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AN EXTENSION OF AMIR–LINDENSTRAUSS THEOREM

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Abstract: In this paper we give an extension of Amir-Lindenstrauss Theorem on weak* sequential compactness as follows: if a locally convex space X has a sequence $K_1 \subset K_2 \subset K_3 \subset \ldots$ of relatively weakly countably compact sets such that $\mathrm{span}(\bigcup_{n=1}^{\infty} K_n)$ is dense in X, then each weak* compact absolutely convex subset of X' is weak* sequentially compact. Using the extension we obtain an improvement of Kalton's closed graph theorem.

1 – Introduction

Let $(X,\mathcal{T})$ be a locally convex, Hausdorff, topological vector space (abbreviated l.c.s.) and X' the topological dual of $(X,\mathcal{T})$. Alaoglu–Bourbaki Theorem says that if U is a neighborhood of 0 in $(X,\mathcal{T})$ then U^0 is $\sigma(X',X)$-compact [1, p. 264 or 2, Th. 9-1-10]. An important and interesting problem is: when is $U^0 \sigma(X',X)$-sequentially compact? Moreover, when is each $\sigma(X',X)$-compact subset of $X' \sigma(X',X)$-sequentially compact? If the above problems are solved, we shall be allowed to use sequences, instead of filters, to study the weak* topology on U^0. As well known, if $(X,\mathcal{T})$ is a separable l.c.s. or a reflexive Banach space, then U^0 is $\sigma(X',X)$-sequentially compact [2, Th. 9-5-3 and Ex. 14-1-11]. Furthermore, the theorem of Amir and Lindenstrauss points out that any WCG (weakly compactly generated) Banach space has a weak* sequentially compact dual ball [3, p. 228]. In this paper we shall find some more general conditions for U^0 to be $\sigma(X',X)$-sequentially compact. Our main result includes the above known facts as particular cases. Besides, our proof is simple and direct. As an application of the main result, we obtain an extension of Kalton's closed graph theorem.

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2 – Main results

The following Lemma 1 is substantially a variant of the theorem of Dieudonn and Schwartz [1, p. 311].

Lemma 1. *Let $(X,\mathcal{T})$ be a l.c.s. and X' its topological dual. If there exists a metrizable locally convex topology η on X' such that $(X',\eta)'\subset X$, then each $\sigma(X',X)$-compact subset M of X' is $\sigma(X',X)$-sequentially compact.*

Proof: Suppose that $(X',\eta)'=H$, then $H\subset X$. Since $(M,\sigma(X',X)\,|\,M)$ is compact and $(M,\sigma(X',H)\,|\,M)$ is Hausdorff and the topology $\sigma(X',X)$ is stronger than $\sigma(X',H)$, we have $(M,\sigma(X',X)\,|\,M)=(M,\sigma(X',H)\,|\,M)$ [4, p. 32]. Thus M is $\sigma(X',H)$-compact. Now M is a weakly compact set in the metrizable l.c.s. (X',η), so M is weakly sequentially compact in (X',η); see [1, p. 311]. This means that M is $\sigma(X',H)$-sequentially compact. Since $\sigma(X',H)\,|\,M=\sigma(X',X)\,|\,M$, M is also $\sigma(X',X)$-sequentially compact. ∎

Theorem 1. *If there exists a sequence $K_1\subset K_2\subset\dots$ of (weakly) compact absolutely convex sets in $(X,\mathcal{T})$ such that $\overline{\operatorname{span}(\bigcup_{n=1}^\infty K_n)}=X$, then each $\sigma(X',X)$-compact subset M of X' is $\sigma(X',X)$-sequentially compact. Particularly, for any neighborhood U of 0 in $(X,\mathcal{T})$, U^0 is $\sigma(X',X)$-sequentially compact.*

Proof: Without loss of generality, we may assume that $2K_n\subset K_{n+1}$ for each $n\in\mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers. Then the polars K_n^0, $n\in\mathbb{N}$, form a base of neighborhoods of 0 for some locally convex topology η on X'. Since $\overline{\operatorname{span}(\bigcup_{n=1}^\infty K_n)}=X$, (X',η) is Hausdorff. And since (X',η) has a countable base of neighborhoods of 0, (X',η) is a metrizable l.c.s. Recalling the definition of the Mackey topology [1, p. 260], we know that η is weaker than the Mackey topology $\tau(X',X)$ and so $(X',\eta)'\subset X$. By Lemma 1, the result follows. This completes the proof. ∎

By using Theorem 1, we obtain immediately the following Corollary 1, which is well-known.

Corollary 1. *If $(X,\mathcal{T})$ is a reflexive Banach space or a separable l.c.s. or a WCG l.c.s. (namely X contains a weakly compact absolutely convex set whose linear span is dense in X), then each $\sigma(X',X)$-compact subset of X' is $\sigma(X',X)$-sequentially compact. Particularly U^0 is $\sigma(X',X)$-sequentially compact for any neighborhood U of 0 in $(X,\mathcal{T})$.*

The following Corollary 2 is also obvious.

Corollary 2. *If $(X,\mathcal{T})$ has a fundamental sequence $K_1 \subset K_2 \subset \ldots$ of absolutely convex (weakly) compact sets, then each $\sigma(X',X)$-compact subset of X' is $\sigma(X',X)$-sequentially compact.*

Inspired by [5], we deduce the following Corollary 3 and 4.

Corollary 3. *Let $(E,\xi) = \operatorname{ind}\lim(E_n,\xi_n)$ be an inductive limit of a sequence of reflexive Banach spaces [see 6, 7, 8] and there exist a continuous linear map $t\colon (E,\xi) \to (X,\mathcal{T})$ such that $\operatorname{range}(t)$ is dense in $(X,\mathcal{T})$, then each $\sigma(X',X)$-compact subset of X' is $\sigma(X',X)$-sequentially compact.*

Proof: Let K_n be the closed unit ball in the reflexive Banach space (E_n,ξ_n), then K_n is $\sigma(E_n,E_n')$-compact. Without loss of generality, we may assume that $K_1 \subset K_2 \subset \ldots$. Or else we may take $K_2' = K_1 + K_2$, $K_3' = K_3 + K_2'$, etc, since each K_n is $\sigma(E_m,E_m')$-compact for any $m \geq n$. We denote by t_n the restriction to E_n of t, then $t_n = t \,|\, E_n : (E_n,\xi_n) \to (X,\mathcal{T})$ is continuous and so is weakly continuous. Thus $t(K_n) = t_n(K_n)$ is $\sigma(X,X')$-compact for each $n \in \mathbb{N}$. Now we have a sequence of weakly compact absolutely convex sets $t(K_1) \subset t(K_2) \subset \ldots$ in $(X,\mathcal{T})$ such that

$$X = \overline{t(E)} = \overline{t\Big(\operatorname{span}\Big(\bigcup_{n=1}^{\infty} K_n\Big)\Big)} = \overline{\operatorname{span}\Big(\bigcup_{n=1}^{\infty} t(K_n)\Big)}\ .$$

By Theorem 1, the result follows. ∎

Corollary 4. *Let $(E,\xi) = \operatorname{ind}\lim(E_n,\xi_n)$ be an inductive limit of a sequence of WCG l.c.s. and there exist a continuous linear map $t\colon (E,\xi) \to (X,\mathcal{T})$ such that $\operatorname{range}(t)$ is dense in $(X,\mathcal{T})$, then each $\sigma(X',X)$-compact subset of X' is $\sigma(X',X)$-sequentially compact.*

Proof: Let K_n be $\sigma(E_n,E_n')$-compact absolutely convex set such that $\overline{\operatorname{span}(K_n)}^{E_n} = E_n$, where $\overline{\operatorname{span}(K_n)}^{E_n}$ denotes the closure of $\operatorname{span}(K_n)$ in (E_n,ξ_n). We may as well assume that $K_1 \subset K_2 \subset \ldots$. Obviously $t_n = t \,|\, E_n\colon (E_n,\xi_n) \to (X,\mathcal{T})$ is continuous, so $t(K_n) = t_n(K_n)$ is $\sigma(X,X')$-compact for each $n \in \mathbb{N}$. Remarking that the topology ξ_n is stronger than $\xi \,|\, E_n$, we have

$$\overline{\operatorname{span}\Big(\bigcup_{n=1}^{\infty} K_n\Big)}^{E} \supset \overline{\operatorname{span}(K_n)}^{E} \supset \overline{\operatorname{span}(K_n)}^{E_n} = E_n$$

for every $n \in \mathbb{N}$. This shows that

$$\overline{\operatorname{span}\Big(\bigcup_{n=1}^{\infty} K_n\Big)}^{E} = \bigcup_{n=1}^{\infty} E_n = E\ .$$

Since $t\colon (E,\xi) \to (X,\mathcal{T})$ is continuous, we know that $t^{-1}\Big(\overline{t(\operatorname{span}(\bigcup_{n=1}^{\infty} K_n))}\Big)$ is closed and so it contains $\overline{\operatorname{span}(\bigcup_{n=1}^{\infty} K_n)}^{E} = E$. Thus $t(E) \subset \overline{t(\operatorname{span}(\bigcup_{n=1}^{\infty} K_n))}$, so

$$X = \overline{t(E)} \;\subset\; \overline{t\Big(\operatorname{span}\Big(\bigcup_{n=1}^{\infty} K_n\Big)\Big)} \;=\; \overline{\operatorname{span}\Big(\bigcup_{n=1}^{\infty} t(K_n)\Big)}\ .$$

Now $t(K_1) \subset t(K_2) \subset \ldots$ is a sequence of $\sigma(X,X')$-compact absolutely convex sets such that

$$X = \overline{\operatorname{span}\Big(\bigcup_{n=1}^{\infty} t(K_n)\Big)}\ .$$

Hence the result follows from Theorem 1. ∎

Corollary 5. *Let $T_1 \subset T_2 \subset T_3 \subset \ldots$ be a sequence of totally bounded subsets of $(X,\mathcal{T})$ such that $\overline{\operatorname{span}(\bigcup_{n=1}^{\infty} T_n)} = X$, then for any neighborhood U of 0 in $(X,\mathcal{T})$, U^0 is $\sigma(X',X)$-sequentially compact.*

Proof: Since the absolutely convex hull of a totally bounded set is still totally bounded [2, Th. 7-1-5], we may assume that every T_n is absolutely convex, totally bounded. Denote the completion of $(X,\mathcal{T})$ by $(\widetilde{X},\widetilde{\mathcal{T}})$ and the closure of T_n in $(\widetilde{X},\widetilde{\mathcal{T}})$ by $\widetilde{T}_n$, then $\widetilde{T}_1 \subset \widetilde{T}_2 \subset \widetilde{T}_3 \subset \ldots$ is a sequence of compact absolutely convex subsets of $(\widetilde{X},\widetilde{\mathcal{T}})$ such that $\operatorname{span}(\bigcup_{n=1}^{\infty} \widetilde{T}_n)$ is dense in $(\widetilde{X},\widetilde{\mathcal{T}})$. For any neighborhood U of 0 in $(X,\mathcal{T})$, we denote by $\widetilde{U}$ the closure of U in $(\widetilde{X},\widetilde{\mathcal{T}})$, then $\widetilde{U}$ is a neighborhood of 0 in $(\widetilde{X},\widetilde{\mathcal{T}})$ [1, p. 148] and $U^0 = \widetilde{U}^0 \subset X' = (\widetilde{X},\widetilde{\mathcal{T}})'$. Applying Theorem 1 to $(\widetilde{X},\widetilde{\mathcal{T}})$, we know that $\widetilde{U}^0$ is $\sigma(X',\widetilde{X})$-sequentially compact, i.e. U^0 is $\sigma(X',X)$-sequentially compact. ∎

3 – An extension of Kalton's closed graph theorem

First let's recall the Kalton's closed graph theorem: let E be a Mackey space such that $(E',\sigma(E',E))$ is sequentially complete and F be a separable, B_r-complete l.s.c., then every closed graph linear map $t\colon E \to F$ is continuous [2, Th. 12-5-13 or 9]. By Theorem 1, we can extend the above result as follows.

Theorem 2. *Let E be a Mackey space such that $(E',\sigma(E',E))$ is sequentially complete and F be a B_r-complete l.s.c. If there exists a sequence $K_1 \subset K_2 \subset \ldots$ of $\sigma(F,F')$-compact absolutely convex sets such that $\overline{\operatorname{span}(\cup_{n=1}^{\infty} K_n)} = F$, then every closed graph linear map $t\colon E \to F$ is continuous.*

Proof: By [10], it is sufficient to prove that for any dense vector subspace H of $(F',\sigma(F',F))$, $\overset{\sqcup}{H}{}^{\sigma} \supset F'$ or $\overset{\sqcup}{H}{}^{\sigma} \cap F' = F'$; see the following Appendix.

Here $\overset{\sqcup}{H}{}^{\sigma}$ denotes the sequential completion of $(H, \sigma(H, F))$, i.e. the smallest sequentially closed set which contains H in the completion of $(H, \sigma(H, F))$. Since F is B_r-complete, we only need to prove that $\overset{\sqcup}{H}{}^{\sigma} \cap F'$ is a w^*-closed, i.e. for any neighborhood U of 0 in F, $\overset{\sqcup}{H}{}^{\sigma} \cap F' \cap U^0 = \overset{\sqcup}{H}{}^{\sigma} \cap U^0$ is $\sigma(F', F)$-closed. Let η denote the metrizable locally convex topology on F' generated by $\{K_n^0 \colon n \in \mathbb{N}\}$. For any subset A of F', $\overline{A}^{\sigma}$, $\overline{A}^{\tau}$ and $\overline{A}^{\eta}$ denote respectively the closure of A in $(F', \sigma(F', F))$, $(F', \tau(F', F))$ and (F', η). Since $\overset{\sqcup}{H}{}^{\sigma} \cap U^0$ is a convex subset of F' and η is weaker than $\tau(F', F)$,

$$\overline{(\overset{\sqcup}{H}{}^{\sigma} \cap U^0)}^{\sigma} = \overline{(\overset{\sqcup}{H}{}^{\sigma} \cap U^0)}^{\tau} \subset \overline{(\overset{\sqcup}{H}{}^{\sigma} \cap U^0)}^{\eta}.$$

Thus for any $y' \in \overline{(\overset{\sqcup}{H}{}^{\sigma} \cap U^0)}^{\sigma}$, $y' \in \overline{(\overset{\sqcup}{H}{}^{\sigma} \cap U^0)}^{\eta}$. Since η is metrizable, there exists a sequence $\{y'_n\} \subset \overset{\sqcup}{H}{}^{\sigma} \cap U^0$ such that $y'_n \to y'$ in (F', η). Put $S = \operatorname{span}(\bigcup_{n=1}^{\infty} K_n)$, then $y'_n \to y'$ in $(F', \sigma(F', S))$. On the other hand, U^0 is $\sigma(F', F)$-sequentially compact by Theorem 1, so $\overset{\sqcup}{H}{}^{\sigma} \cap U^0$ is $\sigma(F', F)$-sequentially compact. Hence there exists a subsequence $\{y'_{n_i}\}$ of $\{y'_n\}$ such that $y'_{n_i} \to z' \in \overset{\sqcup}{H}{}^{\sigma} \cap U^0$ in $(F', \sigma(F', F))$. Certainly $y'_{n_i} \to z'$ in $(F', \sigma(F', S))$. Since $\sigma(F', S)$ is Hausdorff, $y' = z' \in \overset{\sqcup}{H}{}^{\sigma} \cap U^0$. This means that $\overset{\sqcup}{H}{}^{\sigma} \cap U^0$ is $\sigma(F', F)$-closed. This complete the proof of Theorem 2. ∎

Appendix: At the beginning of the proof of Theorem 2, we have used the following result: let E be a Mackey space such that $(E', \sigma(E', E))$ is sequentially complete and F be a l.c.s. such that $\overset{\sqcup}{H}{}^{\sigma} \supset F'$ for every dense vector subspace H of $(F', \sigma(F', F))$, then every closed graph linear map $t \colon E \to F$ is continuous. Now we outline the proof as follows [for the details, see 10]. Let $t' \colon F' \to E^{\#}$ be the adjoint map of t and $H = t'^{-1}(E')$, then H is dense in $(F', \sigma(F', F))$ since the graph of t is closed [2, Lemma 12-5-2]. Obviously $t' \,|\, H : (H, \sigma(H, F)) \to (E', \sigma(E', E))$ is continuous. Since $(E', \sigma(E', E))$ is sequentially complete, we may prove that $t' \,|\, H$ can be extended to $\overset{\sqcup}{H}{}^{\sigma}$ by using the extension theorem [1, p. 297]. Thus we have a continuous linear extension $\overbracket{t' \,|\, H} : (\overset{\sqcup}{H}{}^{\sigma}, \sigma(\overset{\sqcup}{H}{}^{\sigma}, F)) \to (E', \sigma(E', E))$. By the hypothesis, $\overset{\sqcup}{H}{}^{\sigma} \supset F'$. Then we can prove that $\overbracket{t' \,|\, H}$ and t' coincide on F'. Hence $t'(F') = \overbracket{t' \,|\, H}\,(F') \subset E'$. By Hellinger–Toeplitz Theorem [2, Exa. 11-2-5], we conclude that $t \colon (E, \tau(E, E')) \to (F, \tau(F, F'))$ is continuous and so is $t \colon E \to F$.

4 – Further results

By using the theorem on weak compactness, we can obtain some weaker condition for U^0 to be $\sigma(X',X)$-sequentially compact. First we establish the following lemma.

Lemma 2. *Let $\widetilde{X}$ be the completion of l.c.s. $(X,\mathcal{T})$ and X' the topological dual of $(X,\mathcal{T})$. If there exists a metrizable locally convex topology η on X' such that $(X',\eta)' \subset \widetilde{X}$, then U^0 is $\sigma(X',X)$-sequentially compact for any neighborhood U of 0 in $(X,\mathcal{T})$.*

Proof: On U^0, the topology $\sigma(X',X)$ and $\sigma(X',\widetilde{X})$ coincide [1, p. 264]. Put $H=(X',\eta)'$, then $H \subset \widetilde{X}$ and the topology $\sigma(X',\widetilde{X})$ is stronger than $\sigma(X',H)$. Hence

$$(U^0,\sigma(X',X)\,|\,U^0) = (U^0,\sigma(X',\widetilde{X})\,|\,U^0) = (U^0,\sigma(X',H)\,|\,U^0)$$

is compact. Thus U^0 is a weakly compact set in the metrizable l.c.s. (X',η), so U^0 is weakly sequentially compact. Namely U^0 is $\sigma(X',H)$-sequentially compact and is also $\sigma(X',X)$-sequentially compact. ∎

Remark. Let $\widetilde{X}^\tau$ denote the completion of $(X,\tau(X,X'))$. If we substitute $(X,\mathcal{T})$ by $(X,\tau(X,X'))$, then Lemma 2 becomes as follows: if there exists a metrizable locally convex topology η on X' such that $(X',\eta)' \subset \widetilde{X}^\tau$, then each $\sigma(X',X)$-compact absolutely convex subset of X' is $\sigma(X',X)$-sequentially compact.

Theorem 3. *If there exists a sequence $K_1 \subset K_2 \subset \ldots$ of relatively weakly countably compact subsets of $(X,\mathcal{T})$ such that $\overline{\operatorname{span}(\bigcup_{n=1}^\infty K_n)} = X$, then U^0 is $\sigma(X',X)$-sequentially compact for any neighborhood U of 0 in $(X,\mathcal{T})$. Moreover, each $\sigma(X',X)$-compact absolutely convex subset of X' is $\sigma(X',X)$-sequentially compact.*

Proof: Denote by $(\widetilde{X},\widetilde{\mathcal{T}})$ the completion of $(X,\mathcal{T})$ and denote by C_n the absolutely convex closure of K_n in $(\widetilde{X},\widetilde{\mathcal{T}})$ for every $n \in \mathbb{N}$. By using the theorem on weak compactness [4, p. 159], we conclude that each C_n is $\sigma(\widetilde{X},X')$-compact. Denote by η the topology on X' generated by $\{C_n^0\colon\, n \in \mathbb{N}\}$, then (X',η) is a metrizable l.c.s. and $(X',\eta)' \subset \widetilde{X}$. By Lemma 2, U^0 is $\sigma(X',X)$-sequentially compact for any neighborhood U of 0 in $(X,\mathcal{T})$. Since weakly countable compactness and the closure of $\operatorname{span}(\bigcup_{n=1}^\infty K_n)$ in $(X,\mathcal{T})$ only depend on the dual pair $\langle X,X'\rangle$, we can substitute $(X,\mathcal{T})$ by the Mackey space $(X,\tau(X,X'))$. Thus we know that each $\sigma(X',X)$-compact absolutely convex subset of X' is $\sigma(X',X)$-sequentially compact. ∎

Finally we point out that Theorem 2 can be rewritten as the following form, which is more convenient for application.

Theorem 4. *Let E be a Mackey space such that $(E', \sigma(E', E))$ is sequentially complete and F be a B_r-complete l.c.s. If there exists a sequence $K_1 \subset K_2 \subset \dots$ of relatively weakly countably compact subsets of F such that $\overline{\mathrm{span}(\bigcup_{n=1}^{\infty} K_n)} = F$, then every closed graph linear map $t\colon\ E \to F$ is continuous.*

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