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ON GENERALIZED LANDAU IDENTITIES

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Abstract: The identity

$$\sum_{d|r} \frac{\mu^2(d)}{\phi(d)} = \frac{r}{\phi(r)} ,$$

where μ is the Möbius function and ϕ is the Euler function, is known in the literature as the Landau identity. The present paper collects several extensive generalizations of that identity given by the author and P.J. McCarthy. Some of the extensive generalizations are further generalized. Also a large number of known special cases of the identities here are listed.

1 – Introduction

In 1900, Landau ([14], p. 182) proved that for each positive integer r

$$\sum_{d} \frac{1}{\phi(d)} = \frac{r}{\phi(r)} , \tag{1}$$

where the summation is over such divisors d of r that d is square-free or $d = 1$. Here ϕ denotes the Euler function. Clearly (1) can be written as

$$\sum_{d|r} \frac{\mu^2(d)}{\phi(d)} = \frac{r}{\phi(r)} , \tag{2}$$

where μ is the Möbius function. The identity (1) (or (2)) is called the Landau identity.

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In some recent papers the author and P.J. McCarthy have given several generalized Landau identities. The purpose of the present paper is to survey the generalizations and further generalize some of them.

In 1956, Cohen ([2], Theorem 9) proved that

$$\sum_{\substack{d \mid r \\ (d,n)=1}} \frac{\mu^2(d)}{\phi_k(d)} = \frac{r^k}{(n,r)^k} \frac{\phi_k((n,r))}{\phi_k(r)} , \tag{3}$$

where ϕ_k is the Jordan totient. For $n = k = 1$, (3) reduces to (2). With a principle similar to (3), the Landau identity has been generalized by Cohen ([4], Theorem 9; see also [5], Theorem 3.1) again, Liberatore and Tomasini ([15], Theorem 3.1) and Scheid ([22], formula (50)). So-called unitary analogues of this type of identities have been given by McCarthy ([17], formula (17)), Nageswara Rao ([20], Lemma 2) and Scheid ([23], formula (38)). Eugeni and Rizzi ([8], Prop. 3.1) have also given an identity of type (3), but their identity does not generalize (2). Theorem 1 here contains all these identities. Theorem 1 is Theorem 9 of [12] given in the setting of free Abelian semigroups.

In 1956, Cohen ([3], Theorem 6) also proved that

$$\sum_{d \mid r} \frac{C_k\Big((r/d_1)^k; d\Big)\, C_k\Big((r/d_2)^k; d\Big)}{\phi_k(d)} = \begin{cases} \dfrac{r^k}{\phi_k(e)} & \text{if } d_1 = d_2 = e, \\ 0 & \text{otherwise ,} \end{cases} \tag{4}$$

where $C_k(n;r)$ is the generalized Ramanujan sum given by

$$C_k(n;r) = \sum_{\substack{m (\mathrm{mod}\, r^k) \\ (m, r^k)_k = 1}} \exp(2\pi i\, m\, n/r^k) = \sum_{\substack{d \mid r \\ d^k \mid n}} d^k \mu(r/d) ,$$

where $(m, r^k)_k$ denotes the greatest k-th power divisor of r^k that divides m. For $d_1 = d_2 = r$ and $k = 1$, (4) reduces to the Landau identity (2). Note that for $k = 1$, $C_k(n;r)$ reduces to the Ramanujan sum $C(n;r)$.

Generalizations of the Landau identity involving generalized Ramanujan sums have also been given by Cohen ([7], formula (4.5)) again, McCarthy ([16], formula (18); [18], Theorem 4), Sita Ramaiah ([24], Theorem 7.4) and Sivaramakrishnan ([25], Theorem 3.4). Cohen ([6], Theorem 3.3) has also given a unitary analogue of (4) in the case $k = 1$, Theorem 10 of [11] contains all these identities, except for that by McCarthy [16]. We shall represent Theorem 10 of [11] here in Theorem 2. McCarthy's [16] identity will be generalized in Theorem 3. Finally, in Theorem 4 we shall present an identity generalizing both the Landau and the Brauer– Rademacher [1] identity

$$\phi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} \frac{d}{\phi(d)} \mu(r/d) = \mu(r) \sum_{d \mid (n,r)} d\mu(r/d) .$$

Theorem 4 is a kind of combination of identity (1) of [12] (see also [10], Chapter 3.1) and identity (11) of [13].

Lattice-theoretic generalizations of the Landau identity have been presented by McDonald ([19], Theorem 4.7) and Smith ([26], Theorem 3). Also a different kind of generalization of the Landau identity has been given by Hanumanthachari and Subrahmanyasastri ([19], Theorem 3.3). We do not consider these identities here.

If λ is the Liouville function, then multiplying both sides of (2) by $\lambda(r)$ and noting that $\lambda(r) = \lambda(r/d)\,\lambda(d)$ and $\lambda(d)\,\mu^2(d) = \mu(d)$ for all $d\,|\,r$ we obtain the following modified Landau identity:

$$\sum_{d|r} \frac{\lambda(r/d)\,\mu(d)}{\phi(d)} = \frac{r\,\lambda(r)}{\phi(r)}\ .$$

This kind of modification could be made with respect to generalized Landau identities as well.

2 – Preliminaries

Throughout the paper G will denote a free Abelian semigroup with a countable number of primes p. By an arithmetical function we mean a function of G into $\mathbb{C}$, the set of complex numbers.

For $n \in G$ let $A(n)$ be a non-empty subset of the set of divisors of n. Then the A-convolution [21] of two arithmetical functions f and g is defined by

$$(f A\, g)(n) = \sum_{d\in A(n)} f(d)\, g(n/d)\ .$$

An A-convolution is said to be regular if

i) The set of arithmetical functions forms a commutative ring with identity with respect to the ordinary addition and the A-convolution;

ii) The multiplicativity of f and g implies the multiplicativity of $f A\, g$;

iii) The function E defined by $E(n) = 1$ for all $n \in G$, has an inverse μ_A with respect to the A-convolution and $\mu_A(n) = 0$ or -1 whenever n $(\neq 1)$ is a prime power.

The inverse of f with respect to an A-convolution satisfying item (i) is defined by

$$f A f^{-1} = f^{-1} A\, f = E_0\ ,$$

where $E_0 = 1$ and $E_0(n) = 0$ for $n \neq 1$.

The Dirichlet convolution D, defined by $D(n) = \{d\colon\ d\,|\,n\}$, and the unitary convolution U, defined by $U(n) = \{d\colon\ d\,|\,n,\ (d, n/d) = 1\}$, are regular convolutions.

It is known [21] that an A-convolution is regular if, and only if,

(a) $A(mn) = \{de\colon\ d \in A(m),\ e \in A(n)\}$ whenever $(m, n) = 1$,

(b) For each prime power p^a ($a \in \mathbb{N}$) there exists a positive integer t ($= \tau_A(p^a)$) such that

$$A(p^a) = \{1, p^t, p^{2t}, ..., p^{st}\}\ ,$$

where $st = a$ and

$$A(p^{it}) = \{1, p^t, p^{2t}, ..., p^{it}\}\,, \qquad 0 \le i \le s\ .$$

An element $n \in G$ is said to be A-primitive if $A(n) = \{1, n\}$. For $n, r \in G$, $k \in \mathbb{N}$, the symbol $(n, r)_{A,k}$ denotes the largest k-th power divisor of n which belongs to $A(r)$. Further, denote

$$A_k(n) = \{d\colon\ d^k \in A(n^k)\}\ .$$

It is known [24] that the A_k-convolution is regular whenever the A-convolution is regular.

Throughout this paper A is an arbitrary but fixed regular convolution and k is an arbitrary but fixed positive integer.

An arithmetical function f is said [28] to be A-multiplicative if f is not identically zero and

$$f(n) = f(d)\, f(n/d)$$

whenever $d \in A(n)$. In particular, D-multiplicative functions are completely multiplicative functions and U-multiplicative functions are multiplicative functions.

It is known [28] that a multiplicative function f is A-multiplicative if, and only if,

$$f^{-1} = \mu_A\, f\ .$$

The generalized Möbius function μ_A [21] is the multiplicative function such that

$$\mu_A(p^a) = \begin{cases} 1 & \text{if } a = 0,\\ -1 & \text{if } p^a\ (a \in \mathbb{N}) \text{ is } A\text{-primitive},\\ 0 & \text{otherwise}\ . \end{cases}$$

We define an arithmetical function f to be an A-totient if

$$f = f_T A f_V^{-1} = f_T A(\mu_A\, f_V)\ ,$$

where f_T, the integral component of f, and f_V, the inverse component of f, are A-multiplicative. D-totients are the well-known totients [27].

A pair (f,g) of multiplicative functions f and g is said [12] to be admissible if $(fA_k\,g)(m) \neq 0$ for all $m \in G$ and if

$$\frac{f(p^a)}{(fA_k\,g)(p^a)} = \frac{f(p^t)}{(fA_k\,g)(p^t)}, \qquad t = \tau_{A_k}(p^a)\ ,$$

for all primes p and $a \geq 1$. It is known [12] that if f and g are A_k-totients with $f_V = g_T$, then the pair (f,g) is admissible. A large number of examples of admissible pairs is demonstrated in [12].

The arithmetical representation of Ramanujan's sum

$$C(n;r) = \sum_{d|(n,r)} d\mu(r/d)$$

prompts us to define a generalization of Ramanujan's sum by

$$S_{A,k}^{f,g}(n;r) = \sum_{d^k \in A((n,r^k)_{A,k})} f(d)\,g(r/d)\,, \qquad n,r \in G\ ,$$

where f and g are arithmetical functions. In particular, denote

$$S_{A,k}^{N^{ku},\mu_{A_k}}((n_i);r) = C_{A,k}^{(u)}((n_i);r) = C_{A,k}(n_1,\ldots,n_u;r)\ ,$$

where $N^{ku}(n) = n^{ku}$ for all n, and (n_i) is the greatest common divisor of $n_1,\ldots,n_u$.

It is known [24] that the generalized Ramanujan sum admits the generalized Hölder identity

$$S_{A,k}^{f,\mu_{A_k}g}(n;r) = \frac{\Big(fA_k(\mu_{A_k}g)\Big)(r)\,\Big(\mu_{A_k}g\Big)(m)}{\Big(fA_k(\mu_{A_k}g)\Big)(m)}\ , \tag{5}$$

where $m^k = r^k/(n,r^k)_{A,k}$, f is an A_k-multiplicative function and g is a multiplicative function.

Let $r \in G$ be fixed. An arithmetical function $f(n)$ is said to be even $(\bmod\, r)$ if $f(n) = f((n,r))$ for all n. More generally, an arithmetical function $f(n)$ is said [24] to be A-k-even $(\bmod\, r)$ if for all $n \in G$,

$$f(n) = f\Big((n,r^k)_{A,k}\Big)\ .$$

An arithmetical function $f(n_1,\ldots,n_u)$ is said [13] to be totally A-k-even $(\bmod\, r)$ if there exists an A-k-even function $F(n)$ $(\bmod\, r)$ such that for all $n_1,\ldots,n_u \in G$,

$$f(n_1,\ldots,n_u) = F((n_i))\ .$$

It is known [13] that $f(n_1,...,n_u)$ is totally A-k-even $(\mathrm{mod}\, r)$ if, and only if, it has a unique representation of the form

$$f(n_1,...,n_u) = \sum_{d\in A_k(r)} a(d)\, C_{A,k}(n_1,...,n_u;d)\ ,$$

where

$$a(d) = r^{-ku} \sum_{e\in A_k(r)} F(r^k/e^k)\, C^{(u)}_{A,k}(r^k/d^k;e)\ . \tag{6}$$

It is also known [13] that $f(n_1,...,n_u)$ is totally A-k-even $(\mathrm{mod}\, r)$ if, and only if, there exists an arithmetical function h of two variables such that

$$f(n_1,...,n_u) = \sum_{d^k\in A(((n_i),r^k)_{A,k})} h(d,r/d)\ .$$

In this case

$$a(d) = r^{-ku} \sum_{e\in A_k(r/d)} h(r/e,e)\, e^{ku}\ . \tag{7}$$

Note that some of the above preliminaries have been verified only when $G=\mathbb{N}$. It is easy to see that the preliminaries hold for an arbitrary G.

3 – Identities

Theorem 1. *Let f and g be multiplicative functions. Then*

$$\frac{f\Big(((n,r^k)_{A,k})^{1/k}\Big)}{(fA_k\, g)\Big(((n,r^k)_{A,k})^{1/k}\Big)} \sum_{\substack{d\in A_k(r)\\ (n,d^k)_{A,k}=1}} \frac{g(d)}{(fA_k\, g)(d)}\, \mu_{A_k}(d) = \frac{f(r)}{(fA_k\, g)(r)}$$

for all $n,r\in G$ if, and only if, the pair (f,g) is admissible.

Theorem 2. *Suppose f is an A_k-multiplicative function such that $(fA_k\,\mu_{A_k})(r)\neq 0$ for all $r\in G$. Then for all $m,n,r\in G$*

$$\sum_{d\in A_k(r)} \frac{S^{f,\mu_{A_k}}_{A,k}(m;d)\, S^{f,\mu_{A_k}}_{A,k}(n;d)}{(fA_k\,\mu_{A_k})(d)} = \begin{cases} \dfrac{f(r)}{(fA_k\,\mu_{A_k})(r/\delta)} & \text{if } (m,r^k)_{A,k}=(n,r^k)_{A,k}=\delta^k,\\ 0 & \text{otherwise}\ . \end{cases}$$

Proofs of Theorems 1 and 2 follow applying multiplicativity and considering prime powers. We do not present the details. We recall that Theorem 9 of

[12] is Theorem 1 with $G = \mathbb{N}$ and Theorem 10 of [11] is exactly the same as Theorem 2 here. Theorem 9 of [12] is proved there. The proof of Theorem 10 of [11] is omitted.

Theorem 3. *Suppose s is a complex number and k is a positive integer. Let g be a multiplicative function such that $g(p^t) \neq p^{t(s+ku)}$ for all A_k-primitive prime powers p^t. Then for all $n_1, \ldots, n_u,\ r \in G$,*

$$\sum_{d \in A_k(r)} \frac{(\mu_{A_k} g)(d)}{\left(N^{s+ku}\, A_k(\mu_{A_k} g)\right)(d)}\, C_{A,k}(n_1, \ldots, n_u; d) =$$

$$= r^{ku}\, m^s\, \frac{\left(N^s\, A_k(\mu_{A_k} g)\right)\left(\left(((n_i), r^k)_{A,k}\right)^{1/k}\right)}{\left(N^{s+ku}\, A_k(\mu_{A_k} g)\right)(r)}\ ,$$

where $m^k = r^k/((n_i), r^k)_{A,k}$.

Proof: By (7), we have

$$S_{A,k}^{N^{s+ku}, \mu_{A_k} g}(n_1, \ldots, n_u; r) = \sum_{d \in A_k(r)} \alpha(d)\, C_{A,k}(n_1, \ldots, n_u; d)\ ,$$

where

$$\alpha(d) = r^{-ku} \sum_{e \in A_k(r/d)} (r/e)^{s+ku}\, (\mu_{A_k} g)(e)\, e^{ku} = d^s \left(N^s\, A_k(\mu_{A_k} g)\right)(r/d)\ .$$

Taking $d = m$ gives

$$\alpha(m) = m^s \left(N^s\, A_k(\mu_{A_k} g)\right)\left(\left(((n_i), r^k)_{A,k}\right)^{1/k}\right)\ . \tag{8}$$

On the other hand, by (6),

$$\alpha(d) = r^{-ku} \sum_{e \in A_k(r)} S_{A,k}^{N^{s+ku}, \mu_{A_k} g}(r^k/e^k; r)\, C_{A,k}^{(u)}(r^k/d^k; e)\ ,$$

and, in particular,

$$\alpha(m) = r^{-ku} \sum_{e \in A_k(r)} S_{A,k}^{N^{s+ku}, \mu_{A_k} g}(r^k/e^k; r)\, C_{A,k}^{(u)}((n_i); e)\ .$$

Applying the generalized Hölder identity (5) gives

$$\alpha(m) = r^{-ku} \sum_{e \in A_k(r)} \frac{(N^{s+ku} A_k\, \mu_{A_k} g)(r)\, (\mu_{A_k} g)(e)}{(N^{s+ku} A_k\, \mu_{A_k} g)(e)}\, C_{A,k}^{(u)}((n_i); e)\ . \tag{9}$$

Now, combining (8) and (9) proves Theorem 3. ∎

Remark. Theorem 3 can also be proved applying multiplicativity and considering prime powers.

Theorem 4. *Let f and g be multiplicative functions and let h be an arithmetical function with $h(1) \neq 0$. Then*

$$\sum_{\substack{d \in A_k(r) \\ (n,(d,m)^k)_{A,k}=1}} \frac{f((d,m))}{(f A_k\, g)((d,m))}\, h(r/d) = \tag{10}$$
$$= \sum_{d \in A_k((r,m))} (h A_k E)(r/d)\, \frac{\mu_{A_k}(d)}{(f A_k\, g)(d)}\, S_{A,k}^{f,g}(n;d)$$

for all $m,n,r \in G$ such that $m,r \in A_k(s)$ for some s if, and only if, the pair (f,g) is admissible.

Proof: Identity (10) can be written in the form

$$\left[\frac{f((\cdot,m))}{(f A_k\, g)((\cdot,m))}\, E_0\Big((n,(\cdot,m)^k)_{A,k}\Big) A_k\, h\right](r) =$$
$$= \left[h A_k E A_k\, \chi_{A_k}(m,\cdot)\, \frac{\mu_{A_k}}{f A_k\, g}\, S_{A,k}^{f,g}(n;\cdot)\right](r)\ ,$$

where $\chi_{A_k}(m;d) = 1$ if $d \in A_k(m)$, and $= 0$ otherwise. Since $h(1) \neq 0$ (that is, since the inverse of h with respect to the convolution A_k exists), the above identity is

$$\frac{f((r,m))}{(f A_k\, g)((r,m))}\, E_0\Big((n,(r,m)^k)_{A,k}\Big) = \left[E A_k\, \chi_{A_k}(m;\cdot)\, \frac{\mu_{A_k}}{f A_k\, g}\, S_{A,k}^{f,g}(n;\cdot)\right](r)\ . \tag{11}$$

Applying multiplicativity and considering prime powers it can be proved that (11) holds for all $m,n,r \in G$ such that $m,r \in A_k(s)$ for some s if, and only if, the pair (f,g) is admissible. ∎

Remark. Taking $h = E_0$ in (10) leads to the Landau type identities and $h = \mu_{A_k}$ to the Brauer–Rademacher type identities. On the other hand, taking $m = 1$ in (10) we obtain (1) of [12], and taking $f = N^{ku}$ and $g = \mu_{A_k}$ we obtain (11) of [13].

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