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SOME ASYMPTOTIC RESULTS ASSOCIATED WITH A GENERALIZED GAUSS–KUZMIN-TYPE OPERATOR

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Abstract: Some asymptotic results concerning a random system associated with a generalized Gauss-Kuzmin-type operator are investigated.

1 – Introduction

W. Fluch [1] introduced a *generalized Gauss-Kuzmin-type operator* G_α, $\alpha \geq 1$, as ollows (see Kuzmin [8]). Let $Y = C_0([0,1])$ be the space of all real-valued continuous functions defined on $[0,1]$. Then, for $f \in Y$, the value $G_\alpha f$ is given by

$$(G_\alpha f)(w) = \sum_{x \in N^*} \frac{\alpha^2}{(\alpha x + w)\,(\alpha x + \alpha - 1 + w)}\, f\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right) ,$$

for all $w \in [0,1]$.

Furthermore, let μ be an arbitrary non-atomic probability measure on the σ-algebra $\mathcal{B}_{[0,1]}$ of all Borel subsets of $[0,1]$ and define $F_0(w) = \mu([0,w])$, $w \in [0,1]$. Assuming that the derivative F_0' exists everywhere in $[0,1]$ and is bounded, we define for each $\alpha \geq 1$ the n-th iteration ${}_\alpha F_n' \equiv F_n'$ by the functional equation

$$(1.1) \qquad F_n'(w) = \sum_{x \in N^*} \frac{\alpha^2}{(\alpha x + w)\,(\alpha x + \alpha - 1 + w)}\, F_{n-1}'\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right), \quad w \in [0,1] ,$$

where $n \in N^* = \{1, 2, ...\}$.

Consider further the functions ${}_\alpha f_n \equiv f_n$, $\alpha \geq 1$, $n \in N^*$, defined as

$$f_n(w) = F_n'(w)/\rho(w) , \qquad w \in [0,1] ,$$

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where $\rho(w) = \frac{\alpha}{\alpha+w}$. Then, replacing F_n' by ρf_n in equation (1.1), one obtains that

$$\begin{aligned}f_n(w) = \sum_{x\in N^*} \frac{\alpha(\alpha+w)}{(\alpha x+w)\,(\alpha x+\alpha+w)}\, f_{n-1}\Big(\frac{\alpha}{\alpha x+\alpha-1+w}\Big),\\ w\in[0,1],\quad n\in N^* \;.\end{aligned} \tag{1.2}$$

As it was proved in [2], equation (1.2) suggests the consideration of a parametric random system with complete connections

$$\Big\{(W,\mathcal{W}),(X,\mathcal{X}),u_\alpha,P_\alpha\colon\ \alpha\geq 1\Big\} \tag{1.3}$$

associated with the Gauss–Kuzmin-type operator G_α above, where

$$\begin{aligned}
&W=[0,1]\,,\qquad \mathcal{W}=\mathcal{B}_{[0,1]}\ ,\\
&X=N^*\,,\qquad \mathcal{X}=\mathcal{P}(X)\ \ \text{(the power set)}\ ,\\
&u_\alpha(w,x)=\frac{\alpha}{\alpha x+\alpha-1+w}\ ,\\
&P_\alpha(w,x)=\frac{\alpha(\alpha+w)}{(\alpha x+w)\,(\alpha x+\alpha-1+w)}\,,\qquad w\in W,\quad x\in X\ .
\end{aligned}$$

For a brief presentation of the concept of random system with complete connections (for short rscc) we refer the reader to the appendix added at the end of the present paper.

Applying the existence Theorem 5 of appendix, it follows that for an arbitrary fixed $w_0\in W$ there exists a sequence of random variables $(\zeta_n)_{n\in N}$, with initial distribution concentrated at w_0, named *the associated Markov chain* of the random system with complete connections (1.3). Then its transition operator U_α is given by the equation

$$(U_\alpha f)(w) = \sum_{x\in X} \frac{\alpha(\alpha+w)}{(\alpha x+w)\,(\alpha x+\alpha-1+w)}\, f\Big(\frac{\alpha}{\alpha x+\alpha-1+w}\Big) \tag{1.4}$$

for any $f\in B(W,\mathcal{W})$, where $B(W,\mathcal{W})$ denotes the Banach space of all $\mathcal{W}$-measurable and bounded real-valued functions defined on W.

The present paper arises as an attempt to prove some asymptotic formulas concerning the associated Markov chain $(\zeta_n)_{n\in N}$.

Our approach will be given in the context of the theory of dependence with complete connections (see [3]) and it is concentrated on some special results related to the strong law of large numbers.

Particularly, these results extend certain asymptotic formulas related with the classic continued fraction expansion of any irrational number y in $[0,1]$. The continued fraction expansions were first investigated, in the context of metrical theory, by Khinchin [7].

The paper is organized as follows. Section 2 deals with the asymptotic behaviour of the parametric random system (1.3). In Section 3 we obtain two special asymptotic formulas by using the law of large numbers for Markov chains. Finally, we add an appendix comprising certain concepts of the theory of random systems with complete connections according to [3].

In what follows we shall use the notation: I_A = the characteristic function of A, $(X^r, \mathcal{X}^r)$ = the r-fold product measurable space of $(X, \mathcal{X})$, a.e. = almost everywhere.

2 – Auxiliary results

2.1. The ergodic behaviour of the random system with complete connections introduced in (1.3) is expressed by the following statement which is proved in [2].

Theorem 1. *The random system with complete connections (1.3) associated with the generalized Gauss–Kuzmin-type operator G_α is uniformly ergodic.*

Then, on account of Theorem 7 of appedix it follows that for any $\alpha \geq 1$ there exists a unique limiting probability measure γ_α on $\mathcal{W}$ satisfying the equation

$$\gamma_\alpha(B) = \int_W Q_\alpha(w, B)\, \gamma_\alpha(dw) \ ,$$

for all $B \in \mathcal{W}$, where

$$Q_\alpha(w, B) = \sum_{\{x \in X:\, u_\alpha(w,x) \in B\}} P(\alpha)(w, x)\,, \qquad B \in \mathcal{W} \ ,$$

is the transition probability function of the associated Markov chain $(\zeta_n)_{n \in N}$. Our aproach is similar to that used in [4], [5] and [6].

2.2. Next, by virtue of existence Theorem 5 of appendix for any $w \in [0, 1]$ there exists a probability P_w such that the associated Markov chain with the random system with complete connections (1.3) is $(\zeta_n)_{n \in N}$ introduced in Section 1. Then according to Iosifescu [3] a strong law of large numbers for the sequence $(h(\zeta_n))_{n \in N}$, when h is an arbitrary continuous function on $[0, 1]$, may be stated as follows.

Theorem 3. *For any $w \in W$ and for any real-valued continuous function h*

defined on $[0,1]$ *we have*

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} h(\zeta_k) = \int_W h(w)\,\gamma_\alpha(dw)\,, \qquad P_w\text{-a.e.} \ \blacksquare$$

3 – Some asymptotic formulas

Now, we are prepared to prove

Theorem 1.

i) *For any* $w \in [0,1]$ *and* $\alpha \geq 1$ *we have*

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \zeta_k = \int_0^1 w\,\gamma_\alpha(dw)\,, \qquad P_w\text{-a.e.} \tag{3.1}$$

ii) *For* $\alpha = 1$ *the measure* γ_1 *is identical to the classic Gauss's measure* γ, *that is*

$$\gamma_1(A) = \frac{1}{\log 2} \int_A \frac{1}{1+w}\,dw\,, \qquad A \in \mathcal{B}_{[0,1]}\ . \tag{3.2}$$

Furthermore equation (3.1) becomes

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \zeta_k = \frac{1}{\log 2} - 1\,, \qquad \lambda\text{-a.e.}\ , \tag{3.3}$$

where λ *denotes the Lebesgue measure on* $\mathcal{W}$.

Proof: **i)** Equation (3.1) follows by applying Theorem 3 to $h(w) = w$, $w \in [0,1]$.

ii) If $\alpha = 1$, relation (3.3) follows from (3.1) where γ_1 is identical to γ defined in (3.2). $\blacksquare$

Theorem 2.

i) *For any* $w \in [0,1]$ *and* $\alpha \geq 1$, *we have*

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \log\zeta_k = \int_0^1 \log w\,\gamma_\alpha(dw)\,, \qquad P_w\text{-a.e.} \tag{3.4}$$

ii) *For* $\alpha = 1$, *we have*

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \log\zeta_k = -\frac{\pi^2}{12\log 2}\,, \qquad \lambda\text{-a.e.}\ , \tag{3.5}$$

where λ denotes the Lebesgue measure.

Proof: i) Equation (3.4) is obtained by using Theorem 3 for $h(w) = w^{\varepsilon} \log w$, $w \in [0,1]$, $\varepsilon > 0$ and letting $\varepsilon \to 0$.

ii) When $\alpha = 1$, equation (3.5) follows from (3.4) where γ_1 is identical with the classic Gauss's measure γ. ∎

4 – Appendix

Definition 1. A quadruple $\{(W,\mathcal{W}),(X,\mathcal{X}),u,P\}$ is named *a homogeneous random system with complete connections* if

i) $(W,\mathcal{W})$ and $(X,\mathcal{X})$ are arbitrary measurable spaces;

ii) $u\colon W \times X \to W$ is a $(\mathcal{W} \otimes \mathcal{X},\mathcal{W})$-measurable function;

iii) P is a transition probability function from $(W,\mathcal{W})$ to $(X,\mathcal{X})$.

Next, we denote the element $(x_1,x_2,...,x_n) \in X^n$ by $x^{(n)}$.

Definition 2. The functions $u^{(n)}\colon W \times X^n \to W$, $n \in N^*$, are defined as follows

$$u^{(n+1)}(w,x^{(n+1)}) = \begin{cases} u(w,x), & \text{if } n = 0, \\ u\Big(u^{(n)}(w,x^{(n)}),x_{n+1}\Big), & \text{if } n \geq 1 \ . \end{cases}$$

Convention. We shall write $wx^{(n)}$ instead of $u^{(n)}(w,x^{(n)})$. Throughout the paper, we shall consider the discrete case for the space $(X,\mathcal{X})$.

Definition 3. The transition probability function P_r, $r \in N^*$, are defined by

$$P_r(w,A) = \begin{cases} P(w,A), & \text{if } r=1, \\ \displaystyle\sum_{x_1\in X} P(w,x_1) \sum_{x_2 \in X} P(wx_1,x_2) \cdots \sum_{x_r\in X} P(wx^{(r-1)},x_r)\, I_A(x^{(r)}), & \text{if } r>1, \end{cases}$$

for any $w \in W$, $r \in N^*$ and $A \in \mathcal{X}^r$.

Definition 4. Assume that $X^0 \times A = A$. Then we define

$$P_r^n(w,A) = P_{n+r-1}(w,X^{n-1} \times A) \ ,$$

for any $w \in W$, $r \in N^*$ and $A \in \mathcal{X}^r$.

Theorem 5. *Let $\{(W,\mathcal{W}),(X,\mathcal{X}),u,P\}$ a homogeneous random system with complete connections and suppose an arbitrary fixed $w_0 \in W$. There exists* *a probability space* (Ω, K, P_{w_0}) *and two sequences of random variables* $(\xi_n)_{n\in N^*}$ $(\xi_n\colon \Omega \to X)$ *and* $(\zeta_n)_{n\in N}$ $(\zeta_n\colon \Omega \to W)$ *such that*

i) (1) $P_{w_0}\Big([\xi_n, ..., \xi_{n+r-1}]\Big) = P_r^n(w_0, A)$;

(2) $P_{w_0}\Big([\xi_{n+m}, ..., \xi_{n+m+r-1}] \in A/\xi^{(n)}\Big) = P_r^m(w_0\, \xi^{(n)}, A), \quad P_{w_0}\text{-a.e.}$;

(3) $P_{w_0}\Big([\xi_{n+m}, ..., \xi_{n+m+r-1}] \in A/\xi^{(m)}, \zeta^{(n)}\Big) = P_r^m(\zeta_n, A), \quad P_{w_0}\text{-a.e.}$;

for any $n, m, r \in N^*$ *and* $A \in \mathcal{X}^r$, *where* $\xi^{(n)}$, $\zeta^{(n)}$ *are the random vectors* $(\xi_1, \xi_2, ..., \xi_n)$ *and* $(\zeta_1, \zeta_2, ..., \zeta_n)$.

ii) $(\zeta_n)_n$ *is* W*-valued homogeneous Markov chain with initial distribution concentrated at* w_0 *and transition operator* U *given by the equation*

$$U\, f(w) = \sum_{x\in X} P(w,x)\, f(wx) \tag{4.1}$$

for any $f \in B(W, \mathcal{W})$. *(Here* $B(W, \mathcal{W})$ *is the Banach space of all bounded* $\mathcal{W}$*-measurable real-valued functions defined on* W.*)*

Remark. The sequence $(\zeta_n)_{n\in N}$ is called the *associated Markov chain*. For $f(w) = I_A(w)$, $A \in \mathcal{W}$, we obtain by (4.1) the transition probability function of the associated Markov chain given by the equality

$$Q(w, A) = \sum_{x\in X} P(w,x)\, I_B(wx) \quad (= P(w, A_w)) \ , \tag{4.2}$$

for any $w \in W$, $A \in \mathcal{W}$ with $A_w = \{x \in X\colon\ wx \in A\}$.

Definition 6. A random system with complete connections $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ is called *uniformly ergodic*, if for any $r \in N^*$ there exists a probability P_r^∞ on $\mathcal{X}^r$ such that $\lim \varepsilon_n = 0$ as $n \to \infty$, where

$$\varepsilon_n = \sup_{\substack{w\in W,\ r\in N^* \\ A\in \mathcal{X}^r}} \Big| P_r^n(w, A) - P_r^\infty(A) \Big| \ .$$

Theorem 7. *Let* $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ *be a random system with complete connections uniformly ergodic.*

Then there exists a unique limiting probability measure Q^∞ *on* $\mathcal{W}$ *which satisfies the equation*

$$\int_W Q(w, A)\, Q^\infty(dw) = Q^\infty(B) \ ,$$

for any $B \in \mathcal{W}$, $w \in W$, where Q is the probability function defined by (4.2).

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