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DISTRIBUTIVITY AND WELLFOUNDED SEMILATTICES

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Abstract: It is shown that distributivity over two elements is enough to guarantee distributivity over an arbitrary number of elements for wellfounded semilattices.

1 – Basic definitions

Throughout this paper a semilattice shall mean a meet semilattice, i.e. a set S with a function $\wedge\colon S^2 \to S$ such that $\forall\, a,b,c \in S$, $a \wedge a = a$, $a \wedge b = b \wedge a$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$. The semilattice S means $(S, \wedge)$ is a semilattice. If S is a semilattice, $\wedge$ induces a natural partial order $\leq$ on S by $a \leq b$ if and only if $a \wedge b = a$. We use $a < b$ if $a \leq b$ and a is not equal to b.

Definition. A partial order $\leq$ on a set S is wellfounded if there is no infinite sequence $\{s_n\}_{n=0}^{\infty}$ of elements of S such that $\cdots < s_2 < s_1 < s_0$.

Definition. A semilattice S is wellfounded if the induced partial order $\leq$ on S is wellfounded.

If S is a semilattice, $a, b \in S$, $a \vee b$ denotes the least upper bound of a and b under $\leq$ if it exists.

2 – Distributivity

Definition. A semilattice S satisfies the condition (D_n):

$$x \wedge (s_1 \vee s_2 \vee \ldots \vee s_n) = (x \wedge s_1) \vee (x \wedge s_2) \vee \ldots \vee (x \wedge s_n)$$

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if whenever the left-hand side is defined, then so is the right-hand side and they are equal. The class of semilattices satisfying (D_n) is denoted by D_n.

Definition. A prime semilattice is a semilattice S such that when x, $s_1, s_2, ..., s_n \in S$, if $s_1 \vee s_2 \vee s_3 \vee \ldots \vee s_n$ exists, then

$$(x \wedge s_1) \vee (x \wedge s_2) \vee \ldots \vee (x \wedge s_n)$$

also exists and

$$x \wedge (s_1 \vee s_2 \vee \ldots \vee s_n) = (x \wedge s_1) \vee (x \wedge s_2) \vee \ldots \vee (x \wedge s_n) \ .$$

The class of prime semilattices is denoted by D.

Prime semilattices were first introduced in [1] and (D_n) in [2].

Clearly

$$D_2 \supseteq D_3 \supseteq \ldots \supseteq D_n \supseteq D_{n+1} \supseteq \ldots \supseteq D$$

and

$$D = \bigcap_{n=2}^{\infty} D_n \ .$$

In [2] it is stated without example that one can construct a semilattice satisfying (D_m) but not (D_n) with $m < n$. In [4] it is shown that if S is a finite semilattice, then $S \in D_2$ if and only if $S \in D$. This result was also known to Schein ([3]) and raises the problem of determining which type of semilattices have this property.

Theorem. *Let S be a wellfounded semilattice. Then $S \in D_2$ if and only if $S \in D$.*

Proof: The "only if" part is straightforward from the inclusions. To show the "if" portion we shall see $S \in D_n \ \forall n \geq 2$ whenever $S \in D_2$. Fix n, let $x, s_1, s_2 \ldots s_n \in S$ and assume $s_1 \vee s_2 \vee \ldots \vee s_n$ is defined. We only need to show that $s_1 \vee s_2 \vee \ldots \vee s_{k-1}$ exists whenever $s_1 \vee s_2 \vee \ldots \vee s_k$ does for $k \leq n$, as then

$$\begin{aligned}
x \wedge (s_1 \vee s_2 \ldots \vee s_{n-1} \vee s_n) &= x \wedge \Big((s_1 \vee s_2 \vee \ldots \vee s_{n-1}) \vee s_n\Big) \\
&= \Big(x \wedge (s_1 \vee s_2 \vee \ldots \vee s_{n-1})\Big) \vee (x \wedge s_n) \quad \text{by} \ \ (D_2) \\
&= x \wedge \Big((s_1 \vee s_2 \vee \ldots \vee s_{n-2}) \vee s_{n-1}\Big) \vee (x \wedge s_n) \\
&= \Big(x \wedge (s_1 \vee s_2 \vee \ldots \vee s_{n-2}) \vee (x \wedge s_{n-1})\Big) \vee (x \wedge s_n) \\
\text{by } (D_2) \qquad &\ \ \vdots \\
&= (x \wedge s_1) \vee (x \wedge s_2) \vee \ldots \vee (x \wedge s_n) \ .
\end{aligned}$$

The required condition follows immediatly from the following proposition.

Proposition. *Let S be a wellfounded semilattice $s_1, s_2, ..., s_n \in S$ such that $s_1 \vee s_2 \vee \ldots \vee s_n$ is defined, then $s_1 \vee s_2 \vee \ldots \vee s_k$ exists for $k < n$.*

Proof: Let $x_0 = s_1 \vee s_2 \vee \ldots \vee s_n$ and let U be the set of upper bounds of $s_1, s_2, ..., s_k$. U is nonempty since $x_0 \in U$. Given $x_i \in U$ pick $y \in U$ such that $y \leq x_i$ and let $x_{i+1} = y \wedge x_i < x_i$, $x_{i+1} \in U$. If no such y exists, then $x_i = s_1 \vee s_2 \vee \ldots \vee s_k$. Otherwise we obtain a chain $x_0 > x_1 > x_2 > \ldots$. Since S is wellfounded, this chain must be finite and have a least element $x_j \in U$, but then $x_j = s_1 \vee s_2 \vee \ldots \vee s_k$. ∎

Corollary. *If S is a semilattice satisfying one of the following conditions, then $S \in D_2$ if and only if $S \in D$:*

1) *S is finite;*

2) *every element of S has only finitely many predecessors;*

3) *every chain in S is finite.*

We see that counterexamples to (D_2) implying (D_n) have not only to be infinite, but they also require infinitely many elements between $(s_1 \vee s_2 \vee \ldots \vee s_n)$ and the elements $s_1, s_2, ..., s_n$ chainwise as well.

Two interesting questions are to determine which other types of semilattices have the property that (D_2) is equivalent to (D_n) $\forall n$ and what is the minimal set of conditions for a counterexample.

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