PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 4 – 1994

CONVOLUTION OPERATORS IN INFINITE DIMENSION

Nguyen Van Khue and Nguyen Dinh Sang

1 – Introduction

Let E be a complete convex bornological vector space (denoted by the letters b.v.s.). This means that E is an injective algebraic inductive limit of a family $\{E_i\}_{i\in I}$ of Banach spaces E_i, $i \in I$, such that for $i<j$ the canonical linear map from E_i to E_j is continuous. A subset of E is called bounded if it is contained and bounded in a Banach space E_i. We say that E is a Schwartz (resp. weak Schwartz) b.v.s. if the canonical map from E_i to E_j is compact (resp. weakly compact) for every $i<j$.

Given D a subset of E such that $D_i := D \cap E_i$ is open in E_i for every $i \in I$. A function f on D is said to be holomorphic if $f \mid D_i$ is holomorphic for every $i \in I$. By $H(D)$ we denote the space of holomorphic functions on D equipped with the compact-open topology, where as above a subset K of D is called compact if $K_i := K \cap E_i$ is compact. Consider the Fourier–Borel transformation

$$\mathcal{F}_D\colon\ H'(D) \to H(E^+)$$

given by

$$\mathcal{F}_D(\mu)(x^*) = \mu(\exp x^*) \quad \text{for} \ \ \mu \in H'(D) \ \ \text{and} \ \ x^* \in E \ ,$$

where $H'(D)$ denotes the dual space of $H(D)$ equipped with the compact-open topology and

$$E^+ = \Big\{ f \in H(E)\colon\ f \text{ is linear} \Big\} \ .$$

Equip $\operatorname{Im} \mathcal{F}_D$ the quotient topology via $\mathcal{F}_D$. For each $\alpha \in E^+$ define the translation operator τ_α on $H(E^+)$ by the form

$$\tau_\alpha(\phi)(x^*) = \phi(x^* + \alpha)$$

for $x^* \in E^+$ and $\phi \in H(E^+)$.

Received: June 26, 1992; *Revised*: September 17, 1993.

Since

$$\mathcal{F}_D\,\widetilde{\tau}_\alpha = \tau_\alpha\,\mathcal{F}_D\ ,$$

where $\widetilde{\tau}_\alpha\colon\ H'(D) \to H'(D)$ given by

$$(\widetilde{\tau}_\alpha \mu)(\varphi) = \mu(\varphi \exp\alpha)$$

for $\varphi \in H(D)$ and $\mu \in H'(D)$, it follows that

$$\tau_\alpha\colon\ \operatorname{Im}\mathcal{F}_D \to \operatorname{Im}\mathcal{F}_D$$

is continuous.

Now a continuous linear map $\theta:\ \operatorname{Im}\mathcal{F}_D \to \operatorname{Im}\mathcal{F}_D$ is called a convolution operator if it commutes with every translation.

2 – Statement of the results

In this note we always assume that E is a b.v.s. which is separated by E^+ and D is a subset of E such that $D \cap E_i$ is connected and open in E_i for every $i \in I$.

Existence Theorem. *Every non-zero convolution operator on* $\operatorname{Im}\mathcal{F}_D$ *is surjective.*

Approximation Theorem. *Let hold one of the following two conditions*

i) *D is balanced;*

ii) *D is polynomially convex and E is a weak Schwartz b.v.s. such that every E_i has the approximation property.*

Then every solution u of the homogeneous equation $\theta u = 0$ is a limit for the topology of $\operatorname{Im}\mathcal{F}_D$ *of solutions in $\mathcal{P}(E)\,\mathcal{E}\mathrm{xp}(D)$, where $\mathcal{P}(E)$ denotes the set of all continuous polynomials on E and* $\mathcal{E}\mathrm{xp}(D) = \operatorname{span}\{\exp(x)\colon\ x \in D\}$.

In the case E is a Schwartz b.v.s. such that every space E_i has the approximation property and D is a balanced convex open subset of E the above results have established by Colombeau and Perrot [3]. Some particular cases were proved by Boland [1], Dwyer [4], [5], [6] and Gupta [8].

3 – Proof

Let $T \in (\operatorname{Im}\mathcal{F}_D)'$, the dual space of $\operatorname{Im}\mathcal{F}_D$, equipped with the strong topology, and let $T^*\colon\ \operatorname{Im}\mathcal{F}_D \to \operatorname{Im}\mathcal{F}_D$ given by the form

$$(T^*\phi)(\alpha) = T(\tau_\alpha \phi) \quad \text{for}\ \ \phi \in \operatorname{Im}\mathcal{F}_D\ \ \text{and}\ \ \alpha \in E^+\ .$$

Lemma 1. *T^* is a convolution operator on $\operatorname{Im}\mathcal{F}_D$ and conversely each convolution operator on $\operatorname{Im}\mathcal{F}_D$ is a T^* for some T.*

Proof: First observe that $\mathcal{F}'_D\colon (\operatorname{Im}\mathcal{F}_D)' \to H''(D) = H(D)$ (algebraically). Define the continuous linear map U_T from $H'(D)$ to $H'(D)$ by

$$U_T(\mu)(\psi) = \mu(\mathcal{F}'_D(T)\,\psi)$$

for $\mu \in H'(D)$ and $\psi \in H(D)$.

We have

$$\begin{aligned}(T^*\mathcal{F}_D)(\mu)(\alpha) &= (T^*\mathcal{F}_D(\mu))(\alpha) = T(\tau_\alpha\,\mathcal{F}_D(\mu))\\ &= T(\mathcal{F}_D\,\widetilde{\tau}_\alpha(\mu)) = \mathcal{F}'_D(T)(\widetilde{\tau}_\alpha(\mu))\\ &= \mu(\mathcal{F}'_D(T)\exp\alpha) = (\mathcal{F}_D\,U_T(\mu))(\alpha)\end{aligned}$$

for all $\mu \in H'(D)$ and $\alpha \in E^+$.

Thus

$$T^*\,\mathcal{F}_D = \mathcal{F}_D\,U_T\ .$$

This yields the continuity of T^*.

Let $\mathcal{M}$ denote the algebra of all convolution operators on $\operatorname{Im}\mathcal{F}_D$ and let γ be the map from $\mathcal{M}$ to $(\operatorname{Im}\mathcal{F}_D)'$ given by

$$\gamma\colon\ \theta \mapsto (\phi \mapsto \theta^\phi(0))\ .$$

It is easy to see that

$$\gamma(T^*) = T \quad\text{ and }\quad (\gamma\theta)^* = \theta\ .$$

Hence the map $T \mapsto T^*$ is a bijection between $(\operatorname{Im}\mathcal{F}_D)'$ and $\mathcal{M}$. ∎

Lemma 2. *Let F be a Fréchet space and let $C(F)$ denote the set consisting of all compact balanced convex subsets of F. Then for every $K \in C(F)$ there exists $L \in C(F)$ such that the canonical map from the canonical Banach space $F(K)$ spanned by K to $F(L)$ is compact.*

Proof: Let H be a closed separated subspace of F containing K. From a result of Geijler [7] we can find a continuous linear map η from a Fréchet–Montel space Q onto H. Since K is compact in H there exists $B \in C(Q)$ such that $\eta(B) = K$. Observe that the map $\widetilde{\eta}\colon\ Q(B) \to F(K)$ induced by η is open. Thus it suffices to show that there exists $\widetilde{B} \in C(Q)$ such that $B \le \widetilde{B}$ and the canonical map $e(B,\widetilde{B})$ from $Q(B)$ to $Q(\widetilde{B})$ is compact.

Let $\{\|\cdot\|_n\}$ be an increasing sequence of continuous semi-norms defining the topology of Q and let Q_n be the canonical Banach space associated to $\|\cdot\|_n$. Since

Q is reflexive, Q' is bornological [10]. Hence $Q' = \lim\operatorname{ind} Q'_n$. Put $P = \bigoplus_{n\geq 1} Q'_n$. Let α be the canonical map from P onto Q'.

First we find a continuous semi-norm ρ on P such that the map $\widetilde{\alpha}\colon P_\rho \to Q'_{p(K)}$ induced by α is compact, where $p(K)$ denotes the sup-norm on B. Take a sequence $\lambda_j \downarrow 0$ such that $\sum_j \lambda_j \leq 1$ and such that for the unit open ball U_j in Q_j we have $\lambda_j B \subseteq U_j$. Consider the semi-norm ρ on P given by

$$\rho(\{u_j\}) = \sum_j \|u_j\|_j/\lambda_j^2 ,$$

where $u_j \in Q'_j$ and $\|\cdot\|_j$ is the sup-norm on U_j.

Obviously α induces a continuous linear map $\widetilde{\alpha}$ from P_ρ to $Q'_{p(B)}$.

We show that $\widetilde{\alpha}$ is compact.

Indeed let $\{u^{(n)}\}$ be a sequence in P such that

$$M = \sup\{\rho(u^{(n)})\colon n \geq 1\} < \infty .$$

Then for every $m \geq 1$ and for every $x \in B$ we have

$$\sum_{j\geq m} |u_j^{(n)}(x)| = \sum_{j\geq m} \lambda_j |u_j^{(n)}(\lambda_j x)|/\lambda_j^2 \leq M \sum_{j\geq m} \lambda_j$$

and

$$\sup\{\|u_j^{(n)}\|_j\colon n \geq 1\} \leq M\,\lambda_j^2 \quad \text{for every} \quad j \geq 1 .$$

These inequalities show that $\{\widetilde{\alpha}(u_j^{(n)})\}$ is equicontinuous on B. Since B is compact it follows that $\{u^{(n)}\}$ is relatively compact in $Q'_{p(B)}$.

Now by the openness of $\alpha\colon P \to Q'$ there exists $\widetilde{B} \in C(Q)$ containing B such that the canonical map induced by α from P_ρ onto $Q'_{p(B)}$ is open. Hence the canonical map from $Q'_{p(\widetilde{B})}$ to $Q'_{p(B)}$ is compact. This yields from the commutativity of the diagram

$$\begin{array}{ccc} Q(B) & \longrightarrow & Q(B) \\ \downarrow & & \downarrow \\ [Q'_{p(B)}]' & \longrightarrow & [Q'_{p(B)}]' \end{array}$$

in which the maps $Q(B) \hookrightarrow [Q'_{p(B)}]'$ and $Q(\widetilde{B}) \hookrightarrow [Q'_{p(\widetilde{B})}]'$ are canonical embeddings, the compactness of $e(B,\widetilde{B})$. ∎

Lemma 3. *Let θ be a non-zero convolution operator on $\operatorname{Im}\mathcal{F}_D$. Then U_T with $T = \gamma\,\theta$, is surjective.*

Proof: i) Let $\psi \in H(D)$ with $U'_T(\psi) = 0$. Then

$$\mu(\mathcal{F}'_D(T)\,\psi) = (U'_T\,\mu)(\psi) = 0$$

for every $\mu \in H'(D)$.

By the Hahn–Banach Theorem we have $\mathcal{F}'_D(T)\,\psi = 0$. Since $\mathcal{F}'_D(T) \neq 0$ it follows that $\psi = 0$. Thus U'_T is injective.

Assume now that $\{\psi_\alpha\} \subset \operatorname{Im} U'_T$ which is weakly convergent to φ in $H(D)$. Then for every finite dimensional subspace F of E, the sequence $\{\mathcal{F}'_D(T)\,\psi_\alpha \mid F \cap D\}$ is weakly convergent to $\varphi \mid F \cap D$. Since the ideal in $H(F \cap D)$ generated by $\mathcal{F}'_D(T) \mid F \cap D$ is weakly closed in $H(F \cap D)$ it follows that $\varphi \mid F \cap D = \mathcal{F}'_D(T)\,|_{F\cap D}\,\psi_F$ for some $\psi_F \in H(F \cap D)$. By the unique principle the family $\{\psi_F\}$ defines a Gateaux holomorphic function ψ on D such that $\mathcal{F}'_D(T)^{\psi} = \varphi$. This relation yields by the Zorn Theorem the holomorphicity of ψ on D. Hence U'_T has the weakly closed image in $H(D)$.

ii) Let $\phi \in H'(D)$ with $U_T(\phi) \neq 0$ and let μ be an arbitrary element of $H'(D)$. Take $i_0 \in I$ such that $\phi, \mu \in H'(D_{i_0})$. Let us note that the canonical map from $H'(D_{i_0})$ to $H'(D)$ induces a continuous linear map from $\operatorname{Im}\mathcal{F}_{D_{i_0}}$ to $\operatorname{Im}\mathcal{F}_D$ for which the following diagram is commutative

$$\begin{tikzcd}
& H'(D) \arrow[rr, "U_T"] \arrow[dd, "\mathcal{F}_D" near end] & & H'(D) \arrow[dd, "\mathcal{F}_D"] \\
H'(D_{i_0}) \arrow[ur] \arrow[rr, crossing over] \arrow[dd, "\mathcal{F}_{D_{i_0}}"'] & & H'(D_{i_0}) \arrow[ur] & \\
& \operatorname{Im}\mathcal{F}_D \arrow[rr, "T^*" near end] & & \operatorname{Im}\mathcal{F}_D \\
\operatorname{Im}\mathcal{F}_{D_{i_0}} \arrow[ur] \arrow[rr, "T_0^*"'] & & \operatorname{Im}\mathcal{F}_{D_{i_0}} \arrow[ur] \arrow[from=uu, crossing over, "\mathcal{F}_{D_{i_0}}" near end] &
\end{tikzcd} \tag{1}$$

where $T_0 \in (\operatorname{Im}\mathcal{F}_{D_{i_0}})'$ is induced by T.

Hence by Lemma 2 without loss of generality we may assume that E is a Schwartz b.v.s. Take a strictly increasing sequence $\{i_j\}_{j\geq 0}$ in I. Put

$$F = \operatorname{lim\,ind} E_{i_j}\,, \qquad D_0 = D \cap F$$

and

$$T_0 = T \mid \operatorname{Im}\mathcal{F}_{D_0}\ .$$

Consider the commutative diagram (1) in which D_{i_0} is replaced by D_0 with $\phi, \mu \in H'(D_{i_0})$. By i) U'_{T_0} is injective and has the weakly closed image. This implies that $U_{T_0} = U''_{T_0}$ is surjective. Thus $\mu = U_T(\beta)$ for some $\beta \in H'(D)$. ∎

Proof of Existence Theorem: By the relation $T^*\,\mathcal{F}_D = \mathcal{F}_D\,U_T$ we infer that Existence Theorem is an immediate consequence of Lemma 3. ∎

To prove Approximation Theorem we need the following five lemmas.

Lemma 4. *$\mathcal{E}\mathrm{xp}(D)$ is dense in* $\operatorname{Im}\mathcal{F}_D$.

Proof: Let $S \in (\operatorname{Im} \mathcal{F}_D)'$ such that $S(\exp x) = 0$ for every $x \in D$. Then

$$\mathcal{F}'_D(S)(x) = 0 \quad \text{for every} \quad x \in D \ .$$

Since $\mathcal{F}'_D$ is injective, it follows that $S = 0$. ∎

Lemma 5. *Let $f, g \in H(D)$ such that for every finite dimensional subspace F of E on which $g \neq 0$, the function $f \,|\, F \cap D$ is divisible by $g \,|\, F \cap D$. Then f is divisible by g.*

Proof: By the unique principle there exists a Gateaux holomorphic function h on D such that $f = h\, g$. Since h is holomorphic at every $x \in D$ with $g(x) \neq 0$, by the Zorn Theorem h is holomorphic on D. ∎

Lemma 6. *Let $X, T \in (\operatorname{Im} \mathcal{F}_D)'$, $T \neq 0$, such that*

$$\forall\, x \in D,\ \forall\, P \in \mathcal{P}(E)\colon \quad T^*\, P \exp(x) = 0 \ \Rightarrow\ X(P \exp(x)) = 0 \ .$$

Then $\mathcal{F}'_D(X)$ is divisible by $\mathcal{F}'_D(T)$.

Proof: By hypothesis we have

$$\forall\, x \in D\colon \quad \mathcal{F}'_D(T)(x) = 0 \ \Rightarrow\ \mathcal{F}'_D(X)(x) = 0 \ .$$

This implies that $\mathcal{F}'_D(X) \,|\, F \cap D$ is divisible by $\mathcal{F}'_D(T) \,|\, F \cap D$ for every finite dimensional subspace F of E on which $\mathcal{F}'_D(T) \neq 0$. Lemma 5 yields that $\mathcal{F}'_D(X)$ is divisible by $\mathcal{F}'_D(T)$. ∎

Lemma 7. *Let E be a weak Schwartz b.v.s. and let $i < j < k$. Then E^+ is dense in $E'_k \,|\, E_i$.*

Proof: Denote by E_i^+ the completion of $E^+ / \operatorname{Ker} \|\cdot\|_i$, where $\|\cdot\|_i$ is the semi-norm on E^+ defined by the unit open ball U_i in E_i. Since E^+ separates the points of E, it follows that $E^+ \subseteq \operatorname{lim\,ind} E_i^+$ (algebraically). On the other hand, by the weak compactness of the canonical map ω_{ij} from E_i to E_j for every $i < j$ and since U_i is $\sigma(E_i^{+\prime}, E_i^+)$-dense in the unit open ball $U_i^{+\prime}$ in $E_i^{+\prime}$ we have

$$\operatorname{Cl} \omega_{ij}^{+\prime}(U_i^{+\prime}) = \operatorname{Cl} \omega_{ij}^{+\prime}(\operatorname{Cl}_{\sigma(E_i^{+\prime}, E_i^+)} U_i) \leq \operatorname{Cl}_{\sigma(E_j^{+\prime}, E_j^+)} \omega_{ij}^{+\prime}(U_i) = \operatorname{Cl} \omega_{ij}(U_i) \leq E_j \ ,$$

where ω_{ij}^+ is the restriction map from E_j^+ to E_i^+ and U_i^+ is the unit open in E_i^+.

Thus for $i < j < k$ we have by the weak compactness of ω_{ij}^+ the following two commutative diagrams

$$\begin{array}{ccc} E_j & \longrightarrow & E_j^{+\prime} \\ \omega_{jk} \Big| & \swarrow & \Big| \omega_{jk}^{+\prime} \\ E_k & \longrightarrow & E_k^{+\prime} \end{array} \qquad\qquad \begin{array}{ccc} E_j^+ & \longrightarrow & E_j^+ \\ \omega_{jk}^+ \Big| & \swarrow & \Big| \omega_{ij}^{+\prime\prime} \\ E_i^+ & \longrightarrow & E_i^+ \end{array}$$

This implies that E^+ is dense in $E'_k \,|\, E_i$. ∎

Put

$$H_0(D) = \Big\{ f \in H(D)\colon\ D^n f(x) \text{ can be approximated by elements of } E^{\otimes n} \text{ for every } x \in D \Big\} \ .$$

Lemma 8. $\operatorname{Ker} \mathcal{F}_D = [H_0(D)]^\perp$ *and hence* $\operatorname{Im} \mathcal{F}_D \subseteq H_0(D)$.

Proof: Let $\mu \in [H_0(D)]^\perp$. Then

$$(\mathcal{F}_D \mu)(x^*) = \mu(\exp x^*) = \sum_{k \geq 0} (1/k!)\, \mu({x^*}^k) = 0$$

for every $x^* \in E^+$.

Hence $\mu \in \operatorname{Ker} \mathcal{F}_D$. Conversely, assume that $\mathcal{F}_D\, \mu = 0$. By hypothesis on D and on E it follows that $H_0(E)$ is dense in $H_0(D)$. ∎

Proof of Approximation Theorem: If $\theta = 0$, the result is true since $\mathcal{P}(E)\, \mathcal{E}\mathrm{xp}(D)$ is dense in $\operatorname{Im} \mathcal{F}_D$. Let $\theta \neq 0$ and let $X \in (\operatorname{Im} \mathcal{F}_D)'$, $X \,|\, \mathcal{P}(E)\, \mathcal{E}\mathrm{xp}(D) \cap \operatorname{Ker} \theta = 0$. This means that

$$\forall\, P \in \mathcal{P}(E),\ \forall\, x \in D\colon\quad T^*\, P \exp(x) = 0 \ \Rightarrow\ X(P \exp(x)) = 0$$

where $T^* = \theta$.

Lemma 6 implies that $\mathcal{F}'_D(X) = h\, \mathcal{F}'_D(T)$ for some $h \in H(D)$. From the relations $\mathcal{F}'_D(X) \in H_0(D)$ and $\mathcal{F}'_D(T) \in H_0(D)$, it is easy to see that $h \in H_0(D) = (\operatorname{Ker} \mathcal{F}_D)^\perp$. Thus $h = \mathcal{F}'_D(Q)$ for some $Q \in (\operatorname{Im} \mathcal{F}_D)'$. Hence $\mathcal{F}'_D(X) = \mathcal{F}'_D(Q)\, \mathcal{F}'_D(T) = \mathcal{F}'_D(Q^*\, T)$. From the injectivity of $\mathcal{F}'_D$ we have

$$X = Q^*\, T = T^*\, Q = (T^*)'\, Q = \theta'(Q) \ .$$

These equalities imply $X = 0$ on $\operatorname{Ker} \theta$. ∎

REFERENCES

[1] Boland, P.J. – Malgrange Theorem for entire functions on nuclear spaces, *Lecture Notes in Math.*, 364 (1974), 135–144.

[2] Colombeau, J.F. and Perrot, B. – Transformation de Fourier–Borel et noyeaux en dimension infinie, *C.R. Acad. Sci. Paris. Ser. A*, 284 (1977), 963–966.

[3] Colombeau, J.F. and Perrot, B. – Convolution equation in spaces of infinite dimensional entire functions of exponential and related types, *Trans. Amer. Math. Soc.*, 258 (1980), 191–198.

[**4**] Dwyer, T.A.W. – Equations differentielles d'ordre infinie dans des localement convexes, *C.R. Acad. Sci. Paris. Ser. A*, 218 (1975), 163–166.

[**5**] Dwyer, T.A.W. – Differential operators of infinite order on locally convex spaces I, *Rend. Math.*, 10 (1977), 149–179.

[**6**] Dwyer, T.A.W. – Differential operators of infinite order on locally convex spaces II, *Rend. Math.*, 10 (1978), 273–293.

[**7**] Geijler, W. – Extending and lifting continuous linear mappings in topological vector spaces, *Studia Math.*, 62 (1978), 295–303.

[**8**] Gupta, C. – Malgrange Theorem for nuclear entire functions of bounded type on a Banach space, *Idag. Math.*, 32 (1970), 356–358.

[**9**] Noverraz, Ph. – *Pseudo-Convexite, Convexite Polynomiale et Domaines d'Holomorphie en Dimension Infinie*, North-Holland Math. Stud., 3, 1973.

[**10**] Schaefer, H. – *Topological Vector Spaces*, Springer-Verlag, 1971.

Nguyen Van Khue and Nguyen Dinh Sang,
Department of Mathematics, Pedagogical Institute Hanoi I,
Hanoi – VIETNAM