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A CLASS OF QUASI-ADEQUATE TRANSFORMATION SEMIGROUPS

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0 – Introduction

It is well known that $\mathcal{T}_X$, the semigroup of full transformations on a set X contains an isomorhpic copy of every semigroup of order not exceeding $|X|-1$. Therefore, as remarked by Howie [13], there is little point in attempting a complete classification of the subsemigroups of $\mathcal{T}_X$. However, there is some interest in identifying certain special subsemigroups that appear to be of particular interest. See for example, Howie [10, 11, 13] and Umar [15, 16].

In this paper we construct a class of transformation semigroups based on simple modification of Vagner's [17] method of representing the elements of $\mathcal{J}(X)$, the symmetric inverse semigroup as full transformations. In Section 1 we describe our construction while in Section 2, we show that the construction leads to $\mathcal{R}$-unipotent semigroups. (A regular semigroups is $\mathcal{R}$-unipotent if each of its principal right ideals has a unique idempotent generator. Equivalently, an $\mathcal{R}$-unipotent semigroup is a regular semigroup S in which $E(S)$ is a left regular band; i.e., $efe=ef$, for all $e,f\in E(S)$.) Further, we consider the finite case where we obtain expressions for the order of the semigroup and that of its left regular band of idempotents.

In Section 3 we further obtain a subclass of (irregular) quasi-adequate semigroups (these are the analogues of orthodox semigroups in the abundant semigroup [9] theory), from our earlier construction and show that they are indeed $\mathcal{R}^*$-unipotent semigroups. (An $\mathcal{R}^*$-unipotent semigroup is defined as a quasi-adequate semigroup in which each of its principal right $*$-ideals has a unique idempotent generator. Equivalently, an $\mathcal{R}^*$-unipotent semigroup is an abundant semigroup S in which $E(S)$ is a left regular band.) We also consider a finite case, where we obtain expressions for the order of the semigroup and that of its left regular band of idempotents.

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1 – Preliminaries

For standard terms in semigroup theory see [12]. Let $\mathcal{T}_X$ and $\mathcal{I}(X)$ be the full transformation and symmetric inverse semigroups on a set X (finite or infinite) respectively. Vagner represents the element of $\mathcal{J}(X)$ as full transformations by adjoining an extra element 0 to X and defining, for $\alpha \in \mathcal{J}(X)$, the full transformation α^* in $\mathcal{T}_{X\cup\{0\}}$ by

$$x\alpha^* = x\alpha \quad (\text{if } x \in \operatorname{dom}\alpha) \qquad \text{and} \qquad x\alpha^* = 0 \quad (\text{otherwise}) \ .$$

Now, for a given α in $\mathcal{T}_X$ let

$$C(\alpha) = \bigcup\Big\{t\alpha^{-1}\colon\ (t \in \operatorname{Im}\alpha)\ |t\alpha^{-1}| \geq 2\Big\}\,, \qquad F(\alpha) = \Big\{x \in X\colon\ x\alpha = x\Big\}\ .$$

Then clearly $C(\alpha^*)\alpha^* = \{0\}$. If now we replace $\{0\}$ in these expressions, by a set A, we are thus led to the following definition.

Definition 1.1. An element α (in $\mathcal{T}_x$) is called *a Vagner map with respect to a subset* (of X) A (possibly empty) or simply an *A-Vagner map* if $C(\alpha)\alpha \subseteq A = A\alpha$ and $\alpha|_A$ is one-to-one.

Remark. Notice that (in the above definition) if A is finite then $A = A\alpha$ implies $\alpha|_A$ is one-to-one.

The following lemma is crucial in proving that the set of all A-Vagner maps in $\mathcal{T}_X$ is a subsemigroup.

Lemma 1.2. *Let $\alpha, \beta \in \mathcal{T}_X$. Then $C(\alpha\beta) \subseteq C(\alpha) \cup C(\beta)\alpha^{-1}$.*

Proof: For some $t \in \operatorname{Im}\alpha\beta$ and $x, y \in C(\alpha\beta)$, let $x, y \in t(\alpha\beta)^{-1}$ with $x \neq y$. Then $x\alpha\beta = y\alpha\beta$. Now if $x\alpha = y\alpha$ then $x, y \in C(\alpha)$; otherwise $x\alpha$, $y\alpha \in C(\beta)$ so that $x, y \in C(\beta)\alpha^{-1}$. Thus $C(\alpha\beta) \subseteq C(\alpha) \cup C(\beta)\alpha^{-1}$, as required. ∎

Now let F_A be the set of all A-Vagner maps in $\mathcal{T}_x$. Then we have

Lemma 1.3. *F_A is a subsemigroup of $\mathcal{T}_X$.*

Proof: First notice that for all $\alpha, \beta \in F_A$

$$(A\alpha)\beta = A\beta = A\ ,$$

and $\alpha\beta|_A$ is one-to-one if both $\alpha|_A$ and $\beta|_A$ are one-to-one. Moreover,

$$\begin{aligned} C(\alpha\beta)\alpha\beta &\subseteq C(\alpha)\alpha\beta \cup (C(\beta)\alpha^{-1})\alpha\beta \qquad \text{(by Lemma 1.2)} \\ &\subseteq A\beta \cup C(\beta)\beta \\ &\subseteq A \cup A \\ &= A\ . \end{aligned}$$

Hence $\alpha\beta \in F_A$, as required. ∎

Remark. Notice that if $A = \emptyset$, then F_A is the semigroup of one-to-one maps of $\mathcal{T}_x$.

Lemma 1.4. *Let $\alpha \in \mathcal{T}_X$. Then α is an idempotent if and only if for all $t \in \operatorname{Im}\alpha$, $t \in t\alpha^{-1}$, i.e., if and only if $F(\alpha) = \operatorname{Im}\alpha$.*

Proof: This statement is proved in [14] for the finite case and no essential use is made of the finiteness of X. ∎

Lemma 1.5. *Let $\alpha \in F_A$. Then the following statements are equivalent:*

(1) *α is an idempotent;*

(2) *$A \subseteq F(\alpha)$ and $x\alpha = x$ for all $x \notin C(\alpha)$.*

Proof:

(1) ⇒ (2) By Lemma 1.4, it is clear that for any idempotent $\alpha \in F_A$,

$$A = A\alpha \subseteq \operatorname{Im}\alpha = F(\alpha)\ .$$

Moreover, for all $y \in \operatorname{Im}\alpha\backslash A$, $y\alpha = y$ and $y\alpha^{-1} = \{x\}$ with $x \notin C(\alpha)$. Thus

$$x\alpha = y = y\alpha$$

which implies that $x = y = x\alpha$, for all $x \notin C(\alpha)$.

(2) ⇒ (1) Let $x \in C(\alpha)$. Then $x\alpha \in C(\alpha)\alpha \subseteq A \subseteq F(\alpha)$, so that $x\alpha^2 = x\alpha$. And since $x\alpha = x$ for all $x \notin C(\alpha)$ (by (2)), then $x\alpha^2 = x\alpha$ for all x. Thus α is an idempotent. ∎

In view of the remark made after Lemma 1.3, from this point onwards it is assumed that $A \neq \emptyset$.

Lemma 1.6. *F_A is a regular semigroup.*

Proof: Let $\alpha \in F_A$ and let a_0 be a fixed element of A. If $a \in A$, $a\alpha^{-1} \cap A \neq \emptyset$ since $A = A\alpha$. For each $a \in A$ choose an element b_a in $a\alpha^{-1} \cap A$ and for each

$y \in \operatorname{Im}\alpha\backslash A$, let $y\alpha^{-1} = \{x_y\}$. Now define $\alpha' \in \mathcal{T}_X$ by

$$a\alpha' = b_a \quad (a \in A)$$
$$y\alpha' = x_y \quad (y \in \operatorname{Im}\alpha\backslash A)$$
$$x\alpha' = a_0 \quad (x \in X\backslash \operatorname{Im}\alpha) \ .$$

Then clearly $\alpha\alpha'\alpha = \alpha$, $A\alpha' \subseteq A$ and $A \subseteq \{b_a\colon\ a \in A\} = A\alpha'$. Moreover, since $C(\alpha') = X\backslash \operatorname{Im}\alpha$, then

$$C(\alpha')\alpha' = (X\backslash \operatorname{Im}\alpha)\alpha' = \{a_0\} \ ,$$

and it now follows that $\alpha' \in F_A$. Hence F_A is regular. ∎

2 – Orthodox semigroups

Recall that an orthodox semigroup is a regular semigroup whose set of idempotents $E(S)$ forms a subsemigroup. For a detailed account of orthodox semigroups see [12, Chapter VI].

2.1 Green's relations

For the definitions of the Green's relations see, for example, [12]. It is now clear by Lemma 1.6 and [12, Proposition II.4.5 and Ex. II.10] that in the semigroup F_A, for α, $\beta \in F_A$

(2.1) $$(\alpha,\beta) \in \mathcal{L} \quad \text{iff} \quad \operatorname{Im}\alpha = \operatorname{Im}\beta \ ,$$

(2.2) $$(\alpha,\beta) \in \mathcal{R} \quad \text{iff} \quad \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1} \ ,$$

(2.3) $$(\alpha,\beta) \in \mathcal{LH} \quad \text{iff} \quad \operatorname{Im}\alpha = \operatorname{Im}\beta \ \text{ and } \ \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1} \ .$$

Moreover, if $(\alpha,\beta) \in \mathcal{D} = \mathcal{L} \circ \mathcal{R}$ ($= \mathcal{R} \circ \mathcal{L}$), then there exist $\delta \in F_A$ such that $\alpha\mathcal{L}\,\delta\mathcal{R}\,\beta$, so that

$$\operatorname{Im}\alpha = \operatorname{Im}\delta \quad \text{and} \quad \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1} \ .$$

However, $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$ implies that $C(\delta) = C(\beta)$, which in turn implies that $|\operatorname{Im}\delta\backslash A| = |\operatorname{Im}\beta\backslash A|$. Thus

$$|\operatorname{Im}\alpha\backslash A| = |\operatorname{Im}\delta\backslash A| = |\operatorname{Im}\beta\backslash A| \ .$$

Conversely, suppose that $|\operatorname{Im}\alpha\backslash A| = |\operatorname{Im}\beta\backslash A|$. Let θ be a bijection from $\operatorname{Im}\beta\backslash A$ onto $\operatorname{Im}\alpha\backslash A$, and define δ (in F_A) by

$$x\delta = \begin{cases} x\beta\theta & (\text{if } x \notin C(\beta) \cup A) \\ x\beta \in a & (\text{if } x \in C(\beta) \cup A)\ . \end{cases}$$

Then, clearly δ and β coincide on $C(\beta) \cup A$, and since δ is one-to-one otherwise, it follows that

$$C(\delta)\delta = C(\beta)\beta \subseteq A = A\beta = A\delta\ .$$

Moreover, $\operatorname{Im}\delta = \operatorname{Im}\alpha$ and $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$, so that $\alpha \mathcal{L}\, \delta \mathcal{R}\, \beta$, i.e., $\alpha \mathcal{D} \beta$. Thus

$$(2.4) \qquad (\alpha, \beta) \in \mathcal{D} \quad \text{iff} \quad |\operatorname{Im}\alpha\backslash A| = |\operatorname{Im}\beta\backslash A|\ .$$

Also, if $(\alpha, \beta) \in \mathcal{J}$, then there exist $\delta_1, \delta_2, \gamma_1, \gamma_2 \in F_A$ such that

$$\alpha = \delta_1\beta\delta_2 \quad \text{and} \quad \beta = \gamma_1\alpha\gamma_2\ .$$

However, if $|\operatorname{Im}\alpha\backslash A| < |\operatorname{Im}\beta\backslash A|$, then

$$|\operatorname{Im}\gamma_1\alpha\backslash A| < |\operatorname{Im}\beta\backslash A| \quad (\text{since } \operatorname{Im}\gamma_1\alpha \subseteq \operatorname{Im}\alpha)\ ,$$

and hence

$$|\operatorname{Im}\beta\backslash A| = |\operatorname{Im}\gamma_1\alpha\gamma_2\backslash A| < |\operatorname{Im}\beta\backslash A| \quad (\text{since } A \subseteq \operatorname{Im}\gamma_1\alpha \cap \operatorname{Im}\gamma_2)\ ,$$

which is a contradiction. Thus, on the semigroup F_A, $\mathcal{D} = \mathcal{J}$.

Lemma 2.1.1. *Every $\mathcal{R}$-class of F_A contains exactly one idempotent.*

Proof: Let ε, η be two $\mathcal{R}$-related idempotents in F_A, then $C(\varepsilon) = C(\eta)$. Moreover, for all $x \notin C(\varepsilon)$

$$x\varepsilon = x = x\eta \quad (\text{by Lemma 1.5})\ ,$$

and for all $x \in C(\varepsilon)$

$$x\varepsilon, x\eta \in A \subseteq \operatorname{Im}\varepsilon \cap \operatorname{Im}\eta\ ,$$

so that $\operatorname{Im}\varepsilon = \operatorname{Im}\eta$. Thus, $(\varepsilon, \eta) \in \mathcal{L} \cap \mathcal{R} = \mathcal{H}$, and it follows that $\varepsilon = \eta$, as required. ∎

A regular semigroup is said to be $\mathcal{R}$-unipotent if each of its principal right ideals has a unique idempotent generator. (In other words each $\mathcal{R}$-class contains a unique idempotent.) Equivalently, an $\mathcal{R}$-unipotent semigroup is a regular semigroup S in which $E(S)$ is a *left regular band*; i.e., $efe = ef$, for all e, $f \in E(S)$.

An *$\mathcal{L}$-unipotent* semigroup is defined dually. Notice that an $\mathcal{R}(\mathcal{L})$-unipotent semigroup is necessarily orthodox. $\mathcal{R}(\mathcal{L})$-unipotent semigroups have been studied, for example, by Edwards [4] and Venkatesan [18]. $\mathcal{R}$-unipotent semigroups are also known as *left inverse* semigroups in the literature. In view of the above remarks, by Lemmas 1.6 and 2.1.1, we obtain

Theorem 2.1.2. *Let F_A be the semigroup of all A-vagner maps of $\mathcal{T}_X$. Then F_A is an $\mathcal{R}$-unipotent semigroup.*

2.2 The finite case

For any relation $\mathcal{K}$ we shall denote the $\mathcal{K}$-class containing α by K_α. Let $X = \{1, \ldots, n\}$ and $A = \{a_1, a_2, \ldots, a_k\} \subseteq X$ for some $1 \leq k \leq n$. It follows from (2.3) that if $|\operatorname{Im}\alpha| = r$ then there are $k!\,(r-k)!$ elements in H_α. To see this notice that there must be $r-k$ singleton $(\alpha \circ \alpha^{-1})$-classes outside $A\alpha^{-1}$ and that these must map to the $r-k$ elements of $\operatorname{Im}\alpha$ outside A. Hence there are $(r-k)!$ ways of mapping those elements. The remaining $(\alpha \circ \alpha^{-1})$-classes, k in number, all intersect A and must map onto A in a one-one fashion. There are thus $k!$ possibilities. It now follows that $|H_\alpha| = k!\,(r-k)!$.

And from (2.1), we deduce that the number of $\mathcal{L}$-classes in D_α is equal to the number of $(r-k)$-element subsets of $X \backslash A$ (for the image set must contain A). Hence there are $\binom{n-k}{r-k}$ $\mathcal{L}$-classes in D_α. However, the number of $\mathcal{R}$-classes in D_α is less obvious and the next lemma provides the answer.

Lemma 2.2.1. *Let $\alpha \in F_A$ such that $|\operatorname{Im}\alpha| = r$. Then there are $k^{n-r}\binom{n-k}{r-k}$ $\mathcal{R}$-classes in D_α.*

Proof: Since there are $r-k$ elements not in $C(\alpha) \cup A$, then the number of $\mathcal{R}$-classes in D_α is equivalent to the number of partitions of X_n into r subsets subject to the conditions that there are $r-k$ singletons (from $X_n \backslash A$) and of the remaining k subsets each must contain an element of A. However, there are $\binom{n-k}{r-k}$ ways of choosing the $r-k$ singletons from $X_n \backslash A$ and there are k^{n-r} ways of partitioning the remaining $n-r+k$ elements into k subsets, with each subset containing an element of A. Hence there are

$$k^{n-r}\binom{n-k}{r-k}$$

number of partitions as required. ∎

Evidently, we now have

Lemma 2.2.2. *Let $\alpha \in F_A$ such that $|\operatorname{Im}\alpha| = r$. Then*

$$|J_\alpha| = k^{n-r} \binom{n-k}{r-k}^2 k!\,(r-k)!\ .$$

Theorem 2.2.3. *Let F_A be the semigroup of all A-Vagner maps of $\mathcal{T}_X$. Then*

$$|F_A| = \sum_{r=k}^{n} k^{n-r} \binom{n-k}{r-k}^2 k!\,(r-k)!\ .$$

Theorem 2.2.4. *Let F_A be the semigroup of all A-Vagner maps of $\mathcal{T}_X$. Then*

$$|E(F_A)| = \sum_{r=k}^{n} k^{n-r} \binom{n-k}{r-k} = (k+1)^{n-k}\ .$$

Proof: It follows directly from Lemmas 2.1.1 and 2.2.1. ∎

3 – Irregular quasi-adequate semigroups

Let X be a well ordered set and let A be a (non empty) subset of X. Also, let $\mathcal{T}_X$ be the full transformation semigroup on X, and let F_A be the semigroup of all A-Vagner maps of $\mathcal{T}_X$. Consider the subset of F_A denoted by F_A^-

$$F_A^- = \Big\{\alpha \in F_A\colon\ (\forall\, x \in X)\ x\alpha \le x \ \text{ and } \ A \subseteq F(\alpha)\Big\} \tag{3.1}$$

consisting of all order-decreasing maps of F_A for which $A \subseteq F(\alpha)$. Then clearly F_A^- is a subsemigroup of F_A, since for all $\alpha, \beta \in F_A^-$

$$(x\alpha)\beta \le x\alpha \le x \quad \text{ and } \quad A \subseteq F(\alpha) \cap F(\beta) \subseteq F(\alpha\beta)\ .$$

Notice that if A is finite then $A = A\alpha$ and $x\alpha \le x$ (for all $x \in X$) implies $A \subseteq F(\alpha)$.

3.1 Green's and starred Green's relations

Lemma 3.1.1. *F_A^- is $\mathcal{R}$-trivial.*

Proof: Let $(\alpha, \beta) \in \mathcal{R}$. Then there exist δ, γ in F_A^- such that

$$\alpha\delta = \beta \quad \text{ and } \quad \beta\gamma = \alpha\ .$$

However, for all $x \in X$

$$x\beta = x\alpha\delta \leq x\alpha \quad \text{and} \quad x\alpha = x\beta\gamma \leq x\beta$$

so that $x\alpha = x\beta$. Thus $\alpha = \beta$. ∎

Lemma 3.1.2. *Let $\alpha, \beta \in F_A^-$. Then the following are equivalent:*

(1) $(\alpha, \beta) \in \mathcal{L}$;

(2) $\operatorname{Im}\alpha = \operatorname{Im}\beta$ *and* $z\alpha^{-1} = z\beta^{-1}$ *for all* $z \in \operatorname{Im}\alpha \backslash A$.

Proof: Let $(\alpha, \beta) \in \mathcal{L}$. Then certainly $\operatorname{Im}\alpha = \operatorname{Im}\beta$ and there exists δ, γ in F_A^- such that

$$\delta\alpha = \beta \quad \text{and} \quad \gamma\beta = \alpha \ .$$

Let $z \in \operatorname{Im}\alpha \backslash A = \operatorname{Im}\beta \backslash A$ and let $y = z\alpha^{-1}$. Then

$$y\gamma\beta = y\alpha = z$$

and so

$$y\gamma = z\beta^{-1} \ .$$

Hence

$$y \geq y\gamma = z\beta^{-1} \ .$$

That is, $z\alpha^{-1} \geq z\beta^{-1}$, and we can similarly show that

$$z\beta^{-1} \geq z\alpha^{-1} \ .$$

Thus

$$z\beta^{-1} = z\alpha^{-1} \ ,$$

as required.

Conversely, suppose that $\operatorname{Im}\alpha = \operatorname{Im}\beta$ and $z\alpha^{-1} = z\beta^{-1}$ for all $z \in \operatorname{Im}\alpha \backslash A$. Let δ, γ be defined by

$$x\delta = \begin{cases} x\alpha & (\text{if } x \in A\alpha^{-1}) \\ x & (\text{otherwise}) \end{cases}$$

$$x\gamma = \begin{cases} x\beta & (\text{if } y \in A\beta^{-1}) \\ x & (\text{otherwise}) \ . \end{cases}$$

Then, clearly δ and α coincide on $A\alpha^{-1}$, and since

$$C(\alpha) \subseteq C(\alpha)\alpha\alpha^{-1} \subseteq A\alpha^{-1} \quad \text{and} \quad A \subseteq A\alpha^{-1} \ ,$$

it follows that

$$C(\delta)\delta = C(\alpha)\alpha \subseteq A = A\alpha = A\delta \ .$$

Thus $\delta \in F_A^-$. Similarly, $\delta \in F_A^-$ and $\alpha = \delta\beta$, $\beta = \gamma\alpha$. Hence $(\alpha, \beta) \in \mathcal{L}$. ∎

Some immediate consequences of Lemma 3.1.1 are:

Corollary 3.1.3. *On the semigroup F_A^-, $\mathcal{H} = \mathcal{R} = \imath$, the indentity and $\mathcal{L} = \mathcal{D}$.*

Corollary 3.1.4. *F_A^- is either a band or an irregular semigroup.*

Proof: Let x be a regular element of F_A^-. Then there exists x' in F_A^- such that $x = xx'x$ and $(x, xx') \in \mathcal{R}$, so that $x = xx' \in E(F_A^-)$. Thus the only regular elements of F_A^- are its idempotents. ∎

Now in view of the above Corollary it is natural to ask: when is F_A^- a band? To investigate this, first we introduce some new notations and record some basic results about F_A^-. Let us denote by A^- and A^+ the sets

$$\Big\{x \in X\colon\ (\exists\, a \in A)\ x \leq a\Big\}\,, \qquad \Big\{x \in X\colon\ (\exists\, a \in A)\ x \geq a\Big\}$$

respectively. Then clearly $A \subseteq A^- \cap A^+$ and $A^- \cup A^+ = X$.

Lemma 3.1.5. *Let $\alpha \in F_A^-$. Then $A^-\backslash A^+ \subseteq F(\alpha)$.*

Proof: If $A^-\backslash A^+ \not\subseteq F(\alpha)$, then there is a smallest element $c \in A^-\backslash A^+$ such that $c\alpha \neq c$. Then, as $\alpha \in F_A^-$, we have $c\alpha < c$ so that $c\alpha \in A^-\backslash A^+$ and by the choice of c we have $(c\alpha)\alpha = c\alpha$. Hence $c \in C(\alpha)$ and as $\alpha \in F_A$, we have $c\alpha \in A$, a contradiction. ∎

Lemma 3.1.6. *F_A^- is a band if and only if $|A^+\backslash A| \leq 1$.*

Proof: First observe that if $A^+\backslash A = \emptyset$, then $A = A^+$ so that

$$\begin{aligned} X &= A^- \cup A^+ = A^- = (A^-\backslash A) \cup A = (A^-\backslash A^+) \cup A \\ &\subseteq F(\alpha) \quad \text{(by Lemma 3.1.5)} \ . \end{aligned}$$

Thus F_A^- is the trivial semigroup. Next, if $A^+\backslash A = \{y\}$, then $y\alpha = y$ or $y\alpha \in A$, and hence $(y\alpha)\alpha = y\alpha$. Moreover, $x\alpha = x$ for all $x \in (A^-\backslash A^+) \cup A = (A^+\backslash A)'$, by Lemma 3.1.5, so that $\alpha^2 = \alpha$. Thus F_A^- is a band if $|A^+\backslash A| \leq 1$.

Conversely, suppose that F_A^- is a band and $|A^+\backslash A| \geq 2$, then there exist x, $y \in A^+\backslash A$ with $x \neq y$ such that $x > y$. Now choose an element $a_y \in A$ for which $a_y \leq y$ and define β in F_A^- by

$$x\beta = y\,, \qquad y\beta = a_y \qquad \text{and} \qquad z\beta = z \ \ \text{(otherwise)} \ .$$

Then, clearly β is a non-idempotent element in F_A^-, which is a contradiction as F_A^- is a band. Thus if F_A^- is a band then $|A^+ \backslash A| \leq 1$. Hence the proof. ∎

Recall from [9] that on a semigroup S the relation $\mathcal{L}^*$ ($\mathcal{R}^*$) is defined by the rule that $(a, b) \in \mathcal{L}^*$ ($\mathcal{R}^*$) if and only if the elements a, b are related by the Green's relation $\mathcal{L}$ ($\mathcal{R}$) in some oversemigroup of S. The join of the equivalences $\mathcal{L}^*$ and $\mathcal{R}^*$ is denotes by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. A semigroup S in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent is called *abundant*. Of course regular semigroups are abundant (and in this case $\mathcal{K}^* = \mathcal{K}$, for $\mathcal{K}$ any of $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ or $\mathcal{D}$). The starred relations play a role in the theory of abundant semigroups analogous to that of Green's relations in the theory of regular semigroups. As in [9] we introduce $*$-ideals to obtain the starred analogue of the Green's relation $\mathcal{J}$.

The $\mathcal{L}^*$-class containing the element a is denoted by L_a^*. The corresponding notation is used for the class of the other relations. We now define a *left (right) $*$-ideal* of a semigroup S to be a left (right) ideal I of S for which $L_a^* \subseteq I$ ($R_a^* \subseteq I$), for all elements a of I. A subset I of S is a *$*$-ideal* if it is both a left $*$-ideal and a right $*$-ideal. The principal $*$-ideal $J^*(a)$ generated by the element a of S is the intersection of all $*$-ideals of S to which a belongs. The relation $\mathcal{J}^*$ is defined by the rule that: $a\mathcal{J}^*b$ *if and only if* $J^*(a) = J^*(b)$. Again, for a regular semigroup $S\mathcal{J} = \mathcal{J}^*$. In the case of ambiguity we denote a relation $\mathcal{K}$ on S by $\mathcal{K}_S$.

Before we characterize the starred Green's relations we need the following definition and lemmas:

Definition 3.1.7. Let S be a semigroup and let U be a subsemigroup of S. Then U will be called an *inverse ideal* of S if for all $u \in U$, there exists $u' \in S$ such that $uu'u = u$ and uu', $u'u \in U$.

Lemma 3.1.8. *Every inverse ideal U of a semigroup S is abundant.*

Proof: Since for all $u \in U$

$$(u, u'u) \in \mathcal{L}_S \quad \text{and} \quad (u, uu') \in \mathcal{R}_S$$

it follows that

$$(u, u'u) \in \mathcal{L}_U^* \quad \text{and} \quad (u, uu') \in \mathcal{R}_U^* .$$

Hence every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of U contains an idempotent, since uu', $u'u$ are idempotents in U. Thus U is abundant. ∎

Again, recall from [9] that for any subsemigroup U of S

$$\mathcal{L}_S^* \cap (U \times U) \subseteq \mathcal{L}_U^* \quad \text{and} \quad \mathcal{R}_S^* \cap (U \times U) \subseteq \mathcal{R}_U^* .$$

And for any regular elements a, b of a semigroup S

$$(a,b) \in \mathcal{K} \quad \text{iff} \quad (a,b) \in \mathcal{K}^* \ ,$$

where $\mathcal{K}$ is any of $\mathcal{H}$, $\mathcal{L}$ or $\mathcal{R}$. Moreover, in any semigroup S, $\mathcal{K} \subseteq \mathcal{K}^*$. Hence we have

Lemma 3.1.9. *Let U be an inverse ideal of a semigroup S. Then*

(1) $\mathcal{L}^*_U = \mathcal{L}_S \cap (U \times U)$;

(2) $\mathcal{R}^*_U = \mathcal{R}_S \cap (U \times U)$;

(3) $\mathcal{H}^*_U = \mathcal{H}_S \cap (U \times U)$.

Proof:

(1) Certainly,

$$\mathcal{L}_S \cap (U \times U) \subseteq \mathcal{L}^*_U \ .$$

Conversely, suppose that $(a,b) \in \mathcal{L}^*_U$ and a', b' are elements in S such that $aa'a = a$, $bb'b = b$ and aa', $a'a$, bb', $b'b \in U$. Then

$$(a'a, a) \in \mathcal{L}_S \quad \text{and} \quad (b, b'b) \in \mathcal{L}_S \ ,$$

which implies that

$$(a'a, a) \in \mathcal{L}^*_U \quad \text{and} \quad (b, b'b) \in \mathcal{L}^*_U \ .$$

And, by transitivity

$$(a'a, b'b) \in \mathcal{L}^*_U \ ,$$

which is equivalent to

$$(a'a, b'b) \in \mathcal{L}_U \ .$$

Now, since $\mathcal{L}_U \subseteq \mathcal{L}_S \cap (U \times U)$, then

$$(a'a, b'b) \in \mathcal{L}_S$$

and hence,

$$(a,b) \in \mathcal{L}_S \ .$$

So that

$$\mathcal{L}^*_U \subseteq \mathcal{L}_S \cap (U \times U) \ ,$$

and the result follows.

(2) The proof is similar to that of (1).

(3) This is a simple set-theoretic consequence of (1) and (2). ∎

Corollary 3.1.10. *If U is an inverse ideal of a semigroup S, then*

$$\mathcal{L}^*_U = \mathcal{L}^*_S \cap (U \times U), \quad \mathcal{R}^*_U = \mathcal{R}^*_S \cap (U \times U) \quad \text{and} \quad \mathcal{H}^*_U = \mathcal{H}^*_S \cap (U \times U) \ .$$

Lemma 3.1.11. *F_A^- is an inverse ideal of F_A.*

Proof: Let $\alpha \in F_A^-$. Notice that for all $t \in \operatorname{Im}\alpha \backslash A$

$$|t\alpha^{-1}| = 1 \ ,$$

and for all $t \in A$, if $x \in t\alpha^{-1}$ then

$$x \geq x\alpha = t = t\alpha \ .$$

Thus $\min(t\alpha^{-1})$ exists for all $t \in \operatorname{Im}\alpha$. Now, let $a_0 = \min A$ and define α' by

$$t\alpha' = x_t = \min(t\alpha^{-1}) \ \ (t \in \operatorname{Im}\alpha) \, , \qquad y\alpha' = a_0 \ \ (\text{otherwise}) \ .$$

Then, it is clear that $A \subseteq F(\alpha')$ and $C(\alpha') \cap \operatorname{Im}\alpha = \emptyset$. Thus $C(\alpha')\alpha' = \{a_0\}$ and $A\alpha' = A$. It now follows that $\alpha' \in F_A$ and $\alpha\alpha'\alpha = \alpha$. (However notice that α' need not be a decreasing map.) Also,

$$\begin{aligned} C(\alpha\alpha')\alpha\alpha' &\subseteq C(\alpha) \cdot \alpha\alpha' \cup C(\alpha')\alpha^{-1} \cdot \alpha\alpha' \quad \text{(by Lemma 1.2)} \\ &\subseteq A\alpha' \cup A \\ &= A \ , \end{aligned}$$

$$\begin{aligned} C(\alpha'\alpha)\alpha'\alpha &\subseteq C(\alpha') \cdot \alpha'\alpha \cup C(\alpha)(\alpha')^{-1} \cdot \alpha'\alpha \quad \text{(by Lemma 1.2)} \\ &\subseteq A\alpha \cup A \\ &= A \ , \end{aligned}$$

and $A \subseteq F(\alpha) \cap F(\alpha') = F(\alpha') \cap F(\alpha) \subseteq F(\alpha\alpha') \cap F(\alpha'\alpha)$. Moreover, since for all $x \in X$

$$x\alpha\alpha' = (x\alpha)\alpha' = x_{x\alpha} = \min(x\alpha\alpha^{-1}) \leq x \ ,$$

it follows that $\alpha\alpha' \in F_A^-$. To see that $\alpha'\alpha \in F_A^-$, first notice that if $y \notin \operatorname{Im}\alpha$, then $y \notin A^- \backslash A^+$, by Lemma 3.1.5, and hence $y \in A^+ \backslash \operatorname{Im}\alpha$. Thus for all $t \in \operatorname{Im}\alpha$

$$t\alpha'\alpha = x_t\alpha = t \qquad (\text{since } x_t \in t\alpha^{-1}) \ ,$$

for all $y \in A^+ \backslash \operatorname{Im}\alpha$

$$y\alpha'\alpha = a_0\alpha = a_0 < y \ .$$

Hence the proof. ∎

A *quasi-adequate* semigroup is an abundant semigroup in which $E(S)$ is a subsemigroup. Thus the class of quasi-adequate semigroups includes all orthodox semigroups. By contrast with the regular case, an abundant semigroup in which each of its principal right $*$-ideals has a unique idempotent generator need not be quasi-adequate. In fact S_n^-, the semigroup of all decreasing full transformations of $X_n = \{1, \ldots, n\}$ is an abundant semigroup whose each of its principal right $*$-ideals has a unique idempotent generator ([15, Lemma 2.6 and Theorem 2.7]) but it is idempotent-generated ([15, Theorem 1.4]). El-Qallali [5] defines an $\mathcal{R}^*$-*unipotent* semigroup to be a quasi-adequate semigroup in which each of its principal right $*$-ideal has a unique idempotent generator. In other words, an R^*-unipotent semigroup is a quasi-adequate semigroup in which each $\mathcal{R}^*$-class contains a unique idempotent. Also, El-Qallali showed that the latter condition is equivalent to having a right regular band of idempotents ([5, Lemma 1.1]). By Lemmas 2.1.1, 3.1.8, 3.1.9 and 3.1.11 we have

Theorem 3.1.12. *Let F_A^- be as defined in (3.1). Then F_A^- is an $\mathcal{R}^*$-unipotent semigroup.*

By equation (2.1) and (2.2), Corollary 3.1.10 and Lemma 3.1.11 we deduce

Lemma 3.1.13. *Let $(\alpha, \beta) \in F_A^-$. Then*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{Im}\alpha = \operatorname{Im}\beta$;

(2) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$.

To characterize $\mathcal{D}^*$ on F_A^- we let $T = (A^- \backslash A^+) \cup A$ and define a relation $\mathcal{K}$ on F_A^- by the rule

$$(\alpha, \beta) \in \mathcal{K} \quad \text{iff} \quad |\operatorname{Im}\alpha \backslash T| = |\operatorname{Im}\beta \backslash T| \ .$$

Then, clearly, $\mathcal{L}^* \subseteq \mathcal{K}$ and $\mathcal{R}^* \subseteq \mathcal{K}$, since $T \subseteq F(\alpha) \cap F(\beta) \subseteq \operatorname{Im}\alpha \cap \operatorname{Im}\beta$. Also, $\mathcal{D}^* \subseteq \mathcal{K}$, since $\mathcal{D}^*$ is the smallest equivalence containing both $\mathcal{L}^*$ and $\mathcal{R}^*$. We now have

Lemma 3.1.14. $\mathcal{K} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* = \mathcal{D}^*$.

Proof: Suppose that $(\alpha, \beta) \in \mathcal{K}$ so that $|\operatorname{Im}\alpha \backslash T| = |\operatorname{Im}\beta \backslash T|$ and let θ be a bijection from $\operatorname{Im}\alpha$ onto $\operatorname{Im}\beta$ such that, for all $t \in T$, $t\theta = t$. Define $\delta, \gamma \in \mathcal{T}_X$ as follows:

$$x\delta = \min(x\alpha, x\alpha\theta) \ ,$$

$$x\gamma = \min(x\beta, x\beta\theta^{-1}) \ .$$

Then, it is clear that $C(\delta) = C(\alpha)$, and for all $x \in C(\delta)$, $x\delta = \min(x\alpha, x\alpha\theta) = x\alpha \in A$, so that $C(\delta)\delta \subseteq A = A\delta$. Similarly, $C(\gamma)\gamma \subseteq A = A\gamma$. Moreover, δ, γ are decreasing maps for which $\operatorname{Im}\delta = \operatorname{Im}\gamma$, $\delta\circ\delta^{-1} = \alpha\circ\alpha^{-1}$ and $\gamma\circ\gamma^{-1} = \beta\circ\beta^{-1}$. Thus $\delta, \gamma \in F_A^-$ and $\alpha\mathcal{R}^*\,\delta\mathcal{L}^*\,\gamma\mathcal{R}^*\beta$, by Lemma 3.1.13. Thus

$$\mathcal{K} \subseteq \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* \ .$$

Conversely, let $(\alpha, \beta) \in \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*$. Then, there exist $\delta, \gamma \in F_A^-$ such that $\alpha\mathcal{R}^*\,\delta\mathcal{L}^*\,\gamma\mathcal{R}^*\beta$. Since T is contained in $\operatorname{Im}\alpha$, $\operatorname{Im}\beta$, $\operatorname{Im}\gamma$ and $\operatorname{Im}\delta$ we have

$$|\operatorname{Im}\alpha\backslash T| = |\operatorname{Im}\delta\backslash T|, \quad \operatorname{Im}\delta\backslash T = \operatorname{Im}\gamma\backslash T \quad \text{and} \quad |\operatorname{Im}\gamma\backslash T| = |\operatorname{Im}\beta\backslash T| \ ,$$

so that

$$|\operatorname{Im}\alpha\backslash T| = |\operatorname{Im}\beta\backslash T| \ .$$

Thus

$$\mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* \subseteq \mathcal{K} \ .$$

On the other hand, let $(\alpha, \beta) \in \mathcal{K}$ and let $a_0 = \min A$. Also, let $M(\alpha) = \{\max(x\alpha, x\alpha\theta) \colon x \in X\}$ and define δ', $\gamma' \in \mathcal{T}_X$ as follows:

$$x\delta' = \begin{cases} x\alpha & (\text{if } x \in M(\alpha)) \\ a_0 & (\text{otherwise}) \ , \end{cases}$$

$$x\gamma' = \begin{cases} x\alpha\theta & (\text{if } x \in M(\alpha)) \\ a_0 & (\text{otherwise}) \ . \end{cases}$$

Notice that, $A \subseteq T \subseteq M(\alpha)$ and $C(\delta') \cap M(\alpha) = \emptyset$. Thus $C(\delta') \subseteq X\backslash M(\alpha)$, so that $C(\delta')\delta' = \{a_0\}$. Similarly, $C(\gamma')\gamma' = \{a_0\}$. Moreover, δ', γ' are decreasing maps for which $\operatorname{Im}\alpha = \operatorname{Im}\delta'$, $\delta \circ (\delta')^{-1} = \gamma \circ (\gamma')^{-1}$ and $\operatorname{Im}\gamma' = \operatorname{Im}\beta$. Thus $\delta', \gamma' \in F_A^-$ and $\alpha\mathcal{L}^*\,\delta'\mathcal{R}^*\,\gamma'\mathcal{L}^*\beta$, by Lemma 3.1.13. Thus

$$\mathcal{K} \subseteq \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \ .$$

Similarly (from above), we can show that

$$\mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{K} \ .$$

And finally, from the inequality

$$\mathcal{D}^* \subseteq \mathcal{K} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{D}^* \ ,$$

we deduce the result of the lemma. ∎

The following lemma is essential to our next investigation about the properties of $\mathcal{J}^*$.

Lemma 3.1.15 ([9, Lemma 1.7]). *Let a be an element of a semigroup S. Then $b \in J^*(a)$ if and only if there are elements $a_0, a_1, \dots, a_n \in S$, $x_1, \dots, x_n$, $y_1, \dots, y_n \in S^1$ such that $a = a_0$, $b = a_n$, and $(a_i, x_i a_{i-1} y_i) \in \mathcal{D}^*$ for $i = 1, \dots, n$.*

We immediately have:

Lemma 3.1.16. *Let $\alpha \in J^*(\beta)$. Then $|\operatorname{Im}\alpha \backslash T| \leq |\operatorname{Im}\beta \backslash T|$.*

Proof: Let $\alpha \in J^*(\beta)$. Then, by Lemma 3.1.15, there exist $\beta_0, \beta_1, \dots, \beta_n$, $\delta_1, \dots, \delta_n$, $\gamma_1, \dots, \gamma_n \in F_A^-$ such that $\beta = \beta_0$, $\alpha = \beta_n$, and $(\beta_i, \delta_i \beta_{i-1} \gamma_i) \in \mathcal{D}^*$ for $i = 1, \dots, n$. However, by Lemma 3.1.14, this implies that

$$|\operatorname{Im}\beta_i \backslash T| = |\operatorname{Im}(\delta_i \beta_{i-1} \gamma_i) \backslash T| \leq |\operatorname{Im}\beta_{i-1} \backslash T|$$

so that

$$|\operatorname{Im}\alpha \backslash T| \leq |\operatorname{Im}\beta \backslash T| \ ,$$

as required. ∎

Thus we now have the final result of this section:

Lemma 3.1.17. *On the semigroup F_A^-, $\mathcal{D}^* = \mathcal{J}^*$.*

Proof: Notice we need only show that $\mathcal{J}^* \subseteq \mathcal{D}^*$ (since $\mathcal{D}^* \subseteq \mathcal{J}^*$). So, suppose that $(\alpha, \beta) \in \mathcal{J}^*$, then $J^*(\alpha) = J^*(\beta)$, so that $\alpha \in J^*(\beta)$ and $\beta \in J^*(\alpha)$. However, by Lemma 3.1.16, this implies that

$$|\operatorname{Im}\alpha \backslash T| \leq |\operatorname{Im}\beta \backslash T| \quad \text{and} \quad |\operatorname{Im}\beta \backslash T| \leq |\operatorname{Im}\alpha \backslash T|$$

so that

$$|\operatorname{Im}\alpha \backslash T| = |\operatorname{Im}\beta \backslash T| \ .$$

Thus, by Lemma 3.1.14,

$$\mathcal{J}^* \subseteq \mathcal{D}^* \ ,$$

as required. ∎

3.2 The finite case

We aim to find a formula for the order of the semigroup F_A^- in the case where $A = \{1, \dots, k\}$ and $X = \{1, \dots, n\}$. Let

$$J^*(n, r) = \left| \left\{ \alpha \in F_A^- : \ |\operatorname{Im}\alpha| = r \right\} \right| \ .$$

Then $J^*(n,k) = k^{n-k}$, $J^*(n,n) = 1$ and $J^*(n,r) = 0$ if $r = 0$ or $n < r$ or $r < k$.

Lemma 3.2.1. $J^*(n,r) = k\,J^*(n-1,r) + (n-r+1)\,J^*(n-1,r-1)$.

Proof: Maps α for which $|\operatorname{Im}\alpha| = r$ divide naturally into two classes depending upon whether

$$\operatorname{Im}(\alpha\,|\,\{1,\ldots,n-1\}) = \operatorname{Im}\alpha \qquad (1)$$

or

$$\operatorname{Im}(\alpha\,|\,\{1,\ldots,n-1\}) \subset \operatorname{Im}\alpha \qquad (2)\ .$$

In case (1), n must map to one of the k elements in A, and so there are $k\,J^*(n-1,r)$ elements of this kind. In case (2), $|\operatorname{Im}(\alpha\,|\,\{1,\ldots,n-1\})| = r-1$ and n must map to one of the $n-r+1$ elements not in $\operatorname{Im}(\alpha\,|\,\{1,\ldots,n-1\})$. Hence there are $(n-r+1)\,J^*(n-1,r-1)$ elements of this kind. Thus,

$$J^*(n,r) = k\,J^*(n-1,r) + (n-r+1)\,J^*(n-1,r-1)\ ,$$

as required. ∎

Recall that the *Stirling number of the second kind* denoted by $S(n,k)$ is usually defined as

$$S(n,1) = 1 = S(n,n) \quad \text{and} \quad S(n,k) = S(n-1,k-1) + k\,S(n-1,k)\ ,$$

where n, k are natural numbers such that $n \geq k$.

Lemma 3.2.2. $J^*(n,r) = k^{n-r}\,S(n-k+1,n-r+1) \quad (n \geq r \geq k)$.

Proof: Certainly the result is true when $n = k$. Suppose that $k < n$ and that the result is true for all s such that $k \leq s \leq n-1$. Consider $J^*(n,r)$. Clearly the result is true if $r = n$ or $r = k$. Hence we may assume that $k < r < n$. We have

$$J^*(n,r) = k\,J^*(n-1,r) + (n-r+1)\,J^*(n-1,r-1)$$

so that using the induction hypothesis,

$$\begin{aligned} J^*(n,r) &= k \cdot k^{n-r-1}\,S(n-k,n-r) + (n-r+1)\cdot k^{n-r}\,S(n-k,n-r+1) \\ &= k^{n-r}\left\{S(n-k,n-r) + (n-r+1)\,S(n-k,n-r+1)\right\} \\ &= k^{n-r}\,S(n-k+1,n-r+1) \end{aligned}$$

as required. ∎

Then we immediately have:

Theorem 3.2.3. *Let F_A^- be as defined in (3.1). Then*

$$|F_A^-| = \sum_{r=-k}^{n} k^{n-r}\, S(n-k+1, n-r+1) \ .$$

Theorem 3.2.4. *Let F_A^- be as defined in (3.1). Then*

$$|E(F_A^-)| = \sum_{r=k}^{n} k^{n-r} \binom{n-k}{r-k} = (k+1)^{n-k} \ .$$

Proof: The result will follow from Theorem 2.2.4 if we show that $E(F_A^-) = E(F_A)$. Clearly

$$E(F_A^-) \subseteq E(F_A) \ .$$

Conversely, suppose that $\varepsilon \in E(F_A)$. Then $a\varepsilon = a$, for all $a \in A$, by Lemma 1.5. Since $A = \{1, \dots, k\}$, then for all $x \in C(\varepsilon)\backslash A$

$$x\varepsilon \leq k < x \ .$$

Also, by Lemma 1.5, $x\varepsilon = x$ for all $x \notin C(\varepsilon)$. Thus, for any $x \in X$, $x\varepsilon \leq x$ and $A \subseteq F(\alpha)$. Thus $\varepsilon \in F(F_A^-)$. Therefore,

$$E(F_A) \subseteq E(F_A^-) \ .$$

Hence $E(F_A) = E(F_A^-)$ as required. ∎

Remark. Notice that F_A^- is isomorphic to $(I_{n-1}^-)^1$, the semigroup of order-decreasing partial one-to-one transformations on X_{n-1}, when $k = 1$. Thus Lemma 3.2.2 and Theorem 3.2.3 reduce to [2, Proposition 3.1 and Remark 3.6], when $k = 1$.

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