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SOME STABILITY QUESTIONS CONCERNING CAUSTICS FOR DIFFERENT PROPAGATION LAWS*

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Introduction

Caustics, that originally appeared in Geometrical Optics attached to the propagation phenomena governed by the wave equation, can be viewed in a general setting as the image of the singular set of lagrangian maps. These, being defined as the restrictions of lagrangian fibrations to lagrangian submanifolds of a symplectic manifold, can also be obtained from the so-called generating functions. Stable caustics correspond to stable lagrangian maps, and it is well known that appropriate transversality conditions on the generating functions produce stable lagrangian maps. (See [2] for details.)

We consider here different generating functions associated to different propagation laws (like a Riemannian structure or a Hamiltonian) on a manifold, and analyze the problems of genericity and stability of the caustic generated by a fixed initial wave front with respect to perturbations in the propagation rules themselves. One of the geometrical consequences that can be deduced from our analysis is that given any submanifold of a complete Riemannian manifold, its focal set can be made locally stable by a small perturbation in the metric. Moreover if the manifold does not have conjugate points then both the focal set and the cut-locus (in the sense of Thom [10]) of any submanifold can be made globally stable through such a perturbation.

M.A. Buchner treated in [4] the problem of stability of the cut-locus of a point in a manifold with respect to perturbations of the Riemannian matric. The approach we use here is quite different from his.

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1 – Preliminaries and statement of results

Suppose that Y is a complete Riemannian manifold and let $d\colon Y\times Y\to \mathbb{R}$ be the induced distance squared map (i.e. $d(y_1,y_2)$ is the square of the length of the shortest geodesic joining y_1 to y_2). Let h be an embedding of the manifold X into Y. Then $V=h(X)$ is a submanifold that we shall call *initial wave front*. This is justified by the following: if we start at the points of V and walk along the normal geodesics to V during a fixed small enough time t, we obtain a hypersurface V_t called the *wave front at time* t. Then we shall reach a moment at which these V_t begin to have singularities, i.e. they are not smooth hypersurfaces anymore. If we join all these singularities we get the *focal set* that we shall also call the *caustic* C of V (this is the set where all the geodesic rays emanating from V concentrate).

We can also characterize the caustic, in terms of singularities, as follows: Let $S(\Phi)$ be the singular set of the map

$$\begin{array}{ccccc}
\Phi\colon X\times Y & \stackrel{h\times 1_Y}{\longrightarrow} & Y\times Y & \stackrel{d\times 1_Y}{\longrightarrow} & \mathbb{R}\times Y \\
(x,y) & \longmapsto & (h(x),y) & \longmapsto & (d(h(x),y),y)\ .
\end{array}$$

When d comes from the usual euclidean metric on $Y=\mathbb{R}^n$ it is easy to see that $S(\Phi)$ is precisely the normal bundle NV of V in $\mathbb{R}^n$ (see [9]). Now we consider the restriction to $S(\Phi)$ of the projection $\pi_2\colon X\times\mathbb{R}^n\to\mathbb{R}^n$. Then the image by π_2 of its singular set $S(\pi_2\mid S(\Phi))$ is the focal set or caustic of V in $\mathbb{R}^n$.

A caustic is said to be (locally) *versal* if the map Φ (*generating function*) is a (locally) versal unfolding of functions on X with parameters on Y (see [2]). Clearly any (locally) versal caustic, being the bifurcation set of the unfolding Φ, is (locally) stable with respect to perturbations of Φ (as a family of functions) and thus, under perturbations of both the embedding h and the distance squared function d (and hence of the Riemannian metric on Y). Where (locally) stable means that the caustic of a small perturbation $\hat\Phi$ of Φ is (locally) diffeomorphic (or homeomorphic in the case of topological versality) to the original one.

It follows from the Looijenga topological stability theorem that, for a residual set of embeddings of X in $Y=\mathbb{R}^n$, the focal set of the embedded submanifold is topologically stable (or C^∞-stable if we restrict ourselves to low enough dimensions, i.e. $n\le 5$). So, small perturbations of the embeddings give rise to homeomorphic (or diffeomorphic, for $n\le 5$) focal sets and also cut-locus (which are the centres of the hyperspheres of minimal radii having contact of order at least 2 with the submanifold at two or more points, or at least 3 at a single point [10]).

The corresponding result for submanifolds of complete simply connected Riemannian manifolds without conjugate points was obtained by J.W. Bruce and D.J. Hurley [3].

An alternative approach for submanifolds of Euclidean spaces is due to J.A. Montaldi [7], who studied the generic contacts between submanifolds and spheres of $\mathbb{R}^n$ with some geometrical consequences. Some general results related to this can be found in [8]. We state here the following one that will be used later:

Theorem 1.

1) *Suppose that $F: Y \times U \to Z$ is a locally $\mathcal{G}$-versal family of maps with parameters in U. Let $W \subset J^r(X,Z)$ be a $\mathcal{G}$-invariant submanifold, and let $R_W = \{h \in \mathrm{Imm}^\infty(X,Z)/j_1^r\Phi_g \pitchfork W\}$, where $\Phi_g(x,y) = F(g(x),u)$ and $j_1^r\Phi$ means the r-jet with respect to the first argument. Then R_W is residual in the space $\mathrm{Imm}^\infty(X,Y)$ of smooth immersions of X in Y.*

2) *Suppose that F as above is $\mathcal{G}$-versal, and $s \geq 1$. Let $W \subset_s J^r(X,Z)$ be a $\mathcal{G}$-invariant submanifold, and let $R_W = \{g \in \mathrm{Emb}^\infty(X,Z)/_s\, j_1^r\Phi_g \pitchfork W\}$. Then R_W is residual in $\mathrm{Emb}^\infty(X,Y)$.*

In this theorem $\mathcal{G}$ means any of the standard groups arising in Singularity Theory: $\mathcal{R}$, $\mathcal{R}^+$, $\mathcal{L}$, $\mathcal{C}$, $\mathcal{A}$ or $\mathcal{K}$.

We can also adopt a slightly wider viewpoint and consider the distance function d as being locally defined from a P.D.E. or a Hamiltonian function on T^*Y instead of a Riemannian metric on Y. So, suppose that $H: T^*Y - \{0\} \to \mathbb{R}$ is an everywhere positive and positively homogeneous of degree 1 Hamiltonian. Then, there is a locally defined ray length function associated to H (see [5]). We shall also denote this function as $d\colon Y \times Y \to \mathbb{R}$. Observe that we may also include in this category the metric manifolds (called metric spaces in [6]) provided with a lagrangian function $L\colon TY \to \mathbb{R}$ that satisfies $L(x,\xi) > 0$, $\forall\, \xi \neq 0$ and $L(x,\lambda\xi) = |\lambda|\, L(x,\xi)$. A special case of these are the Finsler manifolds defined as manifolds with a non-negative scalar funtion $F(x,y)\colon TM \to \mathbb{R}$ satisfying the following conditions:

i) $F(x,ky) = k\,F(x,y), \quad \forall\, k > 0$,

ii) $F(x,y) > 0$, for $y \neq 0$,

iii) The quadratic form

$$D^2F^2(x,y)(\xi) = \frac{\partial^2 F^2(x,y)}{\partial \xi^i\, \partial \xi^k}\, \xi^i\, \xi^j$$

is positive definite.

In the particular case that $F(x^i, dx^i) = [g_{ij}(x)\, dx^i\, dx^j]^{\frac{1}{2}}$, F induces a Riemannian metric $g_{ij} = D^2F^2(x, y)$.

Now, the caustic obtained from an initial wave front $V = h(X)$ is (locally) defined as the natural generalization of the focal set in the Riemannian case, with the ray length function as distance function ([5]). Then the caustic associated, to the pair (h, H) is said to be (locally) versal provided the composition

$$\Phi\colon\, X \times Y \xrightarrow{h \times 1_Y} Y \times Y \xrightarrow{d} \mathbb{R}$$

is a (locally) stable family of (germs of) functions at the considered points. G. Wassermann [12] proved that the local versality of the caustic associated to the pair (h, H) is equivalent to its local stability under perturbations of the initial wave front (i.e. of h) alone. Moreover, he showed that for a given Hamiltonian H, there is a residual subset of embeddings E, such that $\forall\, h \in E$, the pair (h, H) produces a versal family Φ and hence a locally versal caustic.

The above mentioned results (Wassermann [12]), Bruce and Hurley [3] or Montaldi [8]) can all be, for our purposes, put in the following form:

Theorem 2. *Given any $\mathcal{A}$-invariant submanifold $W \subset J^k(X, \mathbb{R})$ let $R_W = \{h \in \mathrm{Emb}^\infty(X, Y) /\, j_1^r\, \Phi_g \pitchfork W\}$, where*

$$\Phi_h\colon\, X \times Y \xrightarrow{h \times 1_Y} Y \times Y \xrightarrow{d} \mathbb{R}$$

and d is the distance map associated to either a Riemannian metric, a Finsler metric or a Hamiltonian, as above. Then R_W is residual in the space $\mathrm{Emb}^\infty(X, Y)$.

Our aim here is to analyze the opposite case, in which the initial embedding h is fixed and the distance function is allowed to vary. Since this distance may be induced from different structures, we shall consider instead perturbations of these structures. In this sense we shall distinguish among the following possibilities:

a) The distance function is induced from a Riemannian metric on Y.

b) The distance function is induced from a Finsler metric on Y.

c) The distance function is induced from an everywhere positive and positively homogeneous of degree 1 Hamiltonian function on $T^*Y - \{0\}$.

d) The distance function d is the square of a topological distance function ρ with smooth square, such that the functions $d_b : Y \to \mathbb{R}$, defined as $d_b(y) = d(y, b)$ are smooth submersions in $Y - \{b\}$, $\forall\, b \in Y$.

Notice that for this case we also have the concepts of d-normal bundle, d-caustic and d-cut-locus naturally defined, all of them being related to the contacts of a submanifold with the d-spheres $d_b^{-1}(r)$, $b \in Y$, $r \in \mathbb{R}^+$. We shall call d-manifold to a manifold Y with such a function.

We prove the following results for all these cases:

A) Given any fixed embedding $h : X \to Y$, there is an open and dense subset of structures on Y, such that the corresponding family $\varphi = d \circ (h \times 1_Y)$ is topologically stable (or C^∞-stable, for $\dim Y \leq 5$) and thus produces topologically stable (resp. C^∞-stable) caustics (and cut-locus in the appropriate classes).

B) A caustic is stable with respect to perturbations of the initial wave front if and only if it is stable with respect to perturbations of the structure within each one of the classes.

(Here by structure we mean any one of the above classes a, b, c, or d.)

Remark. Under certain assumptions, for instance when d is induced from a Riemannian metric on a complete simply connected manifold without conjugate points in case a) , or a d-manifold, we can use the multijets spaces and obtain a global version of the above theorem.

2 – Proof of results

First of all we observe that there is an action of the group $\mathcal{G}$ of diffeomorphisms (local diffeomorphisms when necessary) of the manifold Y on each one of the following spaces (considered with the appropriate C^∞-Whitney topologies on them):

a) Riemannian metrics on Y;

b) Finsler metrics on Y;

c) Positive and positively homogeneous of degree one Hamiltonian functions on $T^*Y - \{0\}$;

d) Topological distances with smooth square d on Y, such that the functions $d_b \colon Y \to \mathbb{R}$, are smooth submersions in $Y - \{b\}$, $\forall\, b \in Y$.

We define these actions in the following and see how they behave with respect to the caustic set in each case:

a) Given a Riemannian metric, $g(y): T_yY \times T_yY \to \mathbb{R}$, $y \in Y$, and a diffeomorphism $\varphi\colon Y \to Y$, we define $\varphi^*(g)$ by

$$\varphi^*(g)(y_)(v_1, v_2) = g(\varphi(y))(T_y\varphi(v_1), T_y\varphi(v_2)), \quad \forall\, y \in Y, \;\; \forall\, v_1, v_2 \in T_yY \ .$$

It is easy to see that:

i) $\varphi^*(g)$ is a Riemannian metric on Y;

ii) φ is an isometry between $(Y, \varphi^*(g))$ and (Y, g).

And hence $T\varphi$ carries the normal bundle of a submanifold V with respect to the metric $\varphi^*(g)$ diffeomorphically onto the normal bundle of the submanifold $\varphi(V)$ with respect to the metric g. Moreover, φ takes geodesics of $\varphi^*(g)$ to geodesics of g and the diagram

$$\begin{array}{cccc} & N_{\varphi^*(g)}V & \xrightarrow{T\varphi} & N_g\varphi(V) & \\ \exp\varphi^*(g) & \downarrow & & \downarrow & \exp g \\ & Y & \xrightarrow[\varphi]{} & Y & \end{array}$$

commutes, where $N_{\varphi^*(g)}V$ and $N_g\varphi(V)$ are respectively the normal bundles of V with respect to $\varphi^*(g)$ and of $\varphi(V)$ with respect to g. Consequently φ takes the focal set of V with respect to $\varphi^*(g)$ onto the focal set of $\varphi(V)$ with respect of g.

b) A Finsler metric, $g(x, \xi)\colon T_xY \times T_xY \to \mathbb{R}$, $(x, \xi) \in TY$ is characterized by the fact that $g(x, \xi)(\eta, \eta) > 0$, $\forall\, \xi, \eta \in T_xY - \{0\}$, $\forall\, x \in Y$.

Given φ as above, we define

$$(\varphi^*g)(x, \xi)(\eta_1, \eta_2) = g(\varphi(x), T_x\varphi(\xi))\,(T_x\varphi(\eta_1), T_x\varphi(\eta_2)) \ .$$

This is, clearly, another Finsler metric. Furthermore, φ is an isomorphism between $(Y, \varphi^*(g))$ and (Y, g) in the category of Finsler manifolds. Geodesics are defined here in a similar manner to the Riemannian case, and we have that φ maps geodesic of $\varphi^*(g)$ onto geodesics of g and, this being a particular case of the class below, we can also see that it takes the caustic of a submanifold V with respect to $\varphi^*(g)$ onto the caustic of $\varphi(V)$ with respect to g.

c) Given a Hamiltonian function $H\colon T_*Y - \{0\} \to \mathbb{R}$, there is a Hamiltonian vector field $\chi_H\colon T^*Y \to T(T^*Y)$ associated to it. The flow lines of χ_H project through the cotangent bundle projection, $\pi\colon T^*Y \to Y$, onto the Hamiltonian rays of Y. Moreover, due to the homogeneity property of H, χ_H factorizes to a vector field $\tilde{\chi}_H\colon \Sigma T^*Y \to T(\Sigma T^*Y)$, where ΣT^*Y represents the space of oriented lines in the cotangent vector space of Y. Following Jänich [5] we consider the map

$$\begin{aligned} \tau_H\colon \mathbb{R}\times\Sigma T^*Y &\longrightarrow Y\times Y \\ (t,\xi) &\longmapsto (\tilde{\pi}(\xi),\exp_H(t,\xi)) \end{aligned}$$

where $\tilde{\pi}: \Sigma T^*Y \to Y$ is the bundle projection and $\exp_H: \mathbb{R}^+ \times \Sigma T^*Y \to Y$ is the composition of $\tilde{\pi}$ with the flow map of χ_H (observe that this map will, in general, be defined only on a neighbourhood of $0\times\Sigma T^*Y$ in $\mathbb{R}^+\times\Sigma T^*Y$).

Then the ray length function associated to a regular point (t_0,ξ_0) of the above map is given by the composition

$$d\colon Y\times Y \xrightarrow{\tau_H^{-1}} \mathbb{R}^+\times\Sigma T^*Y \xrightarrow{p_1} \mathbb{R}^+$$

on an appropriate neighbourhood of (t_0,ξ_0).

Let now $\varphi\colon Y\to Y$ be a diffeomorphism and define

$$\varphi^*(H)\colon T^*Y-\{0\}\longrightarrow\mathbb{R}$$

by the composition

$$\varphi^*(H)=H\circ T^*\varphi^{-1}\colon T^*Y \xrightarrow{T^*\varphi^{-1}} T^*Y \xrightarrow{H} \mathbb{R}$$

where

$$\begin{aligned} T^*\varphi(\alpha)\colon T^*Y &\longrightarrow \mathbb{R} \\ \xi &\longmapsto \alpha_{\varphi(x)}(T_x\varphi(\xi)) \ . \end{aligned}$$

We also have that $\varphi^*(H)$ is positive and positively homogeneous of degree one.

Now, $T^*\varphi^{-1}$ is a symplectomorphism of T^*Y, consequently (see [1], pg. 194) we have that $(T^*\varphi^{-1})^*(\chi_H)=\chi_{\varphi^*}(\chi_H)$ where $(T^*\varphi^{-1})^*(\chi_H)=(T^*\varphi)_*(\chi_H)=T(T^*\varphi)(\chi_H)(T^*\varphi)^{-1}$ and thus the following diagram commutes:

$$\begin{array}{ccccc} & T^*Y & \xrightarrow{\theta_H} & T(T^*Y) & \\ T^*\varphi & \downarrow & & \downarrow & T(T^*\varphi) \\ & T^*Y & \underset{\theta_{\varphi^*(H)}}{\longrightarrow} & T(T^*Y) & \end{array}$$

Moreover, it factors to a diagram:

$$\begin{array}{ccccc} & \Sigma T^*Y & \xrightarrow{\tilde{\theta}_H} & T(\Sigma T^*Y) & \\ \Sigma T^*\varphi & \downarrow & & \downarrow & T(\Sigma T^*\varphi) \\ & \Sigma T^*Y & \underset{\tilde{\theta}_{\varphi^*(H)}}{\longrightarrow} & T(\Sigma T^*Y) & \end{array}$$

Then it is not difficult to see that the following diagram is commutative too:

$$\begin{array}{ccccc} & \mathbb{R}^+ \times \Sigma T^*Y & \xrightarrow{\tau_H} & Y \times Y & \\ 1 \times \Sigma T^*\varphi & \downarrow & & \downarrow & \varphi^{-1} \times \varphi^{-1} \\ & \mathbb{R}^+ \times \Sigma T^*Y & \underset{\tau_{\varphi^*(H)}}{\longrightarrow} & Y \times Y & \end{array}$$

And from this we obtain that given any initial wave front $V = h(X)$, the diffeomorphism φ^{-1} takes the caustic set of the pair (h, H) to the caustic set of $(\varphi^{-1} \circ h, \varphi^*(H))$, (and it also takes the wavefront of V at time t for H to the wave front of $\varphi^{-1}(V)$ at time t for $\varphi^*(H)$, or in other words, the diffeomorphism φ takes the wave front of V at time t for $\varphi^*(H)$ to the wave front at time t of $\varphi(V)$ for H, and the caustic of V with respect to $\varphi^*(H)$ to the caustic of $\varphi(V)$ with respect to H.

d) Finally, given a topological distance function ρ on Y with smooth square $d\colon Y \times Y \to \mathbb{R}$, and such that d_b is a submersion on $Y - \{0\}$ $\forall b \in Y$, for any diffeomorphism $\varphi\colon Y \to Y$ we have that $\varphi^*(\rho) = \rho \circ (\varphi \times \varphi)$ gives a distance on Y having the same properties as ρ. Moreover, if $V = h(X)$ is an embedded submanifold, we have the following commutative diagram

$$[*] \qquad \begin{array}{ccccc} X \times Y & \xrightarrow{h \times 1_Y} & Y \times Y & \xrightarrow{d} & \mathbb{R} \\ 1 \times \varphi^{-1} \downarrow & & \downarrow \varphi^{-1} \times \varphi^{-1} & & \downarrow 1 \\ X \times Y & \underset{\varphi^{-1} \circ h \times 1_Y}{\longrightarrow} & Y \times Y & \underset{\varphi^*(d)}{\longrightarrow} & \mathbb{R} \end{array}$$

where $\varphi^*(d) = [\varphi^*(\rho)]^2$.

And hence, φ^{-1} maps the d-caustic of V into the $\varphi^*(d)$-caustic of $\varphi^{-1}(V)$.

All this amounts to say that any diffeomorphism φ of Y (take germs if necessary) induces a new element of each one of the classes above and a diagram of type [*], where d is the distance function associated to the original element and $\varphi^*(d)$ corresponds to the one induced from φ. We can then say that the families $\Psi = d \circ (h \times 1_Y)$ and $\varphi^*(\Psi) = \varphi^*(d)(\varphi^{-1} \circ h \times 1_Y)$ are $\mathcal{A}$-equivalent.

Theorem 3. *Given a fixed embedding and an $\mathcal{A}$-invariant submanifold $\Omega \subset J^k(X, \mathbb{R})$, there is a dense subset D_Ω in the class $\mathcal{C}$, such that $\forall c \in D_\Omega$ we have that $j_1^k(d_c \circ (h \times 1_Y)) \pitchfork \Omega$, where $\mathcal{C}$ is one of the classes a), b), c), or d) above, and d_c is the distance function associated to the element c of the class $\mathcal{C}$ as previously specified.*

Proof: Let $D_\Omega = \{c \in \mathcal{C} :\ j_i^k(d_c \circ (h \times 1_Y) \pitchfork \Omega\}$. In order to prove the density of D_Ω we show that $\forall c \in \mathcal{C}$, $\exists$ a sequence $\{c_n\} \subset D_\Omega$ such that $c_n \to c$ in the topology of $\mathcal{C}$.

Write

$$\Psi_c\colon X\times Y \overset{h\times 1_Y}{\longrightarrow} Y\times Y \overset{d_c}{\longrightarrow} \mathbb{R}$$

then either $j_1^k\Psi \pitchfork \Omega$ in which case $\Psi_c \in D_\Omega$ and there is nothing to prove, or $j_1^k\Psi$ is not transversal to Ω. In this last case, we know from Montaldi's theorem that there is a sequence $\{h_t\}$ converging to h in the Whitney C^∞-topology on $\mathrm{Emb}^\infty(X,Y)$ such that $j_1^k(d_c\circ(h_t\times 1)) \pitchfork \Omega$, $\forall t$. We can now work in small enough neighbourhoods of the embedding h such that there are diffeomorphisms $\varphi_t\colon Y\to Y$ with $h=\varphi_t\circ h_t$ and such that $\{\varphi_t\}$ converges to 1_Y. In fact, we can take φ_t to be the identity off some closed neighbourhood of $h(X)$. Then we get that the families

$$\Psi_t\colon X\times Y \overset{h_t\times 1_Y}{\longrightarrow} Y\times Y \overset{d_c}{\longrightarrow} \mathbb{R}$$

satisfy $j_1^k\Psi_t \pitchfork \Omega$, $\forall t$.

But from the considerations above we know that these families are respectively $\mathcal{A}$-equivalent to the following

$$\varphi_t^*(\Psi)\colon X\times Y \overset{h\times 1_Y}{\longrightarrow} Y\times Y \overset{d_{\varphi_t^*(c)}}{\longrightarrow} \mathbb{R}\ .$$

Consequently $j_1^k\varphi_t^*(\Psi) \pitchfork \Omega$, $\forall t$.

Moreover, since $\{\varphi_t\}$ converges to 1_Y it is not difficult to see that $\{\varphi_t^*(c)\}$ converges to c, and thus we are done. ∎

The next is the result B) stated in Section I.

Theorem 4. *Given $h\in \mathrm{Emb}^\infty(X,Y)$ and $c\in\mathcal{C}$, the caustic $C(h,c)$ associated to them is locally stable with respect to perturbations of h in $\mathrm{Emb}^\infty(X,Y)$ if and only if it is locally stable with respect to perturbations of c in $\mathcal{C}$.*

Proof: Since the result is local we may put $Y=\mathbb{R}^n$ without loss of generality.

Wassermann [13] proved that $C(h,c)$ is stable with respect to perturbations of h if and only if the family $\Psi_{h,c}=\{d_c\circ(h\times 1_Y))$ is $\mathcal{A}$-stable as a family of functions. So, it is enough to prove that the stability of $C(h,c)$ with respect to perturbations in c within $\mathcal{C}$ implies stability of $C(h,c)$ with respect to perturbations of the embedding h.

Consider thus the family

$$\Psi_{h,c}\colon X\times Y \overset{h\times 1_Y}{\longrightarrow} Y\times Y \overset{d_c}{\longrightarrow} \mathbb{R}\ ,$$

and a perturbation

$$\Psi_{\widehat{h},c}\colon X\times Y \overset{\widehat{h}\times 1_Y}{\longrightarrow} Y\times Y \overset{d_c}{\longrightarrow} \mathbb{R}\ ,$$

such that $\widehat{h}$ is near enough to h in $\mathrm{Emb}(X,Y)$. As before, we know that $\widehat{h}$ must be in the $\mathcal{G}$-orbit of h in $\mathrm{Emb}^\infty(X,Y)$ and thus $\exists\varphi\in\mathcal{G}$ that can be taken near enough to 1_Y such that $\widehat{h}=\varphi\circ h$. Then $\widehat{c}=\varphi^*(c)\in\mathcal{C}$ will be as near to c as desired. Now, by stability of $\Psi_{h,c}$ with respect to perturbations in c we have that $\Psi_{h,c}\sim_{\mathcal{A}}\Psi_{h,\widehat{c}}$. But clearly $\Psi_{h,\widehat{c}}\sim_{\mathcal{A}}\Psi_{\widehat{h},c}$ and the proof is finished. ∎

Finally, we would like to make some remark concerning the focal sets of generic closed curves from a global viewpoint. It is known that the 4-vertex theorem holds for closed curves on surfaces with constant negative curvature [11]. Now, suppose that $\alpha:\ S^1\to N$ is a generic curve (in the sense that it belongs to the open and dense set defined by Theorem 3) on a complete simply connected surface N provided with a Riemannian metric g which is near enough to another metric $\tilde{g}$ for which N has constant negative curvature and no conjugate points. In this case the exponential maps of the metrics involved are globally defined and so are the distance functions on N. Moreover, we can use the multijets version of the Theorem 1 above (or also Theorem 3 in [3, pg. 212] applies under our assumptions) and get that, α being generic, its caustic (or focal set) in (N,g) must be stable and hence diffeomorphic to the caustic of α in $(N,\tilde{g})$ (for g near enough to $\tilde{g}$). But since the 4-vertex theorem applies in this later case, we obtain as a consequence a 4-vertex theorem for the previous more general one. Therefore we can state the following:

Given any generic closed curve on the hyperbolic plane, its focal set with respect to any small enough perturbation of the surface (more precisely, of its metric) has at least 4 cuspidal points.

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