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CONTROLLABILITY INDICES OF PARTIALLY PRESCRIBED PAIRS OF MATRICES (*)

Isabel Cabral(**) and Fernando C. Silva

Abstract: We give necessary and sufficient conditions for the existence of a completely controllable pair (A_1, A_2) with prescribed controllability indices and with a prescribed submatrix of $[A_1\ A_2]$ that does not contain principal entries of A_1.

1 – Introduction

Let F be a field. In control theory, a pair of matrices (A_1, A_2), where $A_1 \in F^{m\times m}$, $A_2 \in F^{m\times(n-m)}$, is said to be completely controllable (c.c.) if:

$$\operatorname{rank}\Big[A_2\quad A_1A_2\quad A_1^2A_2\quad \cdots\quad A_1^{m-1}A_2\Big] = m\ .$$

Alternative characterizations of complete controllability are known. For example, (A_1, A_2) is c.c. if and only if:

$$\min_{\lambda\in\overline{F}} \operatorname{rank}\Big[\lambda I_m - A_1 \mid -A_2\Big] = m\ , \tag{1.1}$$

where $\overline{F}$ is an algebraically closed extension of F, if and only if all the invariant factors of the polynomial matrix

$$\Big[xI_m - A_1 \mid -A_2\Big]$$

are equal to 1.

Two matrices $A = [A_1\ A_2]$, $B = [B_1\ B_2] \in F^{m\times n}$, where $A_1, B_1 \in F^{m\times m}$, are said to be *block-similar* if there exists a nonsingular matrix:

$$P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix} \in F^{n\times n}, \qquad \text{with}\ \ P_{11} \in F^{m\times m}\ ,$$

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such that:

$$B = P_{11}^{-1} A P .$$

It is easy to see that A and B are block-similar if and only if the matrix pencils $[xI_m - A_1 \mid -A_2]$ and $[xI_m - B_1 \mid -B_2]$ are strictly equivalent. Using the language of control theory, we call controllability indices of (A_1, A_2) to the column minimal indices of the pencil $[xI_m - A_1 \mid -A_2]$. For a definition of strict equivalence and minimal indices, see [5].

Complete controllability is invariant under block-similarity. More precisely, if $[A_1\ A_2]$ and $[B_1\ B_2]$ are block-similar then (A_1, A_2) is c.c. if and only if (B_1, B_2) is c.c., and the pairs (A_1, A_2), (B_1, B_2) have the same controllability indices.

Several papers study the existence of a c.c. pair (A_1, A_2) with several prescribed entries. For example, [11], [13] and [14] concerned this question when the prescribed entries are equal to zero. The papers [9], [10] and [17] study this question when the prescribed entries are all the entries of the main diagonal and above the main diagonal. In [2] this problem was studied when an arbitrary submatrix of $[A_1\ A_2]$ is prescribed.

Now let us consider the problem of the existence of a c.c. pair (A_1, A_2) with prescribed controllability indices and with a prescribed submatrix of $[A_1\ A_2]$. As we are not able to solve this problem for an arbitrary submatrix of $[A_1\ A_2]$, we turned our attention to the case where the prescribed submatrix does not contain principal entries of A_1. Theorems 1 and 2 solve this case. Note that, as controllability indices are invariant under permutation block-similarity, we may assume, without loss of generality, that the prescribed submatrix corresponds to rows $1, ..., p$ and columns $m - q_1 + 1, ..., m, ..., m + q_2$, where p, q_1 and q_2 are nonnegative integers such that $p + q_1 \leq m$, $q_2 \leq n - m$.

A more general problem would be to study the existence of a matrix $[A_1\ A_2]$ with prescribed block similarity class and a prescribed submatrix. Particular cases of this problem are already solved. For example, when the prescribed submatrix is: A_1 [20]; A_1 and some columns of A_2 [1]; A_2 [15]; the submatrix of A_1 corresponding to columns $1, ..., p$ and rows $1, ..., p, ..., p+q$, where $p, q \in \{1, ..., m\}$, $p + q \leq m$ [3]. Other known results concern the case where A_2 has zero columns and, therefore, block similarity coincides with similarity [4, 7, 8, 12, 16, 19, 21].

2 – Main results

Let $v_1, ..., v_{n-m}$ be nonnegative integers, where $n > m \geq 1$, such that:

$$v_1 \geq ... \geq v_\beta > v_{\beta+1} = ... = v_{n-m}\ (= 0)$$

$$v_1 + ... + v_\beta = m .$$

Let $C_1 \in F^{p\times q_1}$, $C_2 \in F^{p\times q_2}$, where $p+q_1 \leq m$ and $q_2 \leq n-m$, and

$$\rho_1 = \operatorname{rank}[C_1\ C_2] - \operatorname{rank} C_2\ ,$$

$$\rho_2 = \operatorname{rank} C_2\ .$$

In Theorems 1 and 2, we present conditions equivalent to:

(**a**) *There exist matrices:*

$$A_1 = \begin{bmatrix} A_{11} & A_{12} & C_1 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \in F^{m\times m}\ , \qquad A_2 = \begin{bmatrix} C_2 & A_{15} \\ A_{24} & A_{25} \\ A_{34} & A_{35} \end{bmatrix} \in F^{m\times(n-m)}\ ,$$

where $A_{31} \in F^{q_1\times p}$, $A_{34} \in F^{q_1\times q_2}$, such that (A_1, A_2) is completely controllable and has controllability indices $v_1, ..., v_{n-m}$.

Theorem 1 applies to the case where the prescribed blocks have maximal sizes. Theorem 2 applies to the general case.

Theorem 1. *Suppose that $p+q_1 = m$ and $q_2 = n-m$. Then (a) is equivalent to:*

(**b**) *The following conditions are satisfied:*

$$\rho_2 \leq \beta\ , \tag{2.1}$$

$$v_1 + ... + v_{\rho_1+\rho_2} \geq p + \rho_1\ , \tag{2.2}$$

$$v_{\beta-q_1+\rho_1-\rho_2} \geq 2\ . \tag{2.3}$$

Theorem 2. *(a) is equivalent to:*

(**c**) *The following conditions are satisfied:*

$$\rho_2 \leq \beta\ , \tag{2.4}$$

$$v_1 + ... + v_{\sigma_1+\sigma_2} \geq p + \sigma_1\ , \tag{2.5}$$

$$\rho_1 + \rho_2 \leq \sigma_1 + \sigma_2\ , \tag{2.6}$$

where

$$\sigma_2 = \min\Big\{\beta,\ n-m-q_2+\rho_2,\ p\Big\}\ , \tag{2.7}$$

$$\sigma_1 = \min\Big\{m-p-q_1+\rho_1,\ p-\sigma_2,\ ,m-p-\beta+\sigma_2+\nu\Big\}\ , \tag{2.8}$$

$$\nu = \max\Big\{i\colon\ v_i \geq 2\Big\}\ . \tag{2.9}$$

Remark. Suppose that $p+q_1=m$ and $q_2=n-m$. Then it is easy to see that (b) and (c) are equivalent. In fact, if (c) is satisfied, we can deduce that $\sigma_2=\rho_2$ and $\sigma_1=\rho_1$, and, then, (2.3) results from (2.8). Conversely, if (b) is satisfied, we can also deduce that $\sigma_2=\rho_2$ and $\sigma_1=\rho_1$.

In our proofs, Theorem 1 is a first step to prove Theorem 2.

Conventions. We assume that:

(i) if $\rho=0$, then $v_1+...+v_\rho=0$;

(ii) $v_i=+\infty\geq 2$ whenever $i\leq 0$;

(iii) matrices with zero rows or zero columns exist and have rank equal to zero.

From now on, we are proving Theorems 1 and 2.

3 – General considerations

Definition. Let $C_1,D_1\in F^{p\times q_1}$, $C_2,D_2\in F^{p\times q_2}$. We say that $[C_1\ C_2]$ and $[D_1\ D_2]$ are *q_1-equivalent* if there exist nonsingular matrices $P\in F^{p\times p}$ and

$$Q=\begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix}\in F^{(q_1+q_2)\times(q_1+q_2)} ,$$

with $Q_{11}\in F^{q_1\times q_1}$, such that:

$$[D_1\ D_2]=P\ [C_1\ C_2]\ Q .$$

The following two propositions are easy to prove.

Proposition 3.1. *Let $C_1,D_1\in F^{p\times q_1}$, $C_2,D_2\in F^{p\times q_2}$. Then $[C_1\ C_2]$ and $[D_1\ D_2]$ are q_1-equivalent if and only if* $\operatorname{rank}[C_1\ C_2]=\operatorname{rank}[D_1\ D_2]$, $\operatorname{rank} C_2=\operatorname{rank} D_2$.

Proposition 3.2. *Let $C_1,D_1\in F^{p\times q_1}$, $C_2,D_2\in F^{p\times q_2}$. Suppose that $[C_1\ C_2]$ and $[D_1\ D_2]$ are q_1-equivalent. Then (a) is equivalent to the condition that results from it replacing C_1 with D_1 and C_2 with D_2.*

Given a polynomial:

$$f(x)=x^k-a_{k-1}\,x^{k-1}-...-a_1\,x-a_0\ \in\ F[x] ,$$

we denote by $\mathcal{C}(f)$ the following companion matrix of f:

$$\mathcal{C}(f) = \left[e_2^{(k)} \;\; e_3^{(k)} \;\; \dots \;\; e_k^{(k)} \;\; a\right]^{\mathrm{t}} ,$$

where

$$a = \left[a_0 \;\; a_1 \;\; \dots \;\; a_{k-1}\right]^{\mathrm{t}} ,$$

and $e_i^{(k)}$ is the i-th column of the identity matrix I_k.

We denote the degree of a polynomial $f(x) \in F[x]$ by $d(f)$. The degree of a polynomial vector $\pi(x) \in F[x]^{n+1}$, $d(\pi)$, is the maximum of the degrees of its coordinates.

We take:

$$\rho = \rho_1 + \rho_2 \ .$$

4 – Proof that (a) implies (b)

Let us assume that (a) is satisfied and that $p+q_1 = m$, $q_2 = n-m$. We have:

$$A_1 = \begin{bmatrix} A_{11} & C_1 \\ A_{31} & A_{33} \end{bmatrix} , \qquad A_2 = \begin{bmatrix} C_2 \\ A_{34} \end{bmatrix} \ .$$

Condition (2.1) is trivial:

$$\rho_2 = \operatorname{rank} C_2 \le \operatorname{rank} A_2 = \beta \ .$$

Lemma 4.1. *The pair $(A_{11}, [C_1 \; C_2])$ is completely controllable.*

Proof: Since (A_1, A_2) is c.c., condition (1.1) is satisfied. From (1.1) it results that:

$$\min_{\lambda\in\overline{F}} \operatorname{rank}\left[\lambda I_p - A_{11} \;\; \big| \;\; -C_1 \;\; \big| \;\; -C_2\right] = p ,$$

what proves Lemma 4.1. ∎

Let $k_1 \ge \dots \ge k_{q_1+q_2}$ (≥ 0) be the controllability indices of $(A_{11}, [C_1 \; C_2])$.

Lemma 4.2. *If $q_1 = 1$ and $\beta > \rho_2$, then:*

$$v_i \ge k_i, \qquad \textit{for} \;\; i \in \{1, \dots, q_2\} \ .$$

Proof: It is not hard to see that there exists a nonsingular matrix $V \in F^{q_2\times q_2}$ such that:

$$A_2 V = \begin{bmatrix} C_2' & 0 \\ 0 & 1 \end{bmatrix} ,$$

where $C_2' \in F^{p\times(q_2-1)}$, $\operatorname{rank} C_2' = \operatorname{rank} C_2 = \beta - 1$. Then $[A_1\ A_2]$ is block-similar to:

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} (I_{p+1} \oplus V) \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_{q_2-1} & 0 \\ -A_{31} & -A_{33} & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & C_1 & C_2' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

The submatrix of the last matrix corresponding to the first p rows is block-similar to $[A_{11}\ C_1\ C_2]$. Therefore $(A_{11}, [C_1\ C_2'])$ is c.c. and has controllability indices $k_1, ..., k_{q_2}$. Bearing in mind the definition of column minimal indices (see [5]), there exist nonzero polynomial vectors:

$$\pi_1(x), ..., \pi_{q_2}(x) \in F[x]^{(n-1)\times 1} ,$$

with $d(\pi_i) = k_i$, $i \in \{1, ..., q_2\}$, that are solutions of the equation in X:

$$\begin{bmatrix} xI_p - A_{11} & -C_1 & -C_2' \end{bmatrix} X = 0 , \tag{4.1}$$

π_{q_2} is a solution of least degree of (4.1), and π_i, $i \in \{1, ..., q_2 - 1\}$, is a solution of least degree of (4.1) among those solutions that are linearly independent with $\pi_{i+1}, ..., \pi_{q_2}$. Suppose that:

$$\pi_i(x) = \begin{bmatrix} \pi_{i,1}(x) & \dots & \pi_{i,n-1}(x) \end{bmatrix}^{\mathrm{t}} , \qquad i \in \{1, ..., q_2\} .$$

Without loss of generality, suppose that π_{q_2} is chosen so that:

$$d(\pi_{q_2,p+1}) < d(\pi_{q_2})$$

if there exists a nonzero solution π of (4.1), with $d(\pi) = k_{q_2}$, such that the degree of its $(p+1)$-th coordinate is less than the degree of π. And suppose that $\pi_i(x)$, $i \in \{1, ..., q_2 - 1\}$, is chosen so that:

$$d(\pi_{i,p+1}) < d(\pi_i)$$

if there exists a solution π of (4.1), with $d(\pi) = k_i$, such that $\pi, \pi_{i+1}, ..., \pi_{q_2}$ are linearly independent and the degree of the $(p+1)$-th coordinate of π is less than the degree of π (see [5]).

Take

$$\chi_i(x) = \begin{bmatrix} \pi_i(x) \\ -x\,\pi_{i,p+1}(x) \end{bmatrix} \in F[x]^{n\times 1} , \qquad i \in \{1, ..., q_2\} .$$

The polynomial vectors $\chi_1, ..., \chi_{q_2}$ are solutions of the equation in Y:

$$\begin{bmatrix} xI_p - A_{11} & -C_1 & -C_2' & 0 \\ 0 & x & 0 & 1 \end{bmatrix} Y = 0 . \tag{4.2}$$

Moreover, χ_{q_2} is a solution of least degree among the solutions of (4.2), and χ_i, $i \in \{1, ..., q_2 - 1\}$, is a solution of least degree among the solutions of (4.2) that are linearly independent with $\chi_{i+1}, ..., \chi_{q_2}$. Therefore, the controllability indices of (A_1, A_2) are the degrees of the polynomials χ_i, $i \in \{1, ..., q_2\}$, and

$$v_i \geq k_i\,, \quad i \in \{1, ..., q_2\}\ . \ \blacksquare$$

Lemma 4.3. *$[A_1\ A_2]$ is block-similar to a matrix of the form:*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & I_{\rho_2} & 0 & 0 \\ D_{21} & D_{22} & C_{23} & C_{24} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & D_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\beta-\rho_2} \end{bmatrix},$$

where $D_{21} \in F^{(p-\rho_2)\times\rho_2}$, $D_{22} \in F^{(p-\rho_2)\times(p-\rho_2)}$, $C_{23} \in F^{(p-\rho_2)\times(q_1-\beta+\rho_2)}$,

$$C_{24} = \begin{bmatrix} I_\tau & 0 \\ 0 & 0 \end{bmatrix} \in F^{(p-\rho_2)\times(\beta-\rho_2)}\ ,$$

and

$$\operatorname{rank}[C_{23}\ \ C_{24}] = \rho_1\ .$$

Proof: Attending to Proposition 3.2, we may assume that:

$$C_2 = \begin{bmatrix} I_{\rho_2} & 0 \\ 0 & 0 \end{bmatrix}\ .$$

Let S be the submatrix that we obtain from A_{34} deleting the columns $1, 2, ..., \rho_2$. Since $\operatorname{rank} A_2 = \beta$, we have $\operatorname{rank} S = \beta - \rho_2$. Let U and V be nonsingular matrices such that:

$$USV = \begin{bmatrix} 0 & 0 \\ 0 & I_{\beta-\rho_2} \end{bmatrix}\ .$$

Then $[A_1\ A_2]$ is block-similar to:

$$\begin{aligned}[] [A_1'\ A_2'] &= (I_p \oplus U)\,[A_1\ A_2]\,(I_p \oplus U^{-1} \oplus I_{\rho_2} \oplus V) \\ &= \begin{bmatrix} D_{11}' & D_{12}' & C_{13}' & C_{14}' & I_{\rho_2} & 0 & 0 \\ D_{21}' & D_{22}' & C_{23}' & C_{24}' & 0 & 0 & 0 \\ D_{31}' & D_{32}' & D_{33}' & D_{34}' & D_{35}' & 0 & 0 \\ D_{41}' & D_{42}' & D_{43}' & D_{44}' & D_{45}' & 0 & I_{\beta-\rho_2} \end{bmatrix}, \end{aligned}$$

where $D_{11}' \in F^{\rho_2\times\rho_2}$, $D_{22}' \in F^{(p-\rho_2)\times(p-\rho_2)}$, $D_{33}' \in F^{(q_1-\beta+\rho_2)\times(q_1-\beta+\rho_2)}$, $D_{44}' \in F^{(\beta-\rho_2)\times(\beta-\rho_2)}$.

Let

$$T=\begin{bmatrix}-D'_{35} & 0\\ -D'_{45} & 0\end{bmatrix}\in F^{q_1\times p}\ .$$

The matrix

$$\begin{aligned}[A''_1\ A''_2] &= \begin{bmatrix}I_p & 0\\ T & I_{q_1}\end{bmatrix}[A'_1\ A'_2]\left(\begin{bmatrix}I_p & 0\\ -T & I_{q_1}\end{bmatrix}\oplus I_{q_2}\right)\\ &= \begin{bmatrix}D''_{11} & D'_{12} & C'_{13} & C'_{14} & I_{\rho_2} & 0 & 0\\ D''_{21} & D'_{22} & C'_{23} & C'_{24} & 0 & 0 & 0\\ D''_{31} & D''_{32} & D''_{33} & D''_{34} & 0 & 0 & 0\\ D''_{41} & D''_{42} & D''_{43} & D''_{44} & 0 & 0 & I_{\beta-\rho_2}\end{bmatrix}\end{aligned}$$

is block-similar to $[A_1\ A_2]$. Moreover, the submatrix of $[A''_1\ A''_2]$ lying in rows $1,2,...,p$ and columns $p+1,p+2,...,n$ is q_1-equivalent to $[C_1\ C_2]$. Therefore:

$$\operatorname{rank}[C'_{23}\ C'_{24}]=\rho_1\ .$$

Let $\tau=\operatorname{rank}C'_{24}$. Let U' and V' be nonsingular matrices such that:

$$U'\,C'_{24}\,V'=\begin{bmatrix}I_\tau & 0\\ 0 & 0\end{bmatrix}\ .$$

It is not hard to see that $[A''_1\ A''_2]$ is block-similar to:

$$\begin{bmatrix}0 & 0 & 0 & 0 & I_{\rho_2} & 0 & 0\\ U'\,D''_{21} & U'\,D'_{22}\,U'^{-1} & U'\,C'_{23} & U'\,C'_{24}\,V' & 0 & 0 & 0\\ D'_{31} & D''_{32}\,U'^{-1} & D''_{33} & D''_{34}\,V' & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\beta-\rho_2}\end{bmatrix}$$

and hence the result follows. ∎

Lemma 4.4. *Condition (2.2) is satisfied.*

Proof: By induction on q_1.

Case 1. Suppose that $\beta=\rho_2$. Then:

$$v_1+...+v_\rho=v_1+...+v_\beta=m=p+q_1\geq p+\rho_1\ .$$

Case 2. Suppose that $q_1=1$ and $\beta>\rho_2$. In this case we have $\beta=\rho_2+1$.

Subcase 2.1. Suppose that $\rho_1=1$. Then $\beta=\rho$ and

$$v_1+...+v_\rho=v_1+...+v_\beta=m=p+q_1=p+\rho_1\ .$$

Subcase 2.2. Suppose that $\rho_1 = 0$. Then $\beta = \rho + 1$. From Lemma 4.2, it results that:

$$v_1 + ... + v_\rho \geq k_1 + ... + k_\rho = p = p + \rho_1 \ .$$

Case 3. Suppose that $q_1 > 1$ and $\beta > \rho_2$. Let A' be a matrix block-similar to $[A_1 \ A_2]$ of the form indicated in Lemma 4.3.

Subcase 3.1. Suppose that $\tau = 0$. Consider the submatrix that results from A' deleting the m-th row partitioned as follows:

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \end{bmatrix} ,$$

where $E_{11} \in F^{p \times p}$, $E_{22} \in F^{(q_1-1)\times(q_1-1)}$. Using an argument similar to that used in the proof of Lemma 4.1, we deduce that the pair

$$\left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} , \begin{bmatrix} E_{13} \\ E_{23} \end{bmatrix} \right)$$

is c.c.. Let $u_1 \geq ... \geq u_{q_2+1}$ be its controllability indices. According to the induction assumption, we have:

$$u_1 + ... + u_{\rho'} \geq p + \rho_1' \ ,$$

where $\rho' = \operatorname{rank}[E_{12} \ E_{13}] = \rho$ and $\rho_1' = \operatorname{rank} E_{12} = \rho_1$. From Lemma 4.2 it results that $v_1 + ... + v_\rho \geq u_1 + ... + u_\rho$. Hence, (2.2) is satisfied.

Subcase 3.2. Suppose that $\tau \geq 1$. Let A'' be the matrix that results from A' by permutation of the 3-rd and 4-th rows of blocks followed by permutation of the 3-rd and 4-th columns of blocks. Clearly A'' is block-similar to $[A_1 \ A_2]$. Consider A'' partitioned as follows

$$A'' = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \end{bmatrix} ,$$

where $E_{11} \in F^{(p+1)\times(p+1)}$, $E_{22} \in F^{(q_1-1)\times(q_1-1)}$. According to the induction assumption, we have

$$v_1 + ... + v_{\rho'} \geq (p+1) + \rho_1' \ ,$$

where $\rho' = \operatorname{rank}[E_{12} \ E_{13}] = \rho$ and $\rho_1' = \operatorname{rank} E_{12} = \rho_1 - 1$. Then (2.2) is satisfied. ∎

Lemma 4.5. *Condition (2.3) is satisfied.*

Proof: Let A' be a matrix block-similar to $A = [A_1 \ A_2]$ that has the form indicated in Lemma 4.3. It is easy to see that:

$$\tau \geq \beta - q_1 + \rho_1 - \rho_2 \ .$$

If we compute the column minimal indices of $[xI_m\ 0]-A'$, we deduce that this matrix pencil has at least τ column minimal indices greater than 1. Therefore condition (2.3) is satisfied. ∎

5 – Proof that (b) implies (a)

Suppose that $p+q_1=m$ and $q_2=n-m$. The proof is split into several cases. Bearing in mind Proposition 3.2, we assume, throughout the proof, that we have proved (a) whenever we have proved a condition that can be obtained from (a) replacing C_1 and C_2 with matrices D_1 and D_2 such that $[C_1\ C_2]$ and $[D_1\ D_2]$ are q_1-equivalent.

Lemma 5.1. *If $\rho_2=\beta$, then (a) is satisfied.*

Proof: By induction on m. If $m=1$, we have $\rho_2=\beta=1$, $\rho_1=0$. Then (a) is satisfied with:

$$A_1=0\,,\qquad A_2=[1\ \ 0]\ .$$

If $v_1=1$, we have $m=\beta=\rho_2=p$ and $q_1=0$. Then (a) is satisfied with:

$$A_1=0\,,\qquad A_2=[I_\beta\ \ 0]\ .$$

Now suppose that $m\geq 2$ and $v_1\geq 2$.

Case 1. Suppose that $\rho_1=q_1$ and $\rho<p$. According to the induction assumption, there exist matrices:

$$M_1=\begin{bmatrix} M_{11} & N_1\\ M_{31} & M_{33}\end{bmatrix}\in F^{(m-1)\times(m-1)}\ , \tag{5.1}$$

$$M_2=\begin{bmatrix} N_2\\ M_{34}\end{bmatrix}\in F^{(m-1)\times q_2}\ , \tag{5.2}$$

where

$$N_1=\begin{bmatrix} I_{\rho_1}\\ 0\end{bmatrix}\in F^{(p-1)\times q_1}\ , \tag{5.3}$$

$$N_2=\begin{bmatrix} 0 & 0\\ I_{\rho_2} & 0\end{bmatrix}\in F^{(p-1)\times q_2}\ , \tag{5.4}$$

such that (M_1,M_2) is c.c. and has controllability indices $v_1-1,v_2,...,v_{q_2}$ (note that it is possible that the indices $v_1-1,v_2,...,v_{q_2}$ are not ordered in nonincreasing order). Let

$$U\in F^{(m-1)\times(m-1)}\,,\qquad V=\begin{bmatrix} U^{-1} & 0\\ V_{21} & V_{22}\end{bmatrix}\in F^{(m+q_2-1)\times(m+q_2-1)}$$

be nonsingular matrices such that

$$U\,[M_1\ M_2]\,V = [M_1'\ M_2'] \ ,$$

where

$$M_1' = \mathcal{C}(x^{v_1-1}) \oplus \mathcal{C}(x^{v_2}) \oplus ... \oplus \mathcal{C}(x^{v_\beta}) \ ,$$
$$M_2' = \begin{bmatrix} e^{(m-1)}_{v_1-1} & e^{(m-1)}_{v_1+v_2-1} & \cdots & e^{(m-1)}_{v_1+...+v_\beta-1} & 0 \end{bmatrix} .$$

Then, the pair (A_1', A_2'), where

$$A_1' = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline & & \\ 0 & & M_1' \end{array}\right] , \qquad A_2' = \begin{bmatrix} 0 \\ M_2' \end{bmatrix}$$

is c.c. and has controllability indices $v_1, ..., v_{q_2}$.

The matrix $[A_1'\ A_2']$ is block-similar to

$$[A_1''\ A_2''] = ([1] \oplus U^{-1})\,[A_1'\ A_2']\,([1] \oplus V^{-1}) \ ,$$

where

$$A_1'' = \left[\begin{array}{c|ccc} 0 & a_2 & \cdots & a_m \\ \hline & & & \\ 0 & & M_1 & \end{array}\right] \in F^{m\times m} , \qquad a_2, ..., a_m \in F \ ,$$
$$A_2'' = \begin{bmatrix} 0 \\ M_2 \end{bmatrix} .$$

Let

$$Y = \left[\begin{array}{c|ccc|c} 1 & -a_{p+1} & \cdots & -a_m & 0 \\ \hline & & & & \\ 0 & & I_{q_1} & & 0 \\ 0 & & 0 & & I_{p-1} \end{array}\right] .$$

Then (a) is satisfied with:

$$A_1 = Y\,A_1''\,Y^{-1} \quad \text{and} \quad A_2 = Y\,A_2'' \ .$$

Case 2. Suppose that $\rho_1 = q_1$ and $\rho = p$. Note that it is impossible that $q_1 = 0$, as this situation implies that $\beta = \rho_2 = p = m$ and, therefore, $v_1 = ... = v_\beta = 1$. Then assume that $q_1 \geq 1$. Firstly, we use the arguments that we have used in case 1 up to the definition of A_1'' and A_2'', except that we replace (5.3) with

$$N_1 = \begin{bmatrix} I_{\rho_1-1} & 0 \\ 0 & 0 \end{bmatrix} \in F^{(p-1)\times q_1} .$$

Now let us consider $[A_1'' \; A_2'']$ partitioned as follows

$$[A_1'' \; A_2''] = \left[\begin{array}{c|c|c|c|c|c|c} 0 & a_2 \dots a_{\rho_1} & a_{\rho_1+1} \dots a_p & a_{p+1} \dots a_{m-1} & a_m & 0 & 0 \\ \hline 0 & L_{22} & L_{23} & I_{\rho_1-1} & 0 & 0 & 0 \\ 0 & L_{32} & L_{33} & 0 & 0 & I_{\rho_2} & 0 \\ 0 & L_{42} & L_{43} & L_{44} & L_{45} & L_{46} & L_{47} \\ 0 & L_{52} & L_{53} & L_{54} & L_{55} & L_{56} & L_{57} \end{array}\right],$$

where $L_{22} \in F^{(\rho_1-1)\times(\rho_1-1)}$, $L_{33} \in F^{\rho_2\times\rho_2}$, $L_{44} \in F^{(\rho_1-1)\times(\rho_1-1)}$, $L_{55} \in F$.

If $a_m \neq 0$, it is already clear that (a) is satisfied.

Now suppose that $a_m = 0$. Since (A_1'', A_2'') has $\beta = \rho_2$ nonzero controllability indices, we have $L_{47} = 0$ and $L_{57} = 0$. Let

$$Z = \left[\begin{array}{c|c|ccc} 1 & -a_{p+1} \dots - a_{m-1} & 0 & 0 & 0 \\ \hline 0 & I_{\rho_1-1} & 0 & 0 & 0 \\ 0 & 0 & I_{\rho_2} & 0 & 0 \\ 0 & -L_{44} & -L_{46} & I_{\rho_1-1} & 0 \\ 0 & -L_{54} & -L_{56} & 0 & 1 \end{array}\right].$$

Then $[A_1'' \; A_2'']$ is block-similar to a matrix of the form

$$\begin{aligned} [A_1^{(3)} \; A_2^{(3)}] &= Z\,[A_1'' \; A_2'']\,(Z^{-1} \oplus I_{q_2}) \\ &= \left[\begin{array}{c|c|c|cccc} 0 & a_2' \dots a_{\rho_1}' & a_{\rho_1+1}' \dots a_p' & 0 & 0 & 0 & 0 \\ \hline 0 & L_{22}' & L_{23}' & I_{\rho_1-1} & 0 & 0 & 0 \\ 0 & L_{32}' & L_{33}' & 0 & 0 & I_{\rho_2} & 0 \\ 0 & L_{42}' & L_{43}' & 0 & L_{45}' & 0 & 0 \\ 0 & L_{52}' & L_{53}' & 0 & L_{55}' & 0 & 0 \end{array}\right], \end{aligned}$$

where $A_1^{(3)} \in F^{m\times m}$. Since $(A_1^{(3)}, A_2^{(3)})$ is a c.c. pair, from (1.1) we conclude that there exists $j \in \{2, ..., p\}$ such that $a_j' \neq 0$. In $[A_1^{(3)} \; A_2^{(3)}]$ we add the j-th column to the m-th column and we subtract the m-th row from the j-th row. We get a block-similar matrix $[A_1 \; A_2]$ that shows that (a) is satisfied.

Case 3. Suppose that $\rho_1 < q_1$. We use the arguments that we have used in case 1 up to the definition of A_1'' and A_2'', except that we replace N_1 and N_2 with:

$$N_1 = \begin{bmatrix} I_{\rho_1} & 0 \\ 0 & 0 \end{bmatrix} \in F^{p\times(q_1-1)},$$

$$N_2 = \begin{bmatrix} 0 & 0 \\ I_{\rho_2} & 0 \end{bmatrix} \in F^{p\times q_2}.$$

Let

$$W = \begin{bmatrix} 0 & I_p & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{q_1-1} \end{bmatrix}.$$

Then (a) is satisfied with:

$$A_1 = W\, A_1''\, W^{-1}\,, \qquad A_2 = W\, A_2''\,. \blacksquare$$

Lemma 5.2. *If (2.1) and (2.3) are satisfied and $\rho_1 + \rho_2 \geq \beta$, then (a) is satisfied.*

Proof: Define ν as in (2.9). Let $t = \min\{\nu, \beta - \rho_2\}$. Condition (2.3) implies that $\nu \geq \beta - q_1 + \rho_1 - \rho_2$. Clearly, $t \geq \beta - q_1 + \rho_1 - \rho_2$. Therefore $q_1 - \beta + \rho_2 \geq \rho_1 - t$. We also have $t \leq \beta - \rho_2 \leq \rho_1$ and $t + \rho_2 \leq \beta \leq q_2$. According to Lemma 5.1, there exist matrices:

$$M_1 = \begin{bmatrix} M_{11} & N_1 \\ M_{31} & M_{33} \end{bmatrix} \in F^{(m-\beta+\rho_2)\times(m-\beta+\rho_2)}\,,$$

$$M_2 = \begin{bmatrix} N_2 \\ M_{34} \end{bmatrix} \in F^{(m-\beta+\rho_2)\times q_2}\,,$$

where

$$N_1 = \begin{bmatrix} I_{\rho_1 - t} & 0 \\ 0 & 0 \end{bmatrix} \in F^{p\times(q_1-\beta+\rho_2)}\,,$$

$$N_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{\rho_2} \\ I_t & 0 & 0 \end{bmatrix} \in F^{p\times q_2}\,,$$

such that the pair (M_1, M_2) is c.c. and its nonzero controllability indices are $v_1 - 1, \dots, v_t - 1, v_{\beta-\rho_2+1}, \dots, v_\beta$. Let

$$U \in F^{(m-\beta+\rho_2)\times(m-\beta+\rho_2)}\,, \qquad V = \begin{bmatrix} U^{-1} & 0 \\ V_{21} & V_{22} \end{bmatrix} \in F^{(m-\beta+\rho_2+q_2)\times(m-\beta+\rho_2+q_2)}\,,$$

be nonsingular matrices such that

$$U\,[M_1\ M_2]\,V = [M_1'\ M_2'\ M_3']\,, \tag{5.5}$$

where

$$M_1' = \mathcal{C}(x^{v_1-1}) \oplus \dots \oplus \mathcal{C}(x^{v_t-1}) \oplus \mathcal{C}(x^{v_{\beta-\rho_2+1}}) \oplus \dots \oplus \mathcal{C}(x^{v_\beta})\,,$$

$$M_2' = \begin{bmatrix} e_{v_1-1}^{(m-\beta+\rho_2)} & e_{v_1+v_2-2}^{(m-\beta+\rho_2)} & \dots & e_{v_1+\dots+v_t-t}^{(m-\beta+\rho_2)} & 0 \end{bmatrix} \in F^{(m-\beta+\rho_2)\times(\beta-\rho_2)}\,,$$

$$M_3' = \begin{bmatrix} e_{v_1+\dots+v_t+v_{\beta-\rho_2+1}-t}^{(m-\beta+\rho_2)} & e_{v_1+\dots+v_t+v_{\beta-\rho_2+1}+v_{\beta-\rho_2+2}-t} & \cdots \\ \cdots\ e_{v_1+\dots+v_t+v_{\beta-\rho_2+1}+\dots+v_\beta-t} & 0 \end{bmatrix} \in F^{(m-\beta+\rho_2)\times(q_2-\beta+\rho_2)}\,.$$

Let

$$[K_1\ K_2] = \begin{bmatrix} M_1' & M_2' & M_3' & 0 \\ 0 & 0 & 0 & I_{\beta-\rho_2} \end{bmatrix}, \qquad \text{with } K_1 \in F^{m\times m} .$$

The pair (K_1, K_2) is c.c. and has controllability indices $v_1, ..., v_{q_2}$. From (5.5), we get:

$$[M_1\ M_2 V_{22}] = U^{-1}\,[M_1'\ M_2'\ M_3'] \begin{bmatrix} U & 0 \\ -V_{22}^{-1} V_{21}\, U & I_{q_2} \end{bmatrix} .$$

Let W be the submatrix of

$$\begin{bmatrix} U & 0 \\ -V_{22}^{-1} V_{21}\, U & I_{q_2} \end{bmatrix}$$

lying in rows and columns $1, ..., m$. Then

$$W^{-1} = \begin{bmatrix} U^{-1} & 0 \\ * & I_{\beta-\rho_2} \end{bmatrix}$$

and $[K_1\ K_2]$ is block-similar to

$$\begin{aligned}[K_1'\ K_2'] &= W^{-1}\,[K_1\ K_2] \left(\begin{bmatrix} U & 0 \\ -V_{22}^{-1} V_{21}\, U & I_{q_2} \end{bmatrix} \oplus I_{\beta-\rho_2} \right) \\ &= \begin{bmatrix} M_1 & M_2 V_{22} & 0 \\ * & * & I_{\beta-\rho_2} \end{bmatrix} ,\end{aligned}$$

$K_1' \in F^{m\times m}$. Moreover, $M_2 V_{22} = U^{-1}\,[M_2'\ M_3']$. Consider the matrix $M_2 V_{22}$ partitioned as follows:

$$M_2 V_{22} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \qquad \text{where } L_{11} \in F^{p\times(\beta-\rho_2)} .$$

We have:

$$\operatorname{rank}(M_2 V_{22}) = \operatorname{rank}[M_2'\ M_3'] = \rho_2 + t ,$$

$$\operatorname{rank}[L_{11}\ L_{12}] = \operatorname{rank}(N_2 V_{22}) = \rho_2 + t .$$

Hence the last $q_1 - \beta + \rho_2$ rows of $M_2 V_{22}$ are linear combinations of the first p rows. Therefore, from

$$\operatorname{rank} \begin{bmatrix} L_{12} \\ L_{22} \end{bmatrix} = \operatorname{rank}(U^{-1} M_3') = \rho_2 ,$$

we conclude that $\operatorname{rank} L_{12} = \rho_2$. Moreover:

$$\operatorname{rank}\Big[N_1\ \ L_{11}\ \ L_{12}\Big] = \operatorname{rank}\Big[N_1\ \ N_2 V_{22}\Big] = \rho_1 + \rho_2 .$$

Consequently, the submatrix of $[K_1'\ K_2']$ lying in rows $1, ..., p$ and columns $p+1, ..., m+q_2$ is q_1-equivalent to $[C_1\ C_2]$, and the proof is complete. ∎

Proof that (b) implies (a): By induction on m. If $m = 1$, the result is trivial. Suppose that $m \geq 2$. Bearing in mind Lemma 5.2, we assume that $\rho_1 + \rho_2 < \beta$. From (2.2), we deduce that $v_\beta \leq q_1 - \rho_1$. According to the induction assumption, there exist matrices

$$M_1 = \begin{bmatrix} M_{11} & N_1 \\ M_{31} & M_{33} \end{bmatrix} \in F^{(m-v_\beta)\times(m-v_\beta)} ,$$

$$M_2 = \begin{bmatrix} N_2 \\ M_{34} \end{bmatrix} \in F^{(m-v_\beta)\times(q_2-1)} ,$$

where

$$N_1 = \begin{bmatrix} I_{\rho_1} & 0 \\ 0 & 0 \end{bmatrix} \in F^{p\times(q_1-v_\beta)} ,$$

$$N_2 = \begin{bmatrix} 0 & 0 \\ I_{\rho_2} & 0 \end{bmatrix} \in F^{p\times(q_2-1)} ,$$

such that (M_1, M_2) is c.c. and has controllability indices $v_1, ..., v_{\beta-1}, v_{\beta+1}, ..., v_{q_2}$. Let

$$A_1 = \begin{bmatrix} M_1 & 0 \\ 0 & \mathcal{C}(x^{v_\beta}) \end{bmatrix} \in F^{m\times m} ,$$

$$A_2 = \begin{bmatrix} M_2 & 0 \\ 0 & e_{v_\beta}^{(v_\beta)} \end{bmatrix} \in F^{m\times q_2} .$$

The pair (A_1, A_2) is c.c., has controllability indices $v_1, ..., v_{q_2}$ and the submatrix of $[A_1\ A_2]$ lying in rows $1, ..., p$ and columns $p+1, ..., m+q_2$ is q_1-equivalent to $[C_1\ C_2]$. ∎

6 – Proof of Theorem 2

Suppose that (a) is satisfied. Then (2.4) is satisfied. Let

$$\tau_2 = \operatorname{rank}\Big[C_2\ A_{15}\Big] , \tag{6.1}$$

$$\tau_1 = \operatorname{rank}\Big[A_{12}\ C_1\ C_2\ A_{15}\Big] - \tau_2 . \tag{6.2}$$

The following inequalities are trivial:

$$\rho_2 \leq \tau_2 \leq \min\Big\{n - m - q_2 + \rho_2,\ p\Big\} , \tag{6.3}$$

$$0 \leq \tau_1 \leq \min\Big\{m - p - q_1 + \rho_1,\ p - \tau_2\Big\} . \tag{6.4}$$

According to Theorem 1, we have:

$$\tau_2 \leq \beta \ , \tag{6.5}$$

$$v_1 + ... + v_{\tau_1+\tau_2} \geq p + \tau_1 \ , \tag{6.6}$$

$$v_{\beta-m+p+\tau_1-\tau_2} \geq 2 \ . \tag{6.7}$$

Condition (6.7) is equivalent to

$$\beta - m + p + \tau_1 - \tau_2 \leq \nu \ ,$$

where ν is defined according to (2.9). Therefore:

$$\tau_1 \leq m - p - \beta + \tau_2 + \nu \ . \tag{6.8}$$

From (6.3) and (6.5), we get:

$$\tau_2 \leq \sigma_2 \ . \tag{6.9}$$

If $\sigma_1 = m - p - q_1 + \rho_1$ (respectively, $\sigma_1 = m - p - \beta + \sigma_2 + \nu$) then from (6.4) (respectively, (6.8) and (6.9)) we conclude that:

$$\tau_1 \leq \sigma_1 \ .$$

Otherwise, we have $\sigma_1 = p - \sigma_2$ and from (6.4) it results that:

$$\tau_1 + \tau_2 \leq p = \sigma_1 + \sigma_2 \ .$$

We have always:

$$\tau_1 + \tau_2 \leq \sigma_1 + \sigma_2 \ . \tag{6.10}$$

As $\rho_1 + \rho_2 \leq \tau_1 + \tau_2$, we conclude that (2.6) is satisfied.

If $\sigma_1 + \sigma_2 \geq \beta$ we have:

$$v_1 + ... + v_{\sigma_1+\sigma_2} = m \geq p + m - p - q_1 + \rho_1 \geq p + \sigma_1 \ .$$

Now suppose that $\sigma_1 + \sigma_2 < \beta$. Bearing in mind (6.10), we have:

$$v_1 + ... + v_{\sigma_1+\sigma_2} \geq v_1 + ... + v_{\tau_1+\tau_2} + (\sigma_1 + \sigma_2) - (\tau_1 + \tau_2) \ .$$

Then, from (6.6) and (6.9) we conclude that:

$$v_1 + ... + v_{\sigma_1+\sigma_2} \geq p + \tau_1 + (\sigma_1 + \sigma_2) - (\tau_1 + \tau_2) \geq p + \sigma_1 \ .$$

Conversely, suppose that (c) is satisfied. Let $\tau_2 = \sigma_2$ and $\tau_1 = \sigma_1$. Then (6.3) and (6.4) are satisfied. We only prove that $0 \leq \tau_1$, as the other inequalities of (6.3) and (6.4) are trivial. Suppose that $\sigma_1 = \tau_1 < 0$. Then, from (2.5), we get $v_1 + ... + v_{\sigma_2} \geq p$. Therefore $\beta - \sigma_2 \leq v_{\sigma_2+1} + ... + v_\beta \leq m - p$ and $m-p-\beta+\sigma_2+\nu \geq 0$. Clearly, $m-p-q_1+\rho_1 \geq 0$ and $p-\sigma_2 \geq 0$. Consequently, $\sigma_1 = \tau_1 \geq 0$.

Now let $h_1, ..., h_{\rho_2}$ be a basis of the subspace of $F^{p\times 1}$ generated by the columns of C_2; let $h_1, ..., h_{\rho_2}, ..., h_{\rho_1+\rho_2}$ be a basis of the subspace of $F^{p\times 1}$ generated by the columns of $[C_1\ C_2]$ and let $h_1, ..., h_p$ be a basis of $F^{p\times 1}$.

Let

$$A_{15} = \Big[h_{\rho_2+1}\ \ ...\ \ h_{\rho_2+\alpha_1}\ \ h_{\rho_1+\rho_2+1}\ \ ...\ \ h_{\rho_1+\rho_2+\alpha_2}\ \ 0\Big]\ ,$$

$$A_{12} = \Big[h_{\rho_1+\rho_2+\alpha_2+1}\ \ ...\ \ h_{\rho_1+\rho_2+\alpha_2+\alpha_3}\ \ 0\Big]\ ,$$

where

$$\alpha_2 = \min\Big\{\tau_2 - \rho_2,\ \tau_1 + \tau_2 - \rho_1 - \rho_2\Big\}\ ,$$

$$\alpha_1 = \tau_2 - \rho_2 - \alpha_2\ ,$$

$$\alpha_3 = \tau_1 + \tau_2 - \rho_1 - \rho_2 - \alpha_2\ .$$

Then (6.1) and (6.2) are satisfied. From (2.8), we get (6.8), what is equivalent to (6.7). Note that (6.5) and (6.6) are also satisfied. According to Theorem 1, condition (a) is satisfied. ∎

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Isabel Cabral,
Departamento de Matemática, Universidade Nova de Lisboa,
Faculdade de Ciências e Tecnologia, Quinta da Torre, 2825 Monte da Caparica – PORTUGAL

and

Fernando C. Silva,
Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa,
Rua Ernesto de Vasconcelos, C1, 1700 Lisboa – PORTUGAL