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THE WEAK AND STRONG CONVERGENCE IN METRIZABLE SPACES OVER VALUED FIELDS

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Abstract: Its is proved that a metrizable tvs E over a spherically complete non-archimedean non-trivially valued field is non-archimedean iff every weak null-sequence in E is a null-sequence.

Throughout this paper $\mathbb{K} = (\mathbb{K}, |\ |)$ denotes a spherically complete non-archimedean non-trivially complete valued field. A topological vector space (tvs) (E, τ) over $\mathbb{K}$ is said to be *locally convex* (or F-convex as in [6]) if τ has a basis of *absolutely convex* neighbourhoods of zero, where a subset A of E is called *absolutely convex* (in the sense of Monna [5]) if $ax + by \in A$ provided $x, y \in A$, $a, b \in \mathbb{K}$, $|a| \leq 1$, $|b| \leq 1$. Normed (F-normed) spaces E are defined in a natural way; a norm (F-norm) $\|\ \|$ on E is called *non-archimedean* (n.a.) if $\|x+y\| \leq \max\{\|x\|, \|y\|\}$, $x, y \in E$. A normed (F-normed) space E is called *non-archimedean* (n.a.) if the original topology of E can be defined by a n.a. norm (n.a. F-norm). Clearly every n.a. normed (F-normed) space is locally convex; by Proposition III.4 of [1] every normed space which as a tvs is locally convex must be n.a.; similarly one proves the same for F-normed spaces.

In [5], p. 74, Monna discussed relations between the convergence and the weak convergence of sequences in some normed spaces. It is known that every weakly convergent sequence in a locally convex space (E, τ) over a spherically complete field $\mathbb{K}$ is τ-convergent; this follows directly from Proposition 4.3 of [2] and Theorem 4.12 of [6]. In [4], Theorem 3, Martinez-Maurica and Perez-Garcia improved some results of Monna by showing that a *normed space E over a spherically complete field $\mathbb{K}$ is n.a. iff every weakly convergent sequence in E is convergent*. The assumption concerning the spherical completness (for the definition see [5]) can not be removed, cf. [7], Remark, p. 200.

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In this note we extend last result by showing the following

Theorem. *Let (E, τ) be a metrizable tvs over a spherically complete field $\mathbb{K}$. The following assertions are equivalent :*

(i) *(E, τ) is locally convex;*

(ii) *(E, τ) is a non-archimedean F-normed space;*

(iii) *Every weak null-sequence in E is a null-sequence.*

Proof: We have only to show the equivalence between (ii) and (iii). Let E^* be the topological dual of (E, τ). Assume that every weak null-sequence in E, i.e. in the weak topology $\sigma(E, E^*)$, is a null-sequence in τ. Then (as easily seen) E^* separates points of E, so $\sigma(E, E^*)$ is Hausdorff. Let (U_n) be a countable basis of τ-neighbourhoods of zero in E such that $U_{n+1} \subset U_n$, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ set

$$\Gamma(U_n) = \Big\{ \sum_{k=1}^{m} a_k\, x_k \colon\, x_k \in U_n,\ |a_k| \leq 1,\ m \in \mathbb{N} \Big\}$$

(the convex hull in the sense of Monna [5]). Then the sets $\Gamma(U_n)$ form a fundamental system of absolutely convex neighbourhoods of zero for the strongest locally convex topology $\gamma(\tau)$ on E weaker than τ. Since $\mathbb{K}$ is non-archimedean the topologies $\gamma(\tau)$ and τ have the same topological dual. Hence $\sigma(E, E^*) \leq \gamma(\tau) \leq \tau$. By Theorem 3.45 of [6] $\gamma(\tau)$ is metrizable; in fact there is a n.a. F-norm generating $\gamma(\tau)$. By our assumption the both topologies have the same convergent sequences; hence $\gamma(\tau) = \tau$. We proved that (E, τ) is a n.a. F-normed space. For the converse it is enough to apply Proposition 4.3 of [2] and Theorem 4.12 of [6]. ∎

We conclude this note by showing that the metrizability assumption in our Theorem can not be removed; our example uses some ideas of [3].

Example: Let (E, τ) be an infinite dimensional n.a. Banach space having a regular Schauder basis (y_n) (with coefficient functionals (y_n^*)). Let (a_n) be a sequence in $\mathbb{K}$ of non-zero elements such that $\sum |a_n| < \infty$. Since $y_n^*(x) \to 0$, $x \in E$, then the norm $p(x) = \sum |a_n\, y_n^*(x)|$ is well-defined on E. Define $U_\varepsilon = \{x \in E\colon p(x) \leq 1\}$, $\varepsilon > 0$. Observe that the topology $\mathcal{V}$ generated by p is non-locally convex. In fact, if $\mathcal{V}$ is locally convex, then $\Gamma(U_\varepsilon)$ is contained in U_1 for some $\varepsilon > 0$. Choose $\lambda \in \mathbb{K}$ with $|\lambda| < \varepsilon$ and put $z_n = \lambda\, a_n^{-1} y_n$, $n \in \mathbb{N}$. Then $z_n \in U_\varepsilon$, $n \in \mathbb{N}$, so

$$w_n = \lambda a_1^{-1} y_1 + \lambda a_2^{-1} y_2 + \cdots + \lambda a_n^{-1} y_n \ \in\ \Gamma(U_\varepsilon) + \Gamma(U_\varepsilon) + \cdots + \Gamma(U_\varepsilon) \ \subset\ \Gamma(U_\varepsilon)$$

for all $n \in \mathbb{N}$. On the other hand $p(w_n) = |\lambda|\, n$, a contradiction. Now consider the supremum topology $\xi = \sup\{\sigma(E, E^*), \mathcal{V}\}$, i.e. the weakest vector topology on E which is finer than $\sigma(E, E^*)$ and $\mathcal{V}$. Clearly $\sigma(E, E^*) \le \xi \le \tau$. The same argument used in the proof of Theorem 1 of [3] (which works also in our case) can be used to show that (E, ξ) is non-locally convex. Hence ξ is non-metrizable (otherwise $\xi = \tau$). Nevertheless, (by Theorem 3 of [4]) every $\sigma(E, E^*)$-convergent sequence is ξ-convergent.

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