whether regularity implies completeness for LB-spaces, we shall provide a positive answer in the frame of these weighted LB-spaces of Moscatelli type when either the space $(L, \|\ \|)$ is a step space (in the sense of [13]) satisfying property $(\varepsilon)$ or $(L, \|\ \|) = (1_\infty, \|\ \|_\infty)$ or $(L, \|\ \|) = (c_0, \|\ \|_\infty)$. First we should remember the definition of a step (cf. [13]).

A sequence space $(L, \|\ \|)$ is said to be a step if:

**a)** $(L, \|\ \|)$ is perfect;

**b)** $(1_1, \|\ \|_1) \subset (L, \|\ \|) \subset (1_\infty, \|\ \|_\infty)$;

**c)** $(L, \beta(L, L^x))$ is a Banach space, where $L^x$ denotes the $\alpha$-dual of $L$.

(Normal Banach sequence spaces satisfy always property b), cf. [6]).

**1.6 Proposition.** *If* $(L, \|\ \|)$ *is a step, then* $k(V, L, (X_k), (Z_k))$ *and* $k(\overline{V}, L, (X_k), (Y_k))$ *coincide algebraically, they have the same bounded sets and* $k(V, L, (X_k), (Z_k))$ *is regular.*

**Proof:** Let $\mathcal{B}$ be a bounded set in $K(\overline{V}, L, (X_k), (Y_k))$.

**1)** An argument similar to the one in 3.1 iii) in [4] shows that there is $n \in \mathbb{N}$ such that $x_k \in Z_k$ $(k \geq n)$, $\forall\, (x_k)_{k\in\mathbb{N}} \in \mathcal{B}$.

**2)** Let us see that $((0)_{k<n}, (t_k(x_k))_{k\geq n}) \in \mathrm{ind}_m\, L(v_m)$ $(x \in \mathcal{B})$. Assume there is $x \in B$ such that $((0)_{k<n}, (t_k(x_k))_{k\geq n}) \notin \mathrm{ind}_m\, L(v_m)$. Since $L$ is perfect, for all $m \in \mathbb{N}$, there is $z^m \in L^x$ with $\|z^m\|^x = 1$ and $\sum_{k=n}^{\infty} v_m(k)\, t_k(x_k)\, z_k^m = \infty$.

Take $y^m \in \varphi$, such that $\|y^m\|^x \leq 1$, $\max\{k \in \mathbb{N} : y_k^m \neq 0\} < \min\{k : y_k^{m+1} \neq 0\}$, $y_k^n = 0$ for all $k < n$ and $\sum_{k=n}^{\infty} \overline{v}(k)\, t_k(x_k)\, |y_k^m| > m$. We put $J_m := \{k \in \mathbb{N} : y_k^m \neq 0\}$. Thus $(J_m)_{m \in \mathbb{N}}$ is a sequence of pairwise disjoint subsets of $\mathbb{N}$.

Take $\overline{v} \in \overline{V}$, with $\overline{v}(k) = v_n(k)$ if $k < \min J_1$; $\overline{v}(k) = v_m(k)$ if $\min J_m \leq k < \min J_{m+1}$. For all $k \in \mathbb{N}$ such that $t_k(x_k) \neq 0$, we have

$$x_k \notin \frac{1}{2} t_k(x_k)\, C_k = \frac{1}{2}\, t_k(x_k)\, \overline{v}(k) \frac{1}{\overline{v}(k)}\, C_k \ ,$$

whence there is $\varepsilon_k > 0$ such that

$$x_k \notin \frac{1}{2}\, t_k(x_k)\, \overline{v}(k) \left( \frac{1}{\overline{v}(k)}\, B_k + \varepsilon_k\, A_k \right)$$

and therefore for every $k \in \mathbb{N}$ we can find $\varepsilon_k > 0$ such that

$$P_{\varepsilon_k, \overline{v}}(x_k) \geq \frac{1}{2}\, t_k(x_k)\, \overline{v}(k) \ .$$

Let $U$ be the unit ball in the $\alpha$-dual $L^x$. We have

$$\sup_{y \in U} \left| \left\langle (p_{\varepsilon_k, \overline{v}}(x_k)), y \right\rangle \right| \geq \sup_m \sum_{k=n}^{\infty} v_m(k)\, t_k(x_k)\, |y_k^m| = \infty$$

which contradicts that $x \in K(\overline{V}, L, (X_k), (Y_k))$.

**3**) To end the proof, it suffices to show that there exists an index $m \geq n$ such that $\sup_{x \in B} \sup_{y \in U} \sum_{k=m}^{\infty} v_m(k)\, t_k(x_k)\, |y_k| < \infty$.

Assume the contrary. Then for each $m \geq n$, we can find $y^m \in \mathcal{B} \cap \bigoplus_{k \in \mathbb{N}} Z_k$, and $z^m \in U$ such that

$$\sum_{k=m}^{\infty} v_m(k)\, t_k(y_k^m)\, |z_k^m| > m \,, \qquad y_k^n = 0 \quad (k < n)$$

and

$$\max\{k \in \mathbb{N} \colon y_k^m \neq 0\} < \min\{k \colon y_k^{m+1} \neq 0\} \ .$$

Now the proof finishes as in part 2. ∎

**1.7 Corollary.** *If $(L, |\ \|)$ is a step, then $k(V, L, (X_k), (Y_k))$ is regular if and only if it coincides with $k(V, L, (X_k), (Z_k))$.*

**1.8 Proposition.** *If $(L, \|\ \|)$ is a step with property $(\varepsilon)$, then $k(V,L,(X_k),(Y_k))$ is regular if and only if it is complete.*

**Proof:** Let $(x_\alpha)_{\alpha\in A}$ be a Cauchy net in $k(V,L,(X_k),(Y_k))$, hence a Cauchy net in $K(\overline{V},L,(X_k),(Y_k))$. There is $x \in K(\overline{V},L,(X_k),(Y_k))$ such that $(x_\alpha)_{\alpha\in A}$ converges to $x$ in $K(\overline{V},L,(X_k),(Y_k))$. Since $K(\overline{V},L,(X_k),(Y_k))$ and $k(V,L,(X_k),(Y_k))$ induce the same topology on $\bigoplus_{n=1}^m X_n$ for all $m \in \mathbb{N}$, we obtain that $(p_m(x_\alpha))_{\alpha\in A}$ converges to $p_m(x)$ in $k(V,L,(X_k),(Y_k))$ for each $m\in\mathbb{N}$, where $p_m$ stands for the projection onto the first $m$ coordinates.

Let $p$ be a continuous seminorm on $k(V,L,(X_k),(Y_k))$. Given $\varepsilon>0$, there is $\alpha_0$ in $A$ such that $p(x_\alpha - x_{\alpha'}) < \varepsilon/9$, for $\alpha,\alpha' \geq \alpha_0$.

Fix $\alpha_0$ and find $m_1\in\mathbb{N}$ such that $p(p_m(x_{\alpha_0}) - x_{\alpha_0}) < \varepsilon/9$ $(m \geq m_1)$. Now for $\alpha\geq\alpha_0$ and $m\geq m_1$ we may write:

$$p(x_\alpha - p_m(x_\alpha)) \leq p(x_\alpha - x_{\alpha_0}) + p(x_{\alpha_0} - p_m(x_{\alpha_0})) + p(p_m(x_{\alpha_0}) - p_m(x_\alpha)) < \varepsilon/3 \ .$$

There is $m_2$ such that $p(x-p_m(x)) < \varepsilon/3$ $(m\geq m_2)$. Take $m_0 = \max(m_1,m_2)$. There must be $\alpha_1$ such that $p(p_{m_0}(x) - p_{m_0}(x_\alpha)) < \varepsilon/3$ $(\alpha\geq\alpha_1)$. Choose $\alpha_2\geq\alpha_0$, $\alpha_2\geq\alpha_1$ to obtain

$$p(x-x_\alpha) \leq p(x-p_{m_0}(x)) + p(p_{m_0}(x) - p_{m_0}(x_\alpha)) + p(p_{m_0}(x_\alpha) - x_\alpha) < \varepsilon \ ,$$

which proves the completeness of $k(V,L,(X_k),(Y_k))$. ∎

In the case $L = 1_\infty$ the equivalence between regularity and completeness holds too.

**1.9 Proposition.** *$k(V,1^\infty,(X_k),(Y_k))$ is regular if and only if it is complete.*

**Proof:** Let $(Z_k,t_k)_{k\in\mathbb{N}}$ be as in 1.1. It follows from (1.7) that $k(V,1^\infty,(X_k),(Y_k))$ is regular if and only if it coincides with $k(V,1^\infty,(X_k),(Z_k))$, and that $k(V,1^\infty,(X_k),(Y_k))$ is the bornological space associated to $K(\overline{V},1^\infty,(X_k),(Y_k))$. Assume that $k(V,1^\infty,(X_k),(Y_k))$ is regular.

We shall show – following the technics in [5] – that for every sequence of positive real numbers $(\varepsilon_k)_{k\in\mathbb{N}}$,

$$\overline{\sum_{n=1}^{m} \varepsilon_n\, \mathcal{C}_n} \ \subset\ 2\sum_{n=1}^{m+1} \varepsilon_n\, \mathcal{C}_n \qquad (m\in\mathbb{N}) \ ,$$

where $\mathcal{C}_n$ stands for the unit ball in $1^\infty(v_n,(X_k)_{k<n},(Z_k)_{k\geq n})$ and the closure is taken in $k(V,l^\infty,(X_k),(Z_k))$ with respect to $\prod_{k\in\mathbb{N}}(X_k,r_k)$.

Given $m\in\mathbb{N}$, let $p_m : k(V,l^\infty,(X_k),(Z_k)) \to k(V,l^\infty,(X_k),(Z_k))$ be the projection onto the first $m$ coordinates and $Q_m := \mathrm{Id} - p_m$, then

$$\overline{\sum_{n=1}^{m} \varepsilon_n\, \mathcal{C}_n} \ \subset\ \overline{\sum_{n=1}^{m} \varepsilon_n\, p_m(\mathcal{C}_n)} + \overline{\sum_{n=1}^{m} \varepsilon_n\, Q_m(\mathcal{C}_n)} \ .$$

Since $p_m(\mathcal{C}_{m+1})$ is a 0-neighbourhood in $\prod_{k=1}^m (X_k, r_k)$,

$$\overline{\sum_{n=1}^{m} \varepsilon_n\, p_m(\mathcal{C}_n)} \;\subset\; \sum_{n=1}^{m+1} \varepsilon_n\, p_m(\mathcal{C}_n) \;\subset\; \sum_{n=1}^{m+1} \varepsilon_n\, \mathcal{C}_n \ .$$

Put $D_m := \{(((0)_{k\le m}), (x_k)_{k>m}) :\ x_k \in Z_k,\ t_k(x_k) \le \sum_{n=1}^m \frac{\varepsilon_n}{v_n(k)}\ (k \ge m)\}$. Then $D_m$ is closed in $\prod_{k\in\mathbb{N}} (X_k, r_k)$. We claim that $D_m = \sum_{n=1}^m \varepsilon_n Q_m(\mathcal{C}_n)$. The argument we are going to use is due to Ernst and Schnettler (see [7]).

Given $x = ((0)_{k\le m}, (x_k)_{k>m}) \in D_m$, put $x_k^1 = x_k$ if $t_k(x_k) < \frac{\varepsilon_1}{v_1(k)}$. Otherwise put

$$x_k^1 = \frac{\varepsilon_k}{v_1(k)}\, \frac{x_k}{t_k(x_k)} \ .$$

Then $t_k(x_k^1) \le t_k(x_k)$. Consequently, $x^1 \in k(V, 1^\infty, (X_k), (Y_k))$ and $t_k(x_k^1) \le \frac{\varepsilon_1}{v_1(k)}$ $(k \ge m)$.

Define $x^2 = x - x^1$. Thus $x_k^2 = 0$ whenever $t_k(x_k) \le \frac{\varepsilon_1}{v_1(k)}$. For $t_k(x_k) > \frac{\varepsilon_1}{v_1(k)}$,

$$\begin{aligned} t_k(x_k^2) = t_k\left(x_k - \frac{\varepsilon_1}{v_1(k)}\, \frac{x_k}{t_k(x_k)}\right) &= t_k(x_k) \left| 1 - \frac{\varepsilon_1}{v_1(k) t_k(x_k)} \right| \\ &= t_k(x_k) - \frac{\varepsilon_1}{v_1(k)} \le \sum_{n=2}^{m} \frac{\varepsilon_n}{v_n(k)} \ . \end{aligned}$$

After finitely many times, we obtain $x \in \sum_{n=1}^m \varepsilon_n\, Q_m(\mathcal{C}_n)$.

Accordingly

$$\overline{\sum_{n=1}^{m} \varepsilon_n\, \mathcal{C}_n} \;\subset\; 2 \sum_{n=1}^{m+1} \varepsilon_n\, \mathcal{C}_n$$

and this shows that $k(V, 1^\infty, (X_k), (Y_k))$ is the barrelled space associated to $K(\overline{V}, 1^\infty, (X_k), (Y_k))$, hence complete. ∎

In order to deal with the case $(L, \|\ \|) = (c_0, \|\ \|_\infty)$ in our next proposition, we should remember that a decreasing sequence of strictly positive weights $V = (v_n)_{n\in\mathbb{N}}$ is said to be regularly decreasing (see [2,3]) if for every $n \in \mathbb{N}$, there is $m \ge n$ such that for each $\varepsilon > 0$ there is $\overline{v} \in \overline{V}$ with

$$v_m(i) \le \varepsilon\, v_n(i) \quad \text{ if } \ \overline{v}(i) < v_m(i) \ .$$

**1.10 Proposition.** *$k(V, c_0, (X_k), (Y_k))$ is regular if and only if it is complete.*

*In this case $k(V, c_0, (X_k), (Y_k))$ coincides algebraically and topologically with $K(\overline{V}, c_0, (X_k), (Y_k))$.*

**Proof:** Let $(Z_k, t_k)_{k\in\mathbb{N}}$ be as in 1.1. From 1.2 ii) and 1.3 i) we get that $k(V, c_0, (X_k), (Y_k))$ is a topological subspace of $K(\overline{V}, c_0, (X_k), (Y_k))$ and it is regular if and only if $k(V, c_0, (X_k), (Y_k)) = k(V, c_0, (X_k), (Z_k))$ and $V$ is regularly decreasing.

Assume that $k(V, c_0, (X_k), (Y_k))$ is regular. Let $\mathcal{B}$ be a bounded set in $K(\overline{V}, c_0,\ (X_k), (Y_k))$, hence bounded in $K(\overline{V}, 1_\infty, (X_k), (Y_k))$. There must be $n \in \mathbb{N}$ such that

$$\sup_{x\in\mathcal{B}} \sup_{k\geq n} v_n(k)\, t_k(x_k) = M < \infty \ .$$

Since $V$ is regularly decreasing, there is $m \geq n$ such that for each $\varepsilon > 0$ there is $\overline{v} \in \overline{V}$ satisfying $v_m(i) \leq \frac{\varepsilon}{2M}\, v_n(i)$ if $\overline{v}(i) < v_m(i)$.

If $k \geq m$ and $\overline{v}(k) \geq v_m(k)$, we have

$$x_k \ \notin\ \frac{1}{2}\, t_k(x_k)\, \overline{v}(k)\, \frac{1}{\overline{v}(k)}\, C_n \ .$$

Hence there is $\varepsilon_k > 0$ such that

$$x_k \ \notin\ \frac{1}{2}\, t_k(x_k)\, \overline{v}(k) \left( \frac{1}{\overline{v}(k)}\, B_k + \varepsilon_k A_k \right) .$$

Put $I_0 := \{k \geq n \colon\ \overline{v}(k) \geq v_m(k)\}$. If $k \in I_0$,

$$p_{\varepsilon_k,\overline{v}}(x_k) \geq \frac{1}{2}\, t_k(x_k)\, \overline{v}(k) \geq \frac{1}{2}\, t_k(x_k)\, v_m(k) \ .$$

On the other hand if $\overline{v}(k) < v_m(k)$, then

$$v_m(k)\, t_k(x_k) \leq v_n(k)\, t_k(x_k)\, \frac{\varepsilon}{2M} < \varepsilon \ .$$

Since $(p_{\varepsilon_k,\overline{v}}(x_k))_{k\in\mathbb{N}} \in c_0$, we have $(v_m(k)\, t_k(x_k))_{k\geq m} \in c_0$. Therefore $\mathcal{B}$ is contained in $c_0(v_m, (X_k)_{k<m}, (Z_k)_{k\geq m})$. In particular $K(\overline{V}, c_0(X_k), (Y_k)) = k(V, c_0, (X_k)(Y_k))$. ∎

## 2 – Weighted Fréchet spaces of Moscatelli type

### 2.1 Definitions and preliminaries

In this section 2, we would like to make the following conventions:

– $(L, \|\ \|)$ will be a normal Banach sequence space with property $(\gamma)$.

– $A = (a_n)_{n\in\mathbb{N}}$ will stand for a strictly positive Köthe matrix.

– $(Y_k, s_k)_{k\in\mathbb{N}}$ and $(X_k, r_k)_{k\in\mathbb{N}}$ will represent two sequences of Banach spaces and $f_k\colon Y_k \to X_k$ will be a continuous linear mapping such that $f_k(B_k) \subset A_k$ where $A_k$ (resp. $B_k$) stands for the unit ball of $X_k$ (resp. $Y_k$) $(k \in \mathbb{N})$.

Now, for every $n \in \mathbb{N}$, we define:

$$
\begin{aligned}
G_n &= L\Big(a_n, (Y_k)_{k<n}, (X_k)_{k\geq n}\Big) \\
&= \Big\{(x_k)_{k\in\mathbb{N}} \in \prod_{k<n} Y_k \times \prod_{k\geq n} X_k\colon \Big(\Big(a_n(k)\, s_k(x_k)\Big)_{k<n}, \Big(a_n(k)\, r_k(x_k)\Big)_{k\geq n}\Big) \in L\Big\}
\end{aligned}
$$

provided with the norm:

$$
\|(x_k)_{k\in\mathbb{N}}\|_n = \Big\|\Big((a_n(k)\, s_k(x_k))_{k<n},\, (a_n(k)\, r_k(x_k))_{k\geq n}\Big)\Big\| \,.
$$

Clearly $G_n$ is a Banach space $(n \in \mathbb{N})$. We put $g_n : G_{n+1} \to G_n$, $(x_k)_{k\in\mathbb{N}} \to ((x_k)_{k<n}, f_n(x_n), (x_k)_{k>n})$ $(n \in \mathbb{N})$. Clearly $g_n$ is a continuous linear mapping $(n \in \mathbb{N})$ and we define the *weighted Fréchet space of Moscatelli type* w.r.t. $A$, $(L, \|\ \|)$, $(Y_k, s_k)_{k\in\mathbb{N}}$ $(X_k, r_k)_{k\in\mathbb{N}}$ and $f_k\colon Y_k \to X_k$ $(k \in \mathbb{N})$ by

$$
G = \lambda(A, L, (Y_k), (X_k)) := \mathrm{proj}_{n\in\mathbb{N}}(G_n, g_n) \,.
$$

As in [5], it is easy to check that $G$ coincides algebraically with

$$
\Big\{y = (y_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} Y_k\colon \Big(f_k(y_k))_{k\in\mathbb{N}} \in \mathrm{proj}_n\, L(a_n, (X_k)_{k\in\mathbb{N}})\Big)\Big\}
$$

and $G$ has the initial topology w.r.t. the inclusion $j : G \to \prod_{k\in\mathbb{N}} Y_k$ and the linear mapping $\tilde{f}\colon G \to \mathrm{proj}_n\, L(a_n, (Y_k)_{k\in\mathbb{N}})$, $(x_k)_{k\in\mathbb{N}} \to (f_k(y_k))_{k\in\mathbb{N}}$. We can always assume without loss of generality that $f_k(Y_k)$ is dense in $X_k$ $(k \in \mathbb{N})$.

If $X_k = Y_k = \mathbb{K}$ $(k \in \mathbb{N})$, we shall write $\lambda(A, L)$, following the classical notations.

Recall that if $X_k = Y_k = \mathbb{K}$ $(k \in \mathbb{N})$ and $L = 1_p$ we obtain the Köthe echelon spaces [3] and the case $a_n(k) = 1$ $(k, n \in \mathbb{N})$, we get the LB-spaces of Moscatelli type [4].

Let us investigate when the weighted Fréchet spaces of Moscatelli type are Montel, Schwartz, satisfy property $(\Omega_\varphi)$ and property $(\mathrm{DN}_\varphi)$.

The properties $(\Omega_\varphi)$ and $(\mathrm{DN}_\varphi)$ were introduced by D. Vogt in [14] as follows.

For an increasing continuous function $\varphi : (0, \infty) \to (0, \infty)$ we say that a Fréchet space with a basis of zero-neighbourhoods $\{U_n\}_{n\in\mathbb{N}}$ has property $(\Omega_\varphi)$ if $\forall p\ \exists q\ \forall k\ \exists\, C > 0\ \forall\, r > 0$: $U_q \subset C\varphi(r)\, U_k + r^{-1} U_p$ and a Fréchet space with a fundamental sequence of seminorms $\{\|\ \|_n\}_{n\in\mathbb{N}}$ satisfies property $(\mathrm{DN}_\varphi)$ if:

$$
\exists n_0\ \ \forall m\ \ \exists\, n \in \mathbb{N},\ \ \exists\, C > 0\colon \forall\, x \in F,\ \ \forall\, r > 0\colon\ \ \|x\|_m \leq C\, \varphi(r)\, \|x\|_{n_0} + r^{-1} \|x\|_n \,.
$$

These two conditions play an important role in [10, 15].

By [10], a Fréchet space is quasinormable if and only if it has $(\Omega_\varphi)$ for some $\varphi$.

By [16], a Fréchet space $F$ has $(\Omega_\varphi)$ for $\varphi(k) = 1$ $(k \in \mathbb{N})$ if and only if $F''$ is a quojection and this is equivalent to the fact that $F$ does not satisfy the condition (*) of Bellenot and Dubinsky (cf. [1]).

Property $(\mathrm{DN}_\varphi)$ is related with some normability conditions (see [14]).

**2.2 Lemma.**

**a)** *$\lambda(A, L, (Y_k), (X_k))$ is a complemented subspace of*

$$\lambda(A,L,F):=\Big\{(x^n)_{n\in\mathbb{N}}\in F^{\mathbb{N}}\colon\ (a_m(n)\,r(x^n))_{n\in\mathbb{N}}\in L,\ \forall\, r\in cs(F),\ \forall\, m\in\mathbb{N}\Big\},$$

*where $F$ is the Fréchet space of Moscatelli type w.r.t. $(K, \|\ \|)$, $(Y_k, s_k)_{k\in\mathbb{N}}$ $(X_k, r_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ (see [4]).*

**b)** *The sectional subspace $(\lambda(A,L))_J := \{(\alpha_k)_{k\in\mathbb{N}} \in \lambda(A,L) : \alpha_k = 0$ $\forall\, k \notin J\}$, with $J := \{k \in \mathbb{N} : f_k(Y_k) \not\equiv 0\}$ of the Köthe echelon space is algebraically and topologically isomorphic to a complemented subspace of $\lambda(A, L, (Y_k), (X_k))$.*

**2.3 Proposition.** *Let $J := \{k \in \mathbb{N}\colon f_k(Y_k) \not\equiv 0\}$ and consider the sectional subspace $(\lambda(A,L))_J$. Then,*

**i)** *$\lambda(A, L, (Y_k), (X_k))$ is Montel (resp. Schwartz) if and only if $(\lambda(A,L))_J$ is Montel (resp. Schwartz) and $Y_k$ is finite dimensional for all $k \in \mathbb{N}$.*

**ii)** *$\lambda(A, L, (Y_k), (X_k))$ has property $(DN_\varphi)$ (resp. property $(\Omega_\varphi)$) if and only if $(\lambda(A,L))_J$ and $F$ have property $(DN_\varphi)$ (resp. $\Omega_\varphi)$) where $F$ is the Fréchet space of Moscatelli type w.r.t. $(L, \|\ \|)$, $(Y_k, s_k)_{k\in\mathbb{N}}$ $(X_k, r_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$.*

The proof of the result above is an easy consequence of lemma 2.2 (for the case of the property $(\Omega_\varphi)$ use also 2.9 and 2.10 in [5] and for property $(\mathrm{DN}_\varphi)$ use 2.17 in [5]).

## 3 – Duality

In this section, we shall follow the ideas in [3] and [5] to establish the duality between the weighted Fréchet and LB-spaces of Moscatelli type.

In order to settle such a duality we need the following proposition whose proof follows thoroughly its restricted version to the Köthe sequence spaces appearing in [3].

**3.1 Proposition.** *Let $(L, \|\ \|)$ be a normal Banach sequence space with property $(\gamma)$. Let $(X_k, r_k)_{k\in\mathbb{N}}$ be a sequence of Banach spaces and $A = (a_n)_{n\in\mathbb{N}}$ a strictly positive Köthe matrix. A subset $\mathcal{B} \subset \operatorname{proj}_n L(a_n, (X_k)_{k\in\mathbb{N}})$ is bounded if and only if there exists $\overline{v} \in \overline{V}$ (cf 1.1) such that for every $(x_k)_{k\in\mathbb{N}} \in \mathcal{B}$, $(r_k(x_k))_{k\in\mathbb{N}} \in \overline{v}B_L$, where $B_L$ is the unit ball in $L$.*

Let $(L, \|\ \|)$ be a normal Banach sequence space with property $(\gamma)$. Let $(Y_k, s_k)_{k\in\mathbb{N}}$ $(X_k, r_k)_{k\in\mathbb{N}}$ be two sequences of Banach spaces and $f_k\colon Y_k \to X_k$ a continuous linear mapping with $f_k(B_k) \subset A_k$ $(k \in \mathbb{N})$. Let $A = (a_n)_{n\in\mathbb{N}}$ be a strictly positive Köthe matrix and $G = \lambda(A, L, (Y_k), (X_K))$ the corresponding weighted Fréchet space of Moscatelli type. Let $\overline{v} \in \overline{V}$, $\mu_k > 0$ $(k \in \mathbb{N})$ be given. We consider the set $\overline{v}(k) f_k^{-1}(A_k) \cap \mu_k B_k$ and denote its Minkowski functional by $q_{\overline{v}(k)\mu_k}$. Then $q_{\overline{v}(k)\mu_k}$ is a norm on $Y_k$ and it is equivalent to $s_k$ $(k \in \mathbb{N})$. We define $G_{\overline{v}(k)\mu_k} := L((Y_k, q_{\overline{v}(k)\mu_k})_{k\in\mathbb{N}})$. Clearly $G_{\overline{v}(k)\mu_k}$ is a Banach space and is continuously contained in $G$ ($\overline{v} \in \overline{V}$, $\mu_k > 0$ $(k \in \mathbb{N})$).

**3.2 Proposition.** *Under the hypotheses above, for every bounded set $\mathcal{B}$ in $G$, there are $\overline{v} \in \overline{V}$, $(\mu_k) \in \mathbb{K}^{\mathbb{N}}$, $\mu_k > 0$ $(k \in \mathbb{N})$ such that $\mathcal{B}$ is a bounded set of $G_{\overline{v}(k)\mu_k}$. In particular, $G$ can be represented as an (uncountable) inductive limit:*

$$G = \operatorname{ind}\Big(G_{\overline{v}(k)\mu_k}\colon \overline{v} \in \overline{V},\ (\mu_k) \in \mathbb{K}^{\mathbb{N}},\ \mu_k > 0\ (k \in \mathbb{N})\Big)\ .$$

**Proof:** Let $\mathcal{B}$ be a bounded set in $G$. According to proposition 3.1, there exist $\overline{v} \in \overline{V}$, $(\lambda_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, $\lambda_k > 0$ $(k \in \mathbb{N})$ such that

$$\sup\left\{\left\|\frac{1}{\overline{v}(k)}\, r_k(f_k(y_k)))_{k\in\mathbb{N}}\right\|\colon y = (y_k)_{k\in\mathbb{N}} \in \mathcal{B}\right\} \le 1$$

and

$$\sup\Big\{s_m(y_m)\colon y = (y_k)_{k\in\mathbb{N}} \in \mathcal{B}\Big\} \le \lambda_m \qquad (m \in \mathbb{N})\ .$$

Choose $(\eta_k)_{k\in\mathbb{N}} \in L$ with $\eta_k > 0$ $(k \in \mathbb{N})$ and $\|(\eta_k)_{k\in\mathbb{N}}\| = 1$. Put $\mu_k := \lambda_k\, \eta_k^{-1}$ $(k \in \mathbb{N})$. Let us see that $\mathcal{B}$ is bounded in $G_{\overline{v}(k)\mu_k}$. Let $y \in \mathcal{B}$. Then for all $k \in \mathbb{N}$,

$$y_k \ \in\ r_k(f_k(y_k))\, f_k^{-1}(A_k) \cap s_k(y_k) B_k\ .$$

Call

$$I_1 := \left\{k \in N\colon\ \mu_k^{-1} s_k(y_k) \le \frac{1}{\overline{v}(k)}\, r_k(f_k(y_k))\right\}\ ;$$

$$I_2 := \left\{k \in N\colon\ \frac{1}{\overline{v}(k)}\, r_k(f_k(y_k)) < \mu_k^{-1} s_k(y_k)\right\}\ .$$

Take $k \in I_1$, then $y_k \in \frac{1}{\overline{v}(k)} r_k(f_k(y_k))(\overline{v}(k) f_k^{-1}(A_k) \cap \mu_k B_k)$ and

$$\Big\|\Big((q_{\overline{v}(k)\mu_k}(y_k))_{k\in I_1},\, (0)_{k\in I_2}\Big)\Big\| \le \Big\|\Big(\Big(\frac{1}{\overline{v}(k)}\, r_k(f_k(y_k))\Big)_{k\in I_1},\, (0)_{k\in I_2}\Big)\Big\| \le 1 \ .$$

Take $k \in I_2$, then $y_k \in \mu_k^{-1} s_k(y_k)(\overline{v}(k) f_k^{-1}(A_k) \cap \mu_k B_k)$ and

$$\begin{aligned}\Big\|\Big((0)_{k\in I_1},\, (q_{\overline{v}(k)\mu_k}(y_k))_{k\in I_2}\Big)\Big\| &\le \Big\|\Big((0)_{k\in I_1},\, (\mu_k^{-1} s_k(y_k))_{k\in I_2}\Big)\Big\| \\ &\le \Big\|\Big((0)_{k\in I_1},\, (\eta_k)_{k\in I_2}\Big)\Big\| \le 1 \ .\end{aligned}$$

Thus $\|((q_{\overline{v}(k)\mu_k}(y_k))_{k\in\mathbb{N}}\| \le 2$ and that finishes the proof since $G$ is ultrabornological. ∎

Let $G = \lambda(A, L, (Y_k), (X_k))$ be the weighted Fréchet space of Moscatelli type w.r.t. $(L, \|\ \|)$ with $(\varepsilon)$, $A = (a_n)_{n\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$ $(X_k, r_k)_{k\in\mathbb{N}}$, $(f_k)_{k\in\mathbb{N}}$, each $f_k$ having dense range and $f_k(B_k) \subset A_k$ $(k \in \mathbb{N})$. Because of lemma 2.2 in [5] we may naturally identify (algebraically and topologically) the strong dual of $G_n := L(a_n, (Y_k, s_k)_{k<n}, (X_k, r_k)_{k\ge n})$ with $H_n = L'(v_n, (Y_k', s_k')_{k<n}, (X_k', r_k')_{k\ge n})$ and define the weighted LB space of Moscatelli type $H := \mathrm{ind}_{n\in\mathbb{N}} H_n = k(V, L, (Y_k'), (X_k'))$ w.r.t. the duals. We shall keep these notation all over this section 3.

Next we study the relationship between $G'_\beta$ and $H$.

**3.3 Proposition.** *There is an identity map $I\colon H \to G'_\beta$ which is continuous. Moreover $H$ is the bornological space associates to $G'_\beta$ and $G'_\beta$ coincides topologically with $K(\overline{V}, L', (Y_k'), (X_k'))$.*

**Proof:** Let $\mathcal{B} \in G$ be bounded. By (3.2), we can find $\overline{v} \in \overline{V}$, $\mu_k > 0$ $(k \in \mathbb{N})$ such that $\sup \|(q_{\overline{v}(k)\mu_k}(y_k))_{k\in\mathbb{N}}\|\colon y \in \mathcal{B}\} \le 1$. Take $\varepsilon_k = \mu_k^{-1}$ $(k \in \mathbb{N})$ and consider the Minkowski functional $p_{\varepsilon_k \overline{v}(k)}$ of $\frac{1}{\overline{v}(k)} A_k' + \varepsilon_k B_k'$. Observe that $\frac{1}{2} q'_{\overline{v}(k)\mu_k} \le p_{\varepsilon_k\overline{v}(k)} \le 2 q'_{\overline{v}(k)\mu_k}$ $(k \in \mathbb{N})$. Now

$$\mathcal{U} := \Big\{(f_k)_{k\in\mathbb{N}} \in H\colon\ \Big\|(p_{\varepsilon_k\overline{v}(k)}(f_k))_{k\in\mathbb{N}}\Big\| \le \frac{1}{2}\Big\} \subset \mathcal{B}^\circ \ .$$

Indeed, let $f \in \mathcal{U}$ and $y \in \mathcal{B}$. Then,

$$\begin{aligned}\Big|\sum_{k\in\mathbb{N}} f_k(y_k)\Big| &\le 2 \sum_{k\in\mathbb{N}} p_{\varepsilon_k\overline{v}(k)}(f_k)\, q_{\overline{v}(k)\mu_k}(y_k) \\ &\le \Big\|(p_{\varepsilon_k\overline{v}(k)}(f_k))_{k\in\mathbb{N}}\Big\|'\, \Big\|(q_{\overline{v}(k)\mu_k}(y_k))_{k\in\mathbb{N}}\Big\| \le 1 \ .\end{aligned}$$

Conversely let $\overline{v} \in \overline{V}$, $\varepsilon_k > 0$ $(k \in \mathbb{N})$ be given and put $\mu_k = \varepsilon_k^{-1}$ $(k \in \mathbb{N})$. Then the polar of the bounded set $\mathcal{B} := \{y \in \prod_{k\in\mathbb{N}} Y_k\colon\ \|q_{\overline{v}(k)\mu_k}(y_k))_{k\in\mathbb{N}}\| \le 1\}$

in $G'$ is contained in

$$\mathcal{U} := \Big\{ (f_k)_{k\in\mathbb{N}} \in H \colon \Big\| (p_{\varepsilon_k \overline{v}(k)}(f_k))_{k\in\mathbb{N}} \Big\| \le 2 \Big\} \ .$$

Indeed, let $f \in \mathcal{B}^\circ$. It suffices to show that $\sum_{k\in\mathbb{N}} p_{\varepsilon_k \overline{v}(k)}(f_k)\, \|\alpha_k\| \le 2$ for all $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in L$ such that $\|\alpha\| \le 1$ and $\alpha_k > 0$ $(k \in \mathbb{N})$. Fix one of those $\alpha$ and take $y \in \prod_{k\in\mathbb{N}} Y_k$ with $q_{\overline{v}(k)\mu_k}(y_k) \le 1$ $(k \in \mathbb{N})$. Then $(\alpha_k y_k)_{k\in\mathbb{N}} \in \mathcal{B}$ and therefore

$$\sum_{k\in\mathbb{N}} \alpha_k\, |f_k(y_k)| = \sum_{k\in\mathbb{N}} |f_k(\alpha_k y_k)| \le 1 \ .$$

Since $p_{\varepsilon_k \overline{v}(k)} \le 2\, q'_{\overline{v}(k)\mu_k}$ $(k \in \mathbb{N})$, we conclude

$$\sum_{k\in\mathbb{N}} p_{\varepsilon \overline{v}(k)}(\alpha_k f_k) \le 2 \ . \ \blacksquare$$

**3.4 Corollary.** *$G$ is distinguished if and only if the corresponding weighted LB-space w.r.t. the duals $k(V, L', (Y_k'), (X_k'))$ satisfies $k(V, L', (Y_k'), (X_k')) = K(\overline{V}, L', (Y_k'), (X_k'))$ topologically.*

**3.4 Remark.** Under the above hypotheses, if there is an index $n \in \mathbb{N}$ such that $X_k'$ is a normed subspace of $Y_k'$ $(k \ge n)$, then $k(V, L', (Y_k), (X_k')) = k(\overline{V}, L', (Y_k'), (X_k'))$ topologically if and only if $k(V, L') = K(\overline{V}, L')$ topologically.

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