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OSCILLATION PROPERTIES OF NONLINEAR DIFFERENCE EQUATIONS

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0 – Introduction and preliminary results

This paper is concerned with the oscillation of solutions of the equation:

$$(E) \qquad \Delta^4 V_n + b_{n+2}\, f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) = 0 \ ,$$

where Δ is the forward difference operation, i.e. $\Delta V_n = V_{n+1} - V_n$. It will be assumed throughout that the conditions below are satisfied:

i) $b_n > 0$ for $n > 1$;

ii) $f\colon \mathbb{R}^4 \to \mathbb{R}$ is continuous;

iii) $w f(w,x,y,z) \ge 0$ and $f(w,x,y,z) \ne 0$ whenever $w\,x\,y\,z \ne 0$.

The results in this paper were motivated by the work of W.E. Taylor, Jr. [2] on the equation

$$(e) \qquad \Delta^4 V_n + b_{n+2}\, V_{n+2} = 0 \ .$$

Many of the results herein extend results of [2]; however, some theorems are new since the order of (E) need not be four.

By a solution of (E) we mean a real sequence V satisfying equation (E) for $n = 1,2,3,...$. A solution V of (E) is called *nonoscillatory* if it is eventually positive or eventually negative. Otherwise, a solution V of (E) is called *oscillatory*. A solution V of (E) is called *quickly oscillatory* if $V_n V_{n+1} < 0$ for all n sufficiently large, or equivalently, $V_n = (-1)^n\, a_n$ where a_n is a sequence of positive numbers or negative numbers.

In this paper, we only concern the solutions of (E) which $\Delta^4 V_n \not\equiv 0$ for all n sufficiently large.

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We begin the study of this equation by considering a functional which plays a vital role in the investigation.

Lemma 1.1. *If V_n is a solution of (E), define $F(V_n) = V_{n+1}\Delta^3 V_n - \Delta V_n\,\Delta^2 V_n$. Then $F(V_n)$ is nonincreasing for $n \geq 1$.*

Proof: Let V be any solution of (E). Then

$$\begin{aligned}\Delta F(V_n) &= V_{n+2}\,\Delta^4 V_n + \Delta V_{n+1}\,\Delta^3 V_n - \Delta V_{n+1}\,\Delta^3 V_n - (\Delta^2 V_n)^2\\ &= V_{n+2}\,\Delta^4 V_n - (\Delta^2 V_n)^2\\ &= -b_{n+2}\,V_{n+2}\,f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) - (\Delta^2 V_n)^2 \leq 0\ .\end{aligned}$$

Therefore $F(V_n)$ is nonincreasing for all solutions of (E). ∎

Because of this result, the limit of $F(V_n)$ as $n \to \infty$ exists of $F(V_n) \to -\infty$ as $n \to \infty$. If $\lim_{n\to\infty} F(V_n) \geq 0$, we call the solution V of (E) a *Type I solution*. If $\lim_{n\to\infty} F(V_n) < 0$, we call V a *Type II solution*. Furthermore, if $-\infty < \lim_{n\to\infty} F(V_n) < 0$, we call V a *Type II(a) solution*, and if $\lim_{n\to\infty} F(V_n) = -\infty$, we call V a *Type II(b) solution*.

Theorem 1.2. *If V is a Type I or Type II(a) solution, then*

(1) $\sum^\infty V_{n+2}\,b_{n+2}\,f(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}) < \infty$;

(2) $\sum^\infty (\Delta^2 V_n)^2 < \infty$;

(3) $\sum^\infty (\Delta^3 V_n)^2 < \infty$;

(4) $\lim_{n\to\infty} \Delta^2 V_n = \lim_{n\to\infty} \Delta^3 V_n = 0$.

Proof: For Type I and Type II(a) solutions, $\lim_{n\to\infty} F(V_n) = k > -\infty$ and $F(V_n) \geq k$ for all $n = 0, 1, 2, 3, \ldots$. Then,

$$\begin{aligned}F(V_m) &= \sum_{n=0}^{m-1} \Delta F(V_n) + F(V_0)\\ &= -\sum_{n=0}^{m-1} V_{n+2}\,b_{n+2}\,f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) - \sum_{n=0}^{m-1} (\Delta^2 V_n)^2 + F(V_0)\\ &\geq k\ .\end{aligned}$$

Since $V_{n+2}\,b_{n+2}\,f(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}) \geq 0$ and $(\Delta^2 V_n)^2 \geq 0$, then

$$\sum_{n=0}^{m-1} V_{n+2}\,b_{n+2}\,f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) \leq F(V_0) - k < \infty$$

and

$$\sum_{n=0}^{m-1} (\Delta^2 V_n)^2 \le F(V_0) - k < \infty \ .$$

Let $m \to \infty$, then

$$\sum_{n=0}^{\infty} V_{n+2}\, b_{n+2}\, f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) \le F(V_0) - k < \infty$$

and

$$\sum_{n=0}^{\infty} (\Delta^2 V_n)^2 \le F(V_0) - k < \infty \ .$$

Since $(\Delta^3 V_n)^2 \le 2(\Delta^2 V_{n+1})^2 + 2(\Delta^2 V_n)^2$, then

$$\sum_{n=0}^{\infty} (\Delta^3 V_n)^2 \le 2 \sum_{n=0}^{\infty} (\Delta^2 V_{n+1})^2 + 2 \sum_{n=0}^{\infty} (\Delta^2 V_n)^2 < \infty \ .$$

Since $\sum_{n=0}^{\infty} (\Delta^2 V_n)^2 < \infty$ and $\sum_{n=0}^{\infty} (\Delta^3 V_n)^2 < \infty$, then $(\Delta^2 V_n)^2 \to 0$ and $(\Delta^3 V_n)^2 \to 0$ as $n \to \infty$. Therefore $\lim_{n\to\infty} \Delta^2 V_n = \lim_{n\to\infty} \Delta^3 V_n = 0$. ∎

Theorem 1.3. *If V is a Type II solution of (E), then V_n is unbounded.*

Proof: Consider $H(V_n) = V_n\, \Delta^2 V_n - (\Delta V_n)^2$.

$$\begin{aligned} \Delta H(V_n) &= \Delta(V_n\, \Delta^2 V_n) - \Delta(\Delta V_n)^2 \\ &= V_{n+1}\, \Delta^3 V_n + \Delta V_n\, \Delta^2 V_n - 2\, \Delta V_n\, \Delta^2 V_n - (\Delta^2 V_n)^2 \\ &= F(V_n) - (\Delta^2 V_n)^2 \ . \end{aligned}$$

Since V is a Type II solution of (E), $F(V_n)$ is negative and bounded away from zero for all n sufficiently large; hence $\Delta H(V_n)$ is negative and bounded away from zero. So $H(V_n) \to -\infty$ as $n \to \infty$. Thus at least one of V_n, ΔV_n, $\Delta^2 V_n$ must be unbounded. If $\Delta^2 V_n$ is unbounded, then ΔV_n is unbounded. If ΔV_n is unbounded, then V_n is unbounded. Therefore V_n must be unbounded. ∎

2 – Nonoscillation results

Now we discuss the behavior of nonoscillatory solutions.

Lemma 2.1. *If V is a nonoscillatory solution of (E), then either*

$$\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^4 V_n$$

for all n sufficiently large, or

$$\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^2 V_n = -\operatorname{sgn} \Delta^4 V_n$$

for all n sufficiently large, and $\lim_{n\to\infty} \Delta^2 V_n = \lim_{n\to\infty} \Delta^3 V_n = 0$.

Proof: Assume V is a nonoscillatory solution of (E), where $V_n > 0$ for all n sufficiently large. Since $wf(w,x,y,z) \geq 0$ and $b_n > 0$, then

$$V_{n+2}\, b_{n+2}\, f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) \geq 0 \ .$$

For all n sufficiently large, $V_{n+2} > 0$, so we have

$$\Delta^4 V_n = -b_{n+2}\, f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) \leq 0$$

for all n sufficiently large. Then $\Delta^3 V_n$ is nonincreasing. For all n sufficiently large, since $\Delta^4 V_n \not\equiv 0$, the following two cases must be considered: case I, $\Delta^3 V_n > 0$, case II, $\Delta^3 V_n < 0$.

In case I, $\Delta^2 V_n$ is increasing, then either $\Delta^2 V_n > 0$ or $\Delta^2 V_n < 0$ for all n sufficiently large.

If $\Delta^2 V_n > 0$, then $\Delta^2 V_n > k > 0$ for all n sufficiently large. This means $\Delta V_n \to +\infty$ and $V_n \to +\infty$ as $n \to \infty$; then for n sufficiently large,

$$V_n > 0, \quad \Delta V_n > 0, \quad \Delta^2 V_n > 0, \quad \Delta^3 V_n > 0 \ ,$$

and

$$f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) \neq 0 \ ;$$

then $\Delta^4 V_n < 0$.

If $\Delta^2 V_n < 0$, $\lim_{n\to\infty} \Delta^2 V_n = r \leq 0$ because $\Delta^3 V_n > 0$. If $r < 0$, then $\Delta V_n \to -\infty$ and $V_n \to -\infty$ as $n \to \infty$. This contradicts $V_n > 0$. Therefore $r = 0$. Hence ΔV_n is decreasing. Since $V_n > 0$, then $\Delta V_n > 0$. Therefore for all n sufficiently large,

$$V_n > 0, \quad \Delta V_n > 0, \quad \Delta^2 V_n < 0, \quad \Delta^3 V_n > 0 \ ,$$

and

$$f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) \neq 0 \ ;$$

then $\Delta^4 V_n < 0$. Since $\Delta^3 V_n > 0$, $\Delta^2 V_n < 0$, and $\lim_{n\to\infty} \Delta^2 V_n = 0$, then

$$\lim_{n\to\infty} \Delta^2 V_n = \lim_{n\to\infty} \Delta^3 V_n = 0 \ .$$

In case II, for n sufficiently large, there exists $r < 0$ such that $\Delta^3 V_n < r < 0$ because $\Delta^3 V_n$ is decreasing. This means $\Delta^2 V_n \to -\infty$, $\Delta V_n \to -\infty$, and $V_n \to -\infty$ as $n \to \infty$, which is impossible.

Similarly, when $V_n < 0$ for all n sufficiently large, then either

$$V_n < 0, \quad \Delta V_n < 0, \quad \Delta^2 V_n < 0, \quad \Delta^3 V_n < 0 \ \text{ and } \ \Delta^4 V_n > 0$$

or

$$V_n < 0, \quad \Delta V_n < 0, \quad \Delta^2 V_n > 0, \quad \Delta^3 V_n < 0, \quad \Delta^4 V_n > 0$$

and $\lim_{n\to\infty} \Delta^2 V_n = \lim_{n\to\infty} \Delta^3 V_n = 0$. ∎

Theorem 2.2. *If V is a nonoscillatory solution of (E), then the following statements are equivalent:*

(1) *V_n is a Type I solution;*

(2) *$\sum^\infty (\Delta^2 V_n)^2 < \infty$;*

(3) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^2 V_n = -\operatorname{sgn} \Delta^4 V_n$.

Proof: First we prove (2) from (1). If V is a nonoscillatory Type I solution, by Theorem 1.2, $\sum^\infty (\Delta^2 V_n)^2 < \infty$.

Now we prove (3) from (2). Suppose V is nonoscillatory and $\sum^\infty (\Delta^2 V_n)^2 < \infty$. There are only two cases:

(a) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^4 V_n$;

(b) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^2 V_n = -\operatorname{sgn} \Delta^4 V_n$.

Case (a) is impossible because $\lim_{n\to\infty} \Delta^2 V_n \neq 0$ and $\sum^\infty (\Delta^2 V_n)^2 < \infty$. Thus (b) must be true.

We prove (1) from (3). Suppose V is nonoscillatory and satisfies $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^2 V_n$; then

$$\begin{aligned} F(V_n) &= V_{n+1}\, \Delta^3 V_n - \Delta V_n\, \Delta^2 V_n \\ &= \operatorname{sgn} V_{n+1}\ \operatorname{sgn} \Delta^3 V_n\, |V_{n+1}\, \Delta^3 V_n| - \operatorname{sgn} \Delta V_n\ \operatorname{sgn} \Delta^2 V_n\, |\Delta V_n\, \Delta^2 V_n| \\ &= |V_{n+1}\, \Delta^3 V_n| + |\Delta V_n\, \Delta^2 V_n| \geq 0\ . \end{aligned}$$

Therefore V is a Type I solution. ∎

Lemma 2.3. *If V is a Type II(a) solution of (E), then V cannot be nonoscillatory.*

Proof: Assume a Type II(a) solution V is nonoscillatory and $V_n > 0$ for all $n > N$. The proof is similar if $V_n < 0$ for all $n > N$. According to Lemma 2.1, there are only two possibilities:

(**a**) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^4 V_n$,

(**b**) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^2 V_n = -\operatorname{sgn} \Delta^4 V_n$.

According to Theorem 1.2, $\lim_{n\to\infty} \Delta^2 V_n = \lim_{n\to\infty} \Delta^3 V_n = 0$, so only (b) can occur. By Theorem 2.2, V is a Type I solution; this contradicts V is a Type II(a) solution. Therefore V is oscillatory. ∎

Therefore, if V is a nonoscillatory solution of (E), V is either a Type I solution or a Type II(b) solution.

Theorem 2.4. *If V is a nonoscillatory solution of (E), then the following are equivalent:*

(**1**) *V is a Type II(b) solution;*

(**2**) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^4 V_n$ *for all n sufficiently large;*

(**3**) *There exists $k > 0$ such that $|\Delta^2 V_n| \geq k$ for all n sufficiently large.*

Proof: First we prove (2) from (1). Since V is nonoscillatory, then for n sufficiently large, either

(**a**) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^4 V_n$,

or

(**b**) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^2 V_n = -\operatorname{sgn} \Delta^4 V_n$.

By Theorem 2.3, (b) implies that V is a Type I solution. Therefore, (a) must be true.

Now we prove (3) from (2). Assume $V_n > 0$ for $n \geq N$. Since $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^4 V_n$ for $n \geq N$, then $\Delta^2 V_n > 0$ and $\Delta^3 V_n > 0$ for $n \geq N$. So $\Delta^2 V_n$ is increasing when $n \geq N$. Let $k = \Delta^2 V_N > 0$, then $\Delta^2 V_n > k > 0$ for all $n \geq N$. Similarly, when $V_n < 0$ for $n \geq N$, $\Delta^2 V_n < \Delta^2 V_N < 0$ for $n > N$. (3) is proved.

We prove (1) from (3). Since V is nonoscillatory, then for n sufficiently large, either

(**a**) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^4 V_n$,

or

(**b**) $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^3 V_n = -\operatorname{sgn} \Delta^2 V_n = -\operatorname{sgn} \Delta^4 V_n$

and $\lim_{n\to\infty} \Delta^2 V_n = \lim_{n\to\infty} \Delta^3 V_n = 0$. But $|\Delta^2 V_n| > k > 0$ as $n \to \infty$, so (b) is impossible. Therefore, V cannot be a Type I solution. Also, V cannot be a Type II(a) solution because Type II(a) solutions are oscillatory. Thus V can only be a Type II(b) solution. ∎

By Theorems 2.2 and 2.4, if a solution V of (E) is nonoscillatory, then ΔV_n is eventually increasing or decreasing; if ΔV_n is bounded, then V is Type I, and if ΔV_n is unbounded, then V is Type II(b).

3 – Oscillation results

Combining Theorem 1.3 and Lemma 2.3, we get the following theorem.

Theorem 3.1. *If V is a Type II(a) solution of (E), then V is oscillatory and unbounded.*

Whether Type II(a) solutions actually exist remains an open question.

Theorem 3.2. *If $f(w,x,y,z)/w \geq r > 0$ and $\sum^\infty b_n = \infty$, then (E) cannot have nonoscillatory solutions.*

Proof: Suppose V is a nonoscillatory solution of (E) and V_n is eventually positive. The proof is similar if V_n is eventually negative. From (E), we have

$$\begin{aligned} \Delta^4 V_n &= -b_{n+2}\, f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) \\ &\leq -r\, b_{n+2}\, V_{n+2} < 0 \ . \end{aligned}$$

According to Lemma 2.1, there exists N such that when $n > N$, $V_n > 0$, $\Delta V_n > 0$ and $\Delta^3 V_n > 0$. So V_n is eventually increasing. Therefore, there exist M and α such that when $n > M$, $V_n > \alpha > 0$. Let $m = \max(N, M)$. Then for $n > m$,

$$0 < \Delta^3 V_n = \sum_{i=m}^{n-1} \Delta^4 V_i + \Delta^3 V_m \leq -\sum_{i=m}^{n-1} r\, b_{i+2}\, V_{i+2} + \Delta^3 V_m \ .$$

Hence,

$$\sum_{i=m}^{n-1} r\, b_{i+2}\, V_{i+2} < \Delta^3 V_m \ .$$

Thus,

$$0 < r\,\alpha \sum_{i=m}^{n-1} b_{i+2} < \Delta^3 V_m \ .$$

Let $n \to \infty$, then $0 < \sum_{i=m+2}^\infty b_i \leq \Delta^3 V_m / r\alpha$. This contradicts $\sum^\infty b_n = \infty$. Therefore, V_n cannot be nonoscillatory. ∎

Theorem 3.3. *Equation (E) cannot have a quickly oscillatory solution.*

Proof: Suppose $V_n = (-1)^n a_n$ is a solution of (E) where $a_n > 0$. Then

$$\Delta^4 V_n = (-1)^n \Big(a_{n+4} + 4a_{n+3} + 6a_{n+2} + 4a_{n+1} + a_n\Big) \ .$$

(E) can be written

$$(-1)^n \Big(a_{n+4} + 4a_{n+3} + 6a_{n+2} + 4a_{n+1} + a_n\Big) + \\ + b_{n+2}\, f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) = 0 \ .$$

Thus,

$$\begin{aligned} b_{n+2} &= -(-1)^n \frac{\Big(a_{n+4} + 4a_{n+3} + 6a_{n+2} + 4a_{n+1} + a_n\Big)}{f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big)} \\ &= -\frac{a_{n+2}\Big(a_{n+4} + 4a_{n+3} + 6a_{n+2} + 4a_{n+1} + a_n\Big)}{V_{n+2}\, f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big)} < 0 \ . \end{aligned}$$

This contradicts $b_n > 0$. Thus (E) cannot have a quickly oscillatory solution. ∎

4 – Examples

Example 1. $V_n = n - (1/2)^n$ is a solution of (E), where

$$b_n = 2^{2n+8} / \Big(5(n+2)\, 2^{n+2} - 5\Big) \ ,$$

$$f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) = V_{n+2}\Big((\Delta^2 V_{n+2})^2 + (\Delta^3 V_{n+2})^2\Big) \ .$$

Then,

$$\begin{aligned} V_n &= n - (1/2)^n > 0 \quad \text{(when } n > 0) \ , \\ \Delta V_n &= (1/2)^{n+1} + 1 > 0 \ , \\ \Delta^2 V_n &= -(1/2)^{n+2} < 0 \ , \\ \Delta^3 V_n &= (1/2)^{n+3} > 0 \ , \\ \Delta^4 V_n &= -(1/2)^{n+4} < 0 \ . \end{aligned}$$

Hence this is a nonoscillatory Type I solution. Also $\sum^\infty (\Delta^2 V_n)^2 < \infty$. Actually,

$$\begin{aligned} F(V_n) &= V_{n+1}\, \Delta^3 V_n - \Delta V_n\, \Delta^2 V_n \\ &= \Big(n + 1 - (1/2)^{n+1}\Big)\, (1/2)^{n+3} + \Big((1/2)^{n+1} + 1\Big)\, (1/2)^{n+2} > 0 \ , \end{aligned}$$

and $\lim_{n\to\infty} F(V_n) = 0$. Also, $\sum^\infty (\Delta^2 V_n)^2 = \sum^\infty (1/2)^{2n+4} < \infty$.

Example 2. $V_n = n(n-1) - (1/2)^n$ is a solution of (E), where

$$b_n = 2^{2n+4}/\Big(n(n-1)2^n - 1\Big) ,$$

$$f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) = V_{n+2}(\Delta^3 V_{n+2})^2 .$$

Then

$$V_n = n(n-1) - (1/2)^n > 0 \quad (\text{when } n > 1) ,$$

$$\Delta V_n = 2n + (1/2)^{n+1} > 0 ,$$

$$\Delta^2 V_n = 2 - (1/2)^{n+2} > 0 ,$$

$$\Delta^3 V_n = (1/2)^{n+3} > 0 ,$$

$$\Delta^4 V_n = -(1/2)^{n+4} < 0 .$$

Therefore, V_n is nonoscillatory Type II(b) solution. $\lim_{n\to\infty} \Delta^2 V_n = 2$, and $\lim_{n\to\infty} F(V_n) = -\infty$.

Example 3. $V_n = (\sqrt{2}+1)^n \sin(n\pi/4)$ is an oscillatory solution of (E), where

$$b_n = 2\Big[(\sqrt{2}+1)^{n+1} \sin(n\pi/4 + \pi/4) - (\sqrt{2}+1)^n \sin(n\pi/4)\Big]^{-2} ,$$

$$f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) = V_{n+2}(\Delta V_{n+2})^2 .$$

Then,

$$\begin{aligned} \Delta^2 V_n &= V_{n+2} - 2V_{n+1} + V_n \\ &= (1+\sqrt{2})^n \Big[(3+2\sqrt{2}) \sin(n\pi/4 + \pi/2) \\ &\quad - (2+2\sqrt{2}) \sin(n\pi/4 + \pi/4) + \sin(n\pi/4)\Big] . \end{aligned}$$

When $n = 8i$ (i is a positive integer),

$$\begin{aligned} \Delta^2 V_{8i} &= (1+\sqrt{2})^{8i} (3 + 2\sqrt{2} - \sqrt{2} - 2) \\ &= (1+\sqrt{2})^{8i+1} . \end{aligned}$$

When $i \to \infty$, $\Delta^2 V_{8i} \to \infty$. According to Theorem 1.2 (4), V_n cannot be Type I or Type II(a). Therefore, this V_n is a Type II(b) solution.

Example 4. $V_n = (\sqrt{2}-1)^n \sin(n\pi/4)$ is an oscillatory solution of (E), where

$$b_n = 2\Big[(\sqrt{2}-1)^{n+1} \sin(n\pi/4 + \pi/4) - (\sqrt{2}-1)^n \sin(n\pi/4)\Big]^{-2} ,$$

$$f\Big(V_{n+2}, \Delta V_{n+2}, \Delta^2 V_{n+2}, \Delta^3 V_{n+2}\Big) = V_{n+2}(\Delta V_{n+2})^2 .$$

Since $\lim_{n\to\infty} V_n = 0$, according to Theorem 1.3, V cannot be Type II. Therefore, this V is a Type I solution.

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