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EXISTENCE THEOREMS FOR SOME ELLIPTIC SYSTEMS

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Abstract: We investigate the existence of solutions of systems of semilinear elliptic equations. The proof makes use of the Leray–Schauder degree theory. We also study the corresponding linear problem.

1 – Introduction

In this paper we consider the following elliptic system

$$\begin{cases} -\Delta u_j = f_j(x, u_1, ..., u_m), \quad j = 1, ..., m & \text{in } \Omega, \\ u_j = \psi_j, \quad j = 1, ..., m & \text{on } \partial\Omega \ , \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$, $m \geq 1$ is an integer and $f_j : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$, $j = 1, ..., m$, are locally Hölder continuous functions with exponent α. When $\psi_j \in C^{2,\alpha}(\partial\Omega)$, $j = 1, ..., m$, we seek a solution $u = (u_1, ..., u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$.

Let $0 < \mu_1 < \mu_2 \leq ... \leq \mu_k \leq ...$ be the eigenvalues of the operator $-\Delta$ on Ω with Dirichlet boundary conditions. We shall note φ_1 the positive eigenfunction corresponding to μ_1.

Theorem 1. *Suppose that there are constants $a_{jk} \geq 0$ and $c_j \geq 0$, $j, k = 1, ..., m$ such that*

$$\Big| f_j(x, u_1, ..., u_m) \Big| \leq \sum_{k=1}^{m} a_{jk}\, |u_k| + c_j \ , \tag{1.2}$$

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for $j = 1, ..., m$ *and* $(x, u_1, ..., u_m) \in \overline{\Omega} \times \mathbb{R}^m$, *with*

$$\mu_1 > \rho(A) \ , \tag{1.3}$$

where $\rho(A)$ *denotes the spectral radius of* $A = (a_{jk})_{1 \le j,k \le m}$.

Then for any $(\psi_1, ..., \psi_m) \in (C^{2,\alpha}(\partial\Omega))^m$, *problem (1.1) has a solution* $u = (u_1, ..., u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$.

Remark 1. It will be clear from the proof that, at least in the case $n = 1$, theorem 1 remains true for zero boundary conditions if (1.2) is replaced by

$$u_j \, f_j(x, u_1, ..., u_m) \le \sum_{k=1}^{m} a_{jk} \, |u_j \, u_k| + c_j \, |u_j| \ ,$$

which in some instances may be a weaker growth condition; roughly speaking f_j may contain a term in u_j that is linearly bounded from above or below only, according to the sign of u_j.

In Section 4 we shall give an example showing that our condition is sharp.

When $n = 1$, $m = 2$ and $f_1(x, u_1, u_2) = -u_2$ problem (1.1) reduces to

$$\begin{cases} d^4u/dx^4 = f(x, u, u'') \,, & a < x < b, \\ u(a) = u_a, \quad u(b) = u_b, \quad u''(a) = \overline{u}_a, \quad u''(b) = \overline{u}_b \ , \end{cases} \tag{1.4}$$

where $b - a < +\infty$ and $f \in C([a, b] \times \mathbb{R}^2)$.

Aftabizadeh [1] and Yang [7] roved the existence of a solution of (1.4) (with $a = 0$, $b = 1$) when

$$|f(x, u, v)| \le \alpha \, |u| + \beta \, |v| + \gamma \ ,$$

where $\alpha, \beta, \gamma \ge 0$ are such that $\alpha/\pi^4 + \beta/\pi^2 < 1$.

When $n \ge 1$ and $f_j(x, u_1, ..., u_m) = -u_{j+1}$ for $j = 1, ..., m-1$ (if $m \ge 2$), problem (1.1) reduces to

$$\begin{cases} \Delta^m u = f(x, u, \Delta u, ..., \Delta^{m-1} u) & \text{in } \Omega, \\ \Delta^j u = \psi_j, \quad j = 0, ..., m-1 & \text{on } \partial\Omega \ , \end{cases} \tag{1.5}$$

where $f \colon \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ is a locally Hölder continuous function with exponent α. Chen and Nee [4] proved the existence of a solution of (1.5) under the condition

$$|f(x, u_1, ..., u_m)| \le \sum_{k=1}^{m} a_k \, |u_k| + c \ ,$$

where $a_k \geq 0$, $c \geq 0$ are such that

$$\sum_{k=1}^{m} \frac{a_k}{\mu_1^{m-k+1}} < 1 \ . \tag{1.6}$$

We wish to point out that the condition of solvability in the above examples coincides with that given in theorem 1 (see remark 2).

Remark 2. For problem (1.5) the matrix A defined in theorem 1 is such that, when $m \geq 2$, $a_{jj+1} = 1$ for $1 \leq j \leq m-1$, $a_{jk} = 0$ for $k \neq j+1$, $1 \leq j \leq m-1$, $1 \leq k \leq m$ and $a_{mk} = a_k$ for $1 \leq k \leq m$. In Section 2 we shall show that condition (1.3) is equivalent to condition (1.6).

In both cases the proof makes use of the Leray–Schauder degree theory [2]. Therefore the underlying technique is the establishment of a priori estimates.

Note that we can assume that $\psi_j = 0$ for $j = 1, ..., m$. Indeed let $\chi_j \in C^{2,\alpha}(\overline{\Omega})$ be such that

$$\Delta \chi_j = 0, \quad j = 1, ..., m \quad \text{ in } \ \Omega \ ,$$

$$\chi_j = \psi_j, \quad j = 1, ..., m \quad \text{ on } \ \partial\Omega \ .$$

Define $v_j = u_j - \chi_j$, $j = 1, ..., m$. Then problem (1.1) is equivalent to the following boundary value problem

$$\begin{cases} -\Delta v_j = f_j(x, v_1 + \chi_1, ..., v_m + \chi_m), \quad j = 1, ..., m & \text{in } \Omega, \\ v_j = 0, \quad j = 1, ..., m & \text{on } \partial\Omega \ , \end{cases}$$

and the functions

$$g_j(x, v_1, ..., v_m) = f_j(x, v_1 + \chi_1, ..., v_m + \chi_m) \ , \qquad j = 1, ..., m$$

still satisfy (1.2) with different c_j.

In Section 2, in order that the paper be self-contained, we provide preliminary results from the theory of nonnegative matrices. In Section 3 we prove our a priori bounds. Theorem 1 is proved in Section 4. Finally in Section 5 we study the corresponding linear problem.

2 – Preliminaries

In this section, in order that the paper be self-contained, we provide preliminary results from the theory of nonnegative matrices. We refer the reader to Berman and Plemmons [3] for proofs. We consider the proper cone

$$\mathbb{R}^m_+ = \Big\{ x = (x_1, ..., x_m) \in \mathbb{R}^m; \ x_j \geq 0, \ j = 1, ..., m \Big\} \ .$$

Definition 1. An $m \times m$ matrix M is called $\mathbb{R}_+^m$-monotone if

$$Mx \in \mathbb{R}_+^m \ \Rightarrow \ x \in \mathbb{R}_+^m \ .$$

The following theorems are parts of some results proved in [3] (theorem 1.3.2, p. 6, theorem 1.3.12, p. 10, corollary 2.1.12, p. 28 and theorems 5.2.3, 5.2.6, p. 113).

Theorem 2. *Let N be an $m \times m$ nonnegative matrix (i.e. $N = (n_{jk})_{1 \le j,k \le m}$ with $n_{jk} \ge 0$ for $j,k = 1,...,m$). Then $\rho(N)$ is an eigenvalue of N.*

Theorem 3. *Let $M = \alpha I - N$ where $\alpha \in \mathbb{R}$ and N is an $m \times m$ nonnegative matrix. If $Mx \in \mathbb{R}_+^m$ for some $x \in \int \mathbb{R}_+^m$, then $\rho(N) \le \alpha$.*

Theorem 4. *Let N be an $m \times m$ nonnegative matrix. If x is a positive (i.e. $x = (x_j)_{1 \le j \le m}$ with $x_j > 0$ for $j = 1,...,m$) eigenvector of N then x corresponds to $\rho(N)$.*

Theorem 5. *An $m \times m$ matrix M is $\mathbb{R}_+^m$-monotone if and only if it is nonsingular and M^{-1} is nonnegative.*

Theorem 6. *Let $M = \alpha I - N$ where $\alpha \in \mathbb{R}$ and N is an $m \times m$ nonnegative matrix. Then the following are equivalent:*

i) *The matrix M is $\mathbb{R}_+^m$-monotone.*

ii) *$\rho(N) < \alpha$.*

We conclude this section with the proof of the assertion of remark 2. We first note that condition (1.6) can be written $\det(\mu_1 I - A) > 0$. Then we use the following lemma.

Lemma 1. *Let $N = (n_{jk})_{1 \le j,k \le m}$ be a nonnegative matrix such that, when $m \ge 2$, $n_{jk} = 0$ for $k \ne j+1$, $1 \le j \le m-1$, $1 \le k \le m$. If $\alpha > 0$ the following are equivalent:*

i) *$\det(\alpha I - N) > 0$ (resp. $\det(\alpha I - N) = 0$).*

ii) *$\alpha > \rho(N)$ (resp. $\alpha = \rho(N)$).*

Proof: i)⇒ii): Since the lemma is obvious when $m = 1$, we assume $m \ge 2$. Let $\lambda \in \mathbb{R}$. We have

$$\det(\lambda I - N) = \lambda^m - \left\{ n_{mm}\, \lambda^{m-1} + \sum_{k=1}^{m-1} n_{mk}\, n_{kk+1} \cdots n_{m-1m}\, \lambda^{k-1} \right\} .$$

Suppose first that $n_{m-1m} = 0$. Then $\det(\lambda I - N) = \lambda^{m-1}(\lambda - n_{mm})$. Clearly $\rho(N) = n_{mm}$ and since $\alpha > 0$ the result follows.

Now if $n_{m-1m} > 0$ we claim that we can assume that $n_{jj+1} > 0$ for $j = 1, ..., m-1$. Indded if $n_{jj+1} = 0$ for some $j \in \{1, ..., m-2\}$ (thus necessarily $m \geq 3$), we define $h = \max\{j \in \{1, ..., m-2\};\, n_{jj+1} = 0\}$. Then

$$\det(\lambda I - N) = \lambda^h \det(\lambda I - Q) \ ,$$

where $Q = (q_{jk})_{1 \leq j,k \leq m-h}$ is an $(m-h) \times (m-h)$ nonnegative matrix such that $q_{jj+1} > 0$ for $1 \leq j \leq m-h-1$ and $q_{jk} = 0$ for $k \neq j+1$, $1 \leq j \leq m-h-1$, $1 \leq k \leq m-h$. Clearly $\rho(N) = \rho(Q)$. Since $\alpha > 0$, $\det(\alpha I - N) > 0$ (resp. $\det(\alpha I - N) = 0$) if and only if $\det(\alpha I - Q) > 0$ (resp. $\det(\alpha I - Q) = 0$). Thus our claim is proved. Now let $x_m > 0$ and define the column vector $x = (x_j)_{1 \leq j \leq m}$ by

$$x_j = \alpha^{j-m}\, n_{jj+1} \cdots n_{m-1m}\, x_m \qquad \text{for} \ \ j = 1, ..., m-1 \ .$$

Then $(\alpha I - N)\, x = y = (y_j)_{1 \leq j \leq m}$ where $y_j = 0$ for $j = 1, ..., m-1$ and $y_m = \alpha^{1-m}\, x_m \det(\alpha I - N)$. Using theorem 3 we get $\rho(N) \leq \alpha$. Then the result follows with the help of theorem 2.

ii)⇒i): Since $\rho(N)$ is an eigenvalue of N, the result is clear. ∎

3 – A priori bounds

We first introduce the following problems

$$(3.1)_t \qquad \begin{cases} -\Delta u_j = t\, f_j(x, u_1, ..., u_m), \quad j = 1, ..., m & \text{in } \Omega, \\ u_j = 0, \quad j = 1, ..., m & \text{on } \partial\Omega \ , \end{cases}$$

where $t \in [0,1]$ is the Leray–Schauder homotopy parameter.

Theorem 7. *Under the assumptions of theorem 1, there exists a constant $M > 0$ such that for any $t \in [0,1]$ and any solution $u = (u_1, ..., u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$ of $(3.1)_t$ we have*

$$\|u_j\|_{L^\infty(\Omega)} \leq M \ , \qquad j = 1, ..., m \ .$$

Proof: Multiplying the differential equation in $(3.1)_t$ by u_j, integrating over Ω and using (1.2) we obtain

$$\begin{aligned} \int_\Omega |\nabla u_j|^2\, dx &= t \int_\Omega u_j\, f_j(x, u_1, ..., u_m)\, dx \\ &\leq \sum_{k=1}^m a_{jk} \int_\Omega |u_j\, u_k|\, dx + c_j \int_\Omega |u_j|\, dx \end{aligned}$$

for $j = 1, ..., m$. By first using the Schwarz inequality and then the Poincaré inequality we get

$$\int_\Omega |\nabla u_j|^2 \, dx \leq \sum_{k=1}^{m} a_{jk} \Big(\int_\Omega u_j^2 \, dx \Big)^{1/2} \Big(\int_\Omega u_k^2 dx \Big)^{1/2} + c_j \, |\Omega|^{1/2} \Big(\int_\Omega u_j^2 \, dx \Big)^{1/2} \leq$$
$$\leq \sum_{k=1}^{m} \frac{a_{jk}}{\mu_1} \Big(\int_\Omega |\nabla u_j|^2 \, dx \Big)^{1/2} \Big(\int_\Omega |\nabla u_k|^2 \, dx \Big)^{1/2} + \frac{c_j}{\sqrt{\mu_1}} \, |\Omega|^{1/2} \Big(\int_\Omega |\nabla u_j|^2 \, dx \Big)^{1/2}$$

for $j = 1, ...m$ from which we deduce

$$\|\nabla u_j\|_{L^2(\Omega)} \leq \sum_{k=1}^{m} \frac{a_{jk}}{\mu_1} \, \|\nabla u_k\|_{L^2(\Omega)} + \frac{c_j}{\sqrt{\mu_1}} \, |\Omega|^{1/2} , \quad j = 1, ..., m \; . \tag{3.2}$$

Let x and b denote the column vectors

$$x = \Big(\|\nabla u_j\|_{L^2(\Omega)} \Big)_{1 \leq j \leq m} \quad \text{and} \quad b = \Big(\frac{c_j}{\sqrt{\mu_1}} \, |\Omega|^{1/2} \Big)_{1 \leq j \leq m} .$$

(3.2) can be written

$$b - (I - \mu_1^{-1} A) \, x \; \in \; \mathbb{R}_+^m \; .$$

(1.3) and theorem 6 imply that $I - \mu_1^{-1} A$ is $\mathbb{R}_+^m$-monotone. Hence using theorem 5 we obtain

$$(I - \mu_1^{-1} A)^{-1} \, b - x \; \in \; \mathbb{R}_+^m \; . \tag{3.3}$$

From (3.3) and the Poincaré inequality it follows that

$$\|u_j\|_{W^{1,2}(\Omega)} \leq C \, , \quad j = 1, ..., m \; . \tag{3.4}$$

where C is a positive constant. Now for $1 < p < +\infty$ we have the following estimates

$$\|u_j\|_{W^{2,p}(\Omega)} \leq C \, \|\Delta u_j\|_{L^p(\Omega)} \, , \quad j = 1, ..., m \; , \tag{3.5}$$

([6], lemma 9.17, p. 242) for some positive constant C. Moreover from the differential equations in $(3.1)_t$ and condition (1.2) we deduce

$$\|\Delta u_j\|_{L^p(\Omega)} \leq C \sum_{k=1}^{m} \|u_k\|_{L^p(\Omega)} \, , \quad j = 1, ..., m \; , \tag{3.6}$$

for another positive constant C.

Now if $n = 1$, (3.4) and the Sobolev imbedding theorem imply L^∞ bounds.

If $n = 2$, (3.4) and the Sobolev imbedding theorem imply that, for $1 < p < +\infty$, there exists $C > 0$ such that

$$\|u_j\|_{L^p(\Omega)} \leq C\,, \quad j = 1, ..., m\ . \tag{3.7}$$

Then using (3.5)–(3.7) and the Sobolev imbedding theorem we obtain the L^∞ bounds.

Finally if $n \geq 3$, the conclusion follows from a classical bootstrapping procedure (see [2]) using (3.4)–(3.6) and the Sobolev imbedding theorem. The proof of the theorem is complete. ∎

4 – Proof of theorem 1

We recall from Section 1 that it is sufficient to deal with zero boundary conditions.

We shall note $G(x,y)$ the Green's function of the operator $-\Delta$ on Ω with Dirichlet boundary conditions. Consider the function space $X = (C(\overline{\Omega}))^m$ endowed with the norm

$$\|u\| = \max_{1\leq j\leq m} \Big(\|u_j\|_{L^\infty(\Omega)}\Big) \quad \text{for } \ u = (u_1, ..., u_m) \in X\ .$$

Then X is a Banach space. Regularity theory implies that solving $(3.1)_t$ is equivalent to finding a solution $u = (u_1, ..., u_m) \in X$ of the following system of integral equations

$$u_j(x) = t \int_\Omega G(x,y)\, f_j(y, u_1(y), ..., u_m(y))\, dy\,, \quad j = 1, ..., m\ .$$

Now define a map $T_t\colon\ X \to X$ by $T_t u = v = (v_1, ..., v_m)$ where

$$v_j(x) = t \int_\Omega G(x,y)\, f_j(y, u_1(y), ..., u_m(y))\, dy\,, \quad j = 1, ..., m\ .$$

It is well-known that T_t is continuous and compact for $t \in [0,1]$. Regularity theory implies that solving (1.1) (with $\psi_j = 0$, $j = 1, ..., m$) is equivalent to finding a fixed point of the map T_1 in X. Let M be the constant appearing in theorem 7. Consider the ball B_M in X:

$$B_M = \Big\{u \in X;\ \|u\| < M+1\Big\}\ .$$

Theorem 7 implies that T_t has no fixed point on ∂B_M. Let $I\colon\ X \to X$ be the identity map. By the homotopy invariance of the Leray–Schauder degree we have

$$\deg(I - T_1, B_M, 0) = \deg(I - T_t, B_M, 0) = \deg(I - T_0, B_M, 0) = \deg(I, B_M, 0) = 1\ .$$

Consequently, T_1 has a fixed point in B_M. The theorem is proved. ∎

Remark 3. If there exist constants $a_{jk} \geq 0$, $j,k = 1,...,m$, such that

$$\Big|f_j(x,u_1,...,u_m) - f_j(x,v_1,...,v_m)\Big| \leq \sum_{k=1}^{m} a_{jk}\,|u_k - v_k|$$

for $j = 1,...,m$ and $(x,u_1,...,u_m),(x,v_1,...,v_m) \in \overline{\Omega}\times\mathbb{R}^m$ with $A = (a_{jk})_{1\leq j,k\leq m}$ satisfying (1.3), then the solution of (1.1) is unique. The argument is similar to the proof of theorem 7.

Example 1: Let

$$f_j(x,u_1,...,u_m) = \sum_{k=1}^{m} a_{jk}\,u_k$$

for $j = 1,...,m$ and $(x,u_1,...,u_m) \in \overline{\Omega}\times\mathbb{R}^m$ where $a_{jk} \geq 0$ are constants, $j,k = 1,...,m$. Let b denote the column vector

$$b = \Big(-\int_{\partial\Omega} \psi_j\,\frac{\partial\varphi_1}{\partial\nu}\,ds\Big)_{1\leq j\leq m}$$

and $A = (a_{jk})_{1\leq j,k\leq m}$. Suppose that $\mu_1 = \rho(A)$. By theorem 2 $\det(\mu_1 I - A) = 0$. The Hopf boundary lemma ([6], lemma 3.4, p. 33) implies that $\frac{\partial\varphi_1}{\partial\nu} < 0$ on $\partial\Omega$. Therefore we can choose $\psi_j \in C^{2,\alpha}(\partial\Omega)$, $j = 1,...,m$, such that $b \notin R(\mu_1 I - A)$. Then problem (1.1) has no solution. Indeed, suppose that problem (1.1) has a solution $u = (u_1,...,u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$. Multiplying the differential equation in (1.1) by φ_1 and using Green's formula we obtain

$$\begin{aligned}
-\int_\Omega \varphi_1\,\Delta u_j\,dx &= -\int_\Omega u_j\,\Delta\varphi_1\,dx + \int_{\partial\Omega} \psi_j\,\frac{\partial\varphi_1}{\partial\nu}\,ds \\
&= \mu_1 \int_\Omega u_j\,\varphi_1\,dx + \int_{\partial\Omega} \psi_j\,\frac{\partial\varphi_1}{\partial\nu}\,ds \\
&= \sum_{k=1}^{m} a_{jk} \int_\Omega u_k\,\varphi_1\,dx\,, \qquad j = 1,...,m\;,
\end{aligned}$$

where ν is the unit outward normal to $\partial\Omega$. This yields

$$(\mu_1 I - A)\,x = b\;,$$

where x denotes the column vector

$$x = \Big(\int_\Omega u_j\,\varphi_1\,dx\Big)_{1\leq j\leq m}$$

and we reach a contradiction.

The above example shows that our condition is sharp.

5 – The linear problem

In this section we consider the following boundary value problem:

$$-\delta u_j = \sum_{k=1}^{m} a_{jk}\, u_k\,, \qquad j = 1, ..., m \quad \text{in } \Omega\,, \tag{5.1}$$

$$u_j = 0\,, \qquad j = 1, ..., m \quad \text{on } \partial\Omega\,, \tag{5.2}$$

where $m \geq 1$ and $a_{jk} \in \mathbb{R}$ for $1 \leq j, k \leq m$. We define $A_m = (a_{jk})_{1 \leq j,k \leq m}$. Below $u = (u_1, ..., u_m) \geq 0$ (resp. > 0) means $u_j \geq 0$ (resp. $u_j > 0$) for $j = 1, ..., m$.

Lemma 2. *Let $u = (u_1, ..., u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$ be a nonnegative nontrivial solution of problem (5.1), (5.2). Then $\det(\mu_1 I - A_m) = 0$.*

Proof: Arguing as in example 1 we get

$$(\mu_1 I - A_m)\, x = 0$$

where x is the column vector $x = (\int_\Omega u_j\, \varphi_1\, dx)_{1 \leq j \leq m}$. Since there exists $j \in \{1, ..., m\}$ such that $\int_\Omega u_j\, \varphi_1\, dx \neq 0$, we have necessarily

$$\det(\mu_1 I - A_m) = 0$$

and the lemma is proved. ∎

Lemma 3. *Assume that $a_{jk} \geq 0$ for $j, k = 1, ..., m$. Let $u = (u_1, ..., u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$ be a positive solution of problem (5.1), (5.2). Then $\mu_1 = \rho(A_m)$.*

Proof: Indeed, using the above notations we still get $(\mu_1 I - A_m)\, x = 0$ and the result follows from theorem 4. ∎

Remark 4. Assume that $m = 2$ and that problem (5.1), (5.2) has a positive solution $u = (u_1, u_2) \in (C^{2,\alpha}(\overline{\Omega}))^2$. Then we have

$$\mu_1 = a_{11} \quad (\text{resp. } \mu_1 = a_{22}) \quad \Longleftrightarrow \quad a_{12} = 0 \quad (\text{resp. } a_{21} = 0)\,, \tag{5.3}$$

$$\mu_1 > a_{11} \quad (\text{resp. } \mu_1 > a_{22}) \quad \Longleftrightarrow \quad a_{12} > 0 \quad (\text{resp. } a_{21} > 0)\,. \tag{5.4}$$

Indeed, arguing as in example 1 we get

$$(\mu_1 - a_{11}) \int_\Omega u_1\, \varphi_1\, dx = a_{12} \int_\Omega u_2\, \varphi_1\, dx$$

and

$$(\mu_1 - a_{22}) \int_\Omega u_2\, \varphi_1\, dx = a_{21} \int_\Omega u_1\, \varphi_1\, dx \ ,$$

from which we deduce (5.3) and (5.4).

Now we give two examples.

Example 2: Assume $m = 2$ and $\det A_2 \notin \{\mu_1 \mu_k;\ k \geq 2\}$. Then problem (5.1), (5.2) has a positive solution $u = (u_1, u_2) \in (C^{2,\alpha}(\overline{\Omega}))^2$ if and only if

$$\det(\mu_1 I - A_2) = 0 \tag{5.5}$$

and one of the following conditions holds:

i) $\mu_1 = a_{11}$, $a_{12} = 0$ and $(\mu_1 - a_{22})\, a_{21} > 0$.

Then the solution is given by $u_1 = C\varphi_1$ and $u_2 = \frac{a_{21}}{\mu_1 - a_{22}} C\varphi_1$ for some constant $C > 0$.

ii) $\mu_1 = a_{22}$, $a_{21} = 0$ and $(\mu_1 - a_{11})\, a_{12} > 0$.

Then the solution is given by $u_1 = C\varphi_1$ and $u_2 = \frac{\mu_1 - a_{11}}{a_{12}} C\varphi$ for some constant $C > 0$.

iii) $\mu_1 = a_{11} = a_{22}$, $a_{12} = a_{21} = 0$.

Then the solution is given by $u_1 = C\varphi_1$ and $u_2 = C'\varphi_1$ for some constants $C, C' > 0$.

iv) $(\mu_1 - a_{11})\, a_{12} > 0$ and $(\mu_1 - a_{22})\, a_{21} > 0$.

Then the solution is given by $u_1 = C\varphi_1$ and $u_2 = \frac{a_{21}}{\mu_1 - a_{22}} C\varphi_1 = \frac{\mu_1 - a_{11}}{a_{12}} C\varphi_1$ for some constant $C > 0$.

Proof: Assume that problem (5.1), (5.2) has a positive solution $u = (u_1, u_2) \in (C^{2,\alpha}(\overline{\Omega}))^2$. By lemma 2 condition (5.5) is satisfied.

Define $D(\lambda) = \det(\lambda I - A_2)$. D is a polynomial of degree 2. Since $D(\mu_1) = 0$, the roots of D are real. We denote by μ the other root. Since $\mu\, \mu_1 = \det A_2$, our assumption implies $\mu \neq \mu_k$ for all $k \geq 2$.

Now denote by φ_j the eigenfunction corresponding to μ_j (with $\varphi_1 > 0$ in Ω). These form a complete orthonormal set in $W_0^{1,2}(\Omega)$, hence total in $C^{2,\alpha}$. If $u = (u_1, u_2) \in (C^{2,\alpha}(\overline{\Omega}))^2$ is a solution of problem (5.1), (5.2) the corresponding Fourier coefficients u_{1j} and u_{2j} satisfy the linear system

$$\begin{cases} (\mu_j - a_{11})\, u_{1j} - a_{12}\, u_{2j} = 0, \\ -a_{21}\, u_{1j} + (\mu_j - a_{22})\, u_{2j} = 0 \ , \end{cases} \tag{5.6}$$

from which it immediately follows that $u_{1j} = u_{2j} = 0$ for $j \geq 2$. Using (5.3) and (5.4) of remark 4 we easily verify that one of the conditions i)–iv) holds. The relation between u_{11} and u_{21} is easily checked in each case.

The converse is obvious. ∎

Example 3: Assume $m = 2$. If there exists $k \geq 2$ such that $\det A_2 = \mu_1 \mu_k$, then problem (5.1), (5.2) has a positive solution $u = (u_1, u_2) \in (C^{2,\alpha}(\overline{\Omega}))^2$ if and only if (5.5) is satisfied and one of the following conditions holds:

i) $\mu_1 = a_{11}$, $a_{12} = 0$ and $a_{21} < 0$.

Then the solution is given by $u_1 = C\varphi_1$ and $u_2 = \frac{a_{21}}{\mu_1 - \mu_k} C\varphi_1 + v$ where v is an eigenfunction corresponding to μ_k and $C > 0$ is a constant such that $u_2 > 0$ in Ω.

ii) $\mu_1 = a_{22}$, $a_{21} = 0$ and $a_{12} < 0$.

Then the solution is given by $u_2 = C\varphi_1$ and $u_1 = \frac{a_{12}}{\mu_1 - \mu_k} C\varphi_1 + v$ where v is an eigenfunction corresponding to μ_k and $C > 0$ is a constant such that $u_1 > 0$ in Ω.

iii) $(\mu_1 - a_{11})\, a_{12} > 0$ and $(\mu_1 - a_{22})\, a_{21} > 0$.

Then the solution is given by $u_1 = \frac{\mu_1 - a_{22}}{a_{21}} C\varphi_1 + \frac{\mu_k - a_{22}}{a_{21}} v$ and $u_2 = C\varphi_1 + v$ where v is an eigenfunction corresponding to μ_k and $C > 0$ is a constant such that $u_1 > 0$ and $u_2 > 0$ in Ω.

Proof: Assume that problem (5.1), (5.2) has a positive solution $u = (u_1, u_2) \in (C^{2,\alpha}(\overline{\Omega}))^2$. As in example 2 (5.5) is satisfied. We keep the notations of the proof of example 2. Our assumption implies that $\mu = \mu_k$ for some $k \geq 2$. Using the same argument we obtain (5.6) from which it immediately follows that $u_{1j} = u_{2j} = 0$ except possibly for $j = 1$ and the indices such that $\mu_j = \mu_k$. Using (5.3) and (5.4) of remark 4 we easily show that one of the conditions i)–iii) holds. The relations between the coefficients of the expansions of u_1 and u_2 in the eigenfunctions are easily checked according to the various possibilities i)–iii).

The converse is obvious. ∎

Remark 5. Assume $a_{jk} \geq 0$, $j, k = 1, 2$, and $\mu_1 = \rho(A_2)$.

If problem (5.1), (5.2) has a positive solution $u = (u_1, u_2) \in (C^{2,\alpha}(\overline{\Omega}))^2$ then $\det A_2 \leq \mu_1^2$ since $\det A_2 = \mu\, \mu_1$.

If $\det A_2 = \mu_1^2$, let $a_{jj} = \mu_1$ for $j = 1, 2$ and $a_{12} = a_{21} = 0$. Then iii) of example 2 gives the existence of infinitely many positive solutions.

If $\det A_2 < \mu_1^2$, first let

$$A_2 = \begin{pmatrix} \mu_1 & 0 \\ a_{21} & a_{22} \end{pmatrix} \quad \text{with } \mu_1 > a_{22} \text{ and } a_{21} > 0$$

or

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & \mu_1 \end{pmatrix} \quad \text{with } \mu_1 > a_{11} \text{ and } a_{12} > 0\ .$$

Then i) or ii) of example 2 gives the existence of infinitely many positive solutions. Now let

$$A_2 = \begin{pmatrix} \mu_1 - \varepsilon_1 & a_{12} \\ a_{21} & \mu_1 - \varepsilon_2 \end{pmatrix}$$

with $0 < \varepsilon_j < \mu_1$ for $j = 1, 2$, $a_{12}, a_{21} > 0$ and $\varepsilon_1 \varepsilon_2 = a_{12}\, a_{21}$. Then iv) of example 2 gives the existence of infinitely many positive solutions.

Remark 6. If $a_{jk} \geq 0$, $j, k = 1, 2$, and $\mu_1 > \rho(A_2)$ then the only solution of problem (5.1), (5.2) is the trivial solution (see remark 3). If $\mu_1 = \rho(A_2)$, infinitely many positive solutions may exist by remark 5.

The next result was proved in [5].

Theorem 8. *Assume that a_{jk} in (5.1) are such that*

$$a_{jj+1} = \lambda_{j+1}\,, \qquad j = 1, ..., m-1\ ,$$

$$a_{m1} = \lambda_1\ ,$$

and

$$a_{jk} = 0\,, \qquad \textit{otherwise}\ .$$

Then problem (5.1), (5.2) has a positive solution $u = (u_1, ..., u_m) \in (C^{2,\alpha}(\overline{\Omega}))^m$ if and only if

$$\lambda_j > 0, \quad j = 1, ..., m \qquad \textit{and} \qquad \lambda_1 \cdots \lambda_m = \mu_1^m\ . \tag{5.7}$$

The solution is given by $u_j = c_j\, \varphi_1$ where $c_1 > 0$ is an arbitrary constant and $c_j = c_1 (\lambda_2 \cdots \lambda_j)^{-1}\, (\lambda_1 \cdot \lambda_m)^{(j-1)/m}$ for $j = 2, ..., m$.

Remark 7. By lemma 1 condition (5.7) is equivalent to

$$\lambda_j > 0, \quad j = 1, ..., m \qquad \text{and} \qquad \mu_1 = \rho(A_m)\ .$$

Now with the notations of theorem 8, if $\lambda_j \geq 0$, $j = 1, ..., m$ and $\mu_1 > \rho(A_m)$, then the only solution of problem (5.1), (5.2) is the trivial solution (see remark 3). If $\lambda_j > 0$, $j = 1, ..., m$ and $\mu_1 = \rho(A_m)$, theorem 8 shows that there exist infinitely many positive solutions.

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