PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 4 – 1994

AN EXISTENCE THEOREM FOR HAMMERSTEIN INTEGRAL EQUATIONS*

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1 – Introduction

After the publication of the paper [8] the following (nonlinear) Hammerstein integral equation

$$(HIE) \qquad x(t) = g(t) + \lambda \int_D k(t,s)\, f(s, x(s))\, ds$$

(D a compact subset of $\mathbb{R}^n$, g, k, f functions with values in finite dimensional Banach spaces) has been very often investigated in spaces of summable functions, because of its usefulness in applications (see [5], [9], [10], [12] and references there). The most common hypotheses used in this study have been assumptions of regularity, coercivity, differentiability and monotonicity on k, f and on the superposition operator $(Fy)(\cdot) = f(\cdot, y(\cdot))$ and the linear integral operator $(Kz)(\cdot) = \int_D k(t,s)\, z(s)\, ds$ (note that we shall use the same notations through all the paper) (see [2], [5], [9], [10], [11], [12]).

In the recent paper [7] we have been able to dispense with all of these assumptions just assuming that k and f satisfy Caratheodory conditions; but, as observed by Prof. J. Banas, even if such an hypothesis is completely natural for f (see [9], [12]), it is sometime restrictive when applied to k; for instance, if $k(t,s) = p(t)\, q(s)$ it implies the continuity of q, whereas requiring that q belongs to some L^r-space would be more natural. We take this opportunity to thank Prof. J. Banas for this remark that motivated the present paper.

Here we want to show that actually it is possible to have solutions of (HIE) under this more general hypothesis; indeed, we present a result in which we assume

Received: September 11, 1992; *Revised*: November 6, 1992.
* Work performed under the auspices of G.N.A.F.A. of C.N.R. and partially supported by M.U.R.S.T. of Italy (60 %).

that f is a Caratheodory function such that F maps $L^1(D,X)$ into $L^1(D,Y)$, continuously, and k is a measurable function such that the functions $s \to k(t,s)$ belong to L^∞ and K is a linear, continuous operator from $L^1(D,Y)$ into $L^1(D,X)$, where X, Y are finite dimensional Banach spaces (in the sequel, $\|K\|$ will denote the operator norm of K).

2 – Main result

The main tools in the proof of our Theorem below are the following measures of weak noncompactness and of nonequiabsolute continuity introduced by F.S. De Blasi in [4] and by J. Appell and E. De Pascale in [1], respectively.

Definition 1 ([9]). Let H be a bounded subset of a Banach space B. We call *measure of weak noncompactness* of H the number

$$\beta(H) = \inf\Big\{\varepsilon > 0\colon \text{ there is a weakly compact set } C \text{ s.t. } H \subset C + \varepsilon B_1\Big\}$$

(B_1 the unit ball of B).

β has the following useful properties (see [4] for a proof):

i) $\beta(H) = 0$ if and only if H is relatively weakly compact;

ii) $\beta(H_1) \le \beta(H_2)$ if $H_1 \subseteq H_2$;

iii) $\beta(H) = \beta(\overline{co}(H))$;

iv) $\beta(rH) = |r|\,\beta(H)$, $r \in \mathbb{R}$;

v) If (H_n) is a decreasing sequence of nonempty, bounded, closed and convex subsets of B with $\beta(H_n) \to 0$, then $H = \bigcap H_n$ is nonempty (obviously, it is closed, convex and relatively weakly compact by i) and ii)).

Definition 2 ([1]). Let H be a bounded subset of $L^1(D,X)$, X a Banach space. We call *measure of nonequiabsolute continuity* of H the number

$$\pi_1(H) = \limsup_{\delta\to 0}\Big\{\sup\Big\{\int_{D_0} \|x(t)\|\,dt\colon\ m(D_0) < \delta\Big\}\colon\ x \in H\Big\}\ .$$

As in [1], p. 150, we can prove the following result.

Proposition 1. *Let X be a finite dimensional Banach space and H be a bounded subset of $L^1(D,X)$. Then $\beta(H) = \pi_1(H)$.*

We are now ready to show our result improving the main Theorem of [7] (where A is the operator defined by putting $A(x) = g + \lambda\, KF(x)$).

Theorem. *Let X, Y be finite dimensional Banach spaces. We consider the following hypotheses :*

a) $g \in L^1(D, X)$*;*

b) *$f\colon D \times X \to Y$ verifies Caratheodory hypotheses, i.e. f is strongly measurable with respect to $t \in D$, for all $x \in X$, and continuous with respect to $x \in X$, for almost all $t \in D$;*

c) *there are $a \in L^1(D)$ and $b \geq 0$ such that*

$$\|f(t,x)\| \leq a(t) + b\|x\|\,, \qquad t \in D, \quad x \in X \;;$$

d) *$k:\; D \times D \to L(Y,X)$ (the space of bounded linear operators from Y into X) is strongly measurable and the linear operator K defined in the introduction maps $L^1(D,Y)$ into $L^1(D,X)$ continuously;*

e) *The functions $s \to k(t,s)$ are in $L^\infty(D, L(Y,X))$ for almost all $t \in D$;*

f) $|\lambda|\, b\|K\| < 1$.

Then the equation (HIE) has at least a solution $x \in L^1(D,X)$.

Proof: Let us put $s = (\|g\| + |\lambda|\, \|K\|\, \|a\|)/(1 - |\lambda|\, b\|K\|)$. If B_s denotes the ball of $L^1(D,X)$ centered at θ with radius s, it is very easy to see that A maps B_s into itself, continuously. Now, we observe that for all $D_0 \subseteq D$ we have

$$\int_{D_0} \|Fy(t)\|\, dt \leq \int_{D_0} a(t)\, dt + b \int_{D_0} \|y(t)\|\, dt$$

from which it follows very easily that

$$\pi_1(FH) \leq b\, \pi_1(H) \tag{1}$$

for any bounded subset H of $L^1(D,X)$; on the other hand, for any bounded susbset Z of $L^1(D,Y)$ we have $\beta(KZ) \leq \|K\|\, \beta(Z)$, because of the linearity and continuity of K. Using Proposition 1 we see very easily that

$$\beta(AH) \leq |\lambda|\, b\|K\|\, \beta(H)$$

for any bounded subset H of $L^1(D,E)$ where A is not necessarily compact.

Moreover, we recall that $h = |\lambda|\, b\|K\| < 1$ by virtue of f). Now, define $Y_1 = B_s$, $Y_{n+1} = \overline{co}\, AY_n$ for $n \in \mathbb{N}$. It is easy to see that (Y_n) verifies all of the assumptions in v) because $h < 1$, and so $Y = \bigcap Y_n$ is a nonempty, closed, convex, relatively weakly compact subset of $L^1(D, Y)$ that is easily seen to be invariant under A. So it remains only to prove that AY is relatively compact, in such a way that an application of Schauder fixed point Theorem concludes the proof. Let (y_n) be a sequence in Y; Proposition 1 and (1) give that (Fy_n) has a weak converging subsequence $(Fy_{h(n)})$ in $L^1(D, Y)$. Since, by virtue of e) the operator $z \to \int_D k(t,s)\, z(s)\, ds$ for almost all $t \in D$ is weakly continuous on $L^1(D, Y)$, we obtain that $(KFy_{h(n)})$ and so $(Ay_{h(n)})$ is pointwise converging, for almost all $t \in D$. On the other hand, AY is equiabsolutely continuous in $L^1(D, X)$ since $AY \subseteq Y$; hence it is enough to apply Vitali convergence Theorem (see [6]) to prove that $(Ay_{h(n)})$ must be strongly converging in $L^1(D, X)$. We are done. ∎

At the end, we observe that if one assumes the existence of a third function h verifying assumptions similar to b) and c), it is possible to prove the existence of solutions of the following more general functional-integral equation

$$x(t) = h\Big(t,\, \lambda \int_D k(t,s)\, f(s, x(s))\, ds\Big)$$

recently considered in [3] and [7].

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