PORTUGALIAE MATHEMATICA
Vol. 52 Fasc. 2 – 1995

MOTION OF A PARTICLE SUBMITTED TO DRY FRICTION AND TO NORMAL PERCUSSIONS

M. Laghdir and Manuel D.P. Monteiro Marques

Abstract: We consider the one-dimensional dynamics of a particle moving on a mobile plane with contact being kept and dry friction. The particle is submitted to normal loading described not only by usual forces but also by a countable set of percussions. We prove the existence of a solution to this problem by using some convex analysis methods and measure theory.

1 – Introduction and formulation

We consider a particle moving on a line in a horizontal mobile rigid plane. This plane may be materialized as a vibrating table. We assume that the contact is being kept. The particle is submitted to normal loadings which are described by usual forces and also by a countable set of percussions. These percussions constitute in fact the originality of the problem. Their presence implies that the velocity function $t \to U(t)$ may fail to be differentiable or absolutely continuous. Instead, we only expect it to have bounded variation. Classically, we associate with U a scalar measure on an interval of time I, called differential measure of U, and denoted by dU. The role of the acceleration is then held by a density of the measure dU relative to some nonnegative real measure $d\theta$ as explained in the sequel. The actions exerted on the particle, including the contact effects, are expressed as measures. This allows us to take into account the effect of percussions. If all the actions are smooth and contact is kept, it is known that the velocity of the particle is an absolutely continuous function. In that case, U is a solution of a differential inclusion, which brings together Lagrange's equations and Coulomb's friction law in the sense of J.J. Moreau [3]; the acceleration is constrained to satisfy this inclusion on I except possibly on a Lebesgue null subset of I. In the

Received: November 8, 1993.
Classification (AMS 1991): 70D (70D05), 34A (34A37, 34A60).
Keywords: Friction, percussion, dynamics, contact, measure and convex.

case where the particle is subjected both to smooth and nonsmooth effects, using the traditional equations of the dynamic of percussions, the Lagrange's equations with Coulomb's friction law and the notion of differential measure, J.J. Moreau ([2], [3]) elaborated a synthetic formulation. This formulation gives the preceding differential inclusion on all intervals where the velocity $t \to U(t)$ is sufficiently smooth and it describes the shocks where the discontinuities of U occur. The system which governs the nonsmooth motion of the particle, called also a measure differential inclusion, is formulated as follows (see [3] and [4])

$$(Q) \quad \begin{cases} m\,dU = f\,d\theta + P\,dt, \\ f(t) \in \partial\psi^*_{\nu(t)C}(-U^+(t) + e(t))\ d\theta \ \text{ a.e.}, \\ dR_{\mathrm{T}} = f\,d\theta, \\ dR_{\mathrm{N}} = \nu\,d\theta, \\ U(t_0) = 0\ , \end{cases}$$

where m denotes the mass of the particle, $t \to U(t)$ and $t \to e(t)$ denote respectively the velocity of the particle and the transport velocity of the plane. The latter is supposed known and lipschitzian. P is a given tangential force, measurable and bounded. We denote by $U^+(t)$ and $U^-(t)$ respectively the right and left limits at $t \in I$ (the presence of U^+ in system (Q) is justified in [3]). The functions $t \to f(t)$ and $t \to \nu(t)$ are respectively the density of the tangential component dR_{T} and of the normal component dR_{N} of the reaction dR with respect to the measure $d\theta$ defined on I and specified in the sequel. The normal component of the reaction is imposed, i.e. ν is given and by assumption it is a bounded function of t. The set C is equal to $[-\gamma, \gamma]$, where $\gamma > 0$ represents the friction coefficient (here we consider only the case of isotropic friction). All functions considered in this problem are defined from a time-interval $I = [t_0, t_0 + T]$ with $t_0 \in \mathbb{R}$ and $T > 0$ and take their values in $\mathbb{R}$. The first and the second equation express respectively the fundamental law of percussional dynamics and Coulomb's friction law. The latter is expressed by means of convex analysis where $\partial\psi^*_{\nu(t)C}$ denotes the subdifferential of the support function of the convex set $\nu(t)C$. For some details about the equivalence between the second equation of (Q) and the classical formulation of Coulomb's law see [3]. The unknowns of the problem (Q) are U and f (related through the first equation).

In section 2, we introduce the percussion measure $d\mu$ and approximate it by a finite number of percussions. In section 3, we solve (Q) in the absence of percussions ($d\theta = dt$, Lebesgue measure) and of tangential forces ($P = 0$). In section 4, we show that there are solutions to the approximate problems (Q_n) and then in section 5 we prove that they converge to a solution to (Q). The general case ($P \neq 0$) is evoked in section 6.

2 – Definition of the measure $d\theta$

Let us consider a countable subset D of I containing t_0 and t_0+T. Let $(a_i)_{i\in\mathbb{N}}$ be a summable family of positive real numbers i.e.

$$0 < \sum_{i=0}^{\infty} a_i < +\infty \quad \text{with} \;\; a_i > 0 \; .$$

We define

$$d\mu = \sum_{t_i\in D} a_i\,\delta_{t_i} \; ,$$

where δ_{t_i} is the Dirac measure located at the instant $t_i \in D$ with value a_i. Then $d\mu$ defines a Radon measure in the sense that it is a continuous linear form on the space of all continuous functions which we denote by $C(I,\mathbb{R})$. Furthermore, $d\mu$ is finite and positive. The measure $d\theta$ is then defined by

$$d\theta = d\mu + dt \; ,$$

where dt denotes the Lebesgue measure on I. Let us consider the family $(I_n)_{n\in\mathbb{N}}$ given by

$$I_n = \Big\{t_i \in D \mid i \le n\Big\} \cup \Big\{t_0 + T\Big\} \; .$$

Obviously (I_n) is a nondecreasing sequence with respect to the order defined by the set inclusion and we have, in addition, $\bigcup_n I_n = D$. We denote by $(d\mu_n)$ and $(d\theta_n)$ two sequences of Radon measures defined by

$$d\mu_n = \sum_{t_i\in I_n} a_i\,\delta_{t_i} \quad \text{and} \quad d\theta_n = d\mu_n + dt \; .$$

Remark. The measure $d\mu$ expresses the effects of the percussions on the particle; more precisely, at the instant $t_i \in D$, the value of the percussion imposed on the particle is measured by $a_i\,\nu(t_i)$, where ν is the density of the normal reaction.

Proposition 2.1.

1) $(d\mu_n)$ *is a nondecreasing sequence with respect to the usual ordering of measures;*

2) $$\begin{cases} d\mu_n = \mu_n'\,d\mu & \text{with } \mu_n' \in L^1(I,\mathbb{R},d\mu), \\ \mu_n' = 1 & \text{on } I_n, \\ \mu_n' = 0 & \text{on } D - I_n; \end{cases}$$

3) *The sequence $(d\mu_n)$ converges strongly to $d\mu$ (i.e. relative to the norm defined on the space of all scalar finite measures defined on I).*

Proof: 1) and **2)** are obvious.

3) Notice that

$$\|d\mu_n - d\mu\| = \|\mu'_n\, d\mu - d\mu\| = \|\mu'_n - 1\|_{L^1(I,\mathbb{R},d\mu)}$$

and that, as $n \to \infty$,

$$\int_I |\mu'_n - 1|\, d\mu = \sum_{i=0}^{\infty} a_i\, |\mu'_n(t_i) - 1| = \sum_{n+1}^{\infty} a_i \;\to\; 0$$

since the family (a_i) is summable. ∎

3 – Solution of system (Q) without percussions

The purpose of this section is to study as a preliminary step the case where the particle is submitted only to smooth effects without percussions in some subinterval $[a,b]$. The velocity $t \to V(t)$ of such motion is a smooth function, and it must satisfy the following system (Q_1)

$$(Q_1) \quad \begin{cases} m\,\dfrac{dV}{dt} = g, \\ g(t) \in \partial\psi^*_{\nu(t)C}(-V(t)+e(t))\; dt \;\text{ a.e.}, \\ V(a) = V_0\;, \end{cases}$$

where $t \to g(t)$ and $t \to \nu(t)$ defined on $[a,b]$, are respectively the tangential and normal components of the reaction exerted by the plane (in fact, a line) on the particle. In this case, these components are not measures but usual functions. The system (Q_1) is a particular case of a certain class of parabolic evolution problems whose general formulation is

$$(Q_2) \quad \begin{cases} -\dfrac{dV}{dt} \in \partial F(t,V(t))\; dt \;\text{ a.e.}, \\ V(a) = V_0\;, \end{cases}$$

where $F:\; [a,b] \times \mathbb{R} \to]-\infty,+\infty]$ is a normal convex proper integrand. Let us consider $X = L^1([a,b],\mathbb{R},dt)$ and $Y = L^\infty([a,b],\mathbb{R},dt)$ and their traditional duality denoted by $X\langle\;\rangle Y$. For all $x \in X$ we define

$$I_F(x) = \begin{cases} \displaystyle\int_a^b F(t,x(t))\, dt, & \text{if } t \to F(t,x(t)) \text{ is } dt\text{-integrable}, \\ +\infty, & \text{otherwise}\;. \end{cases}$$

For all $y \in Y$, we define a function Sy by

$$Sy\colon t \to Sy(t) = \int_a^t y(s)\,ds \ .$$

The evolution problem (Q_2), where V is an absolutely continuous function, is equivalent by virtue of [5] to

$$-V \in \partial I_F\Big(V(a) + SV\Big) \ .$$

We introduce here a minimization problem which is equivalent to (Q_2). Let S^* denote the adjoint operator of S with respect to the duality $X\langle\ \rangle Y$ and let K be the positive quadratic form defined by

$$K(y) = \langle Sy, y\rangle = \langle S^* y, y\rangle = \frac{1}{2}\langle (S^* + S)y, y\rangle \ ,$$

with $y \in Y$. K is convex and weakly l.s.c. on Y. Let G be the polar function of F with respect to the duality $X\langle\ \rangle Y$. By virtue of [6] we know that if I_F and I_G are proper (i.e. not identical to $+\infty$), then their polar functions $(I_F)^*$ et $(I_G)^*$ are given by $(I_F)^* = I_G$ and $(I_G)^* = I_F$. I_F and I_G are l.s.c. respectively on X and Y relatively to any convex topology τ compatible with this duality. The minimization problem which is equivalent to (Q_2) was given in [5] and is recalled here.

Proposition 3.1 [5]. *Let $J_1 : Y \to \mathbb{R}$ and $J_2 : Y \to \mathbb{R}$ be two functionals defined respectively by*

$$J_1(y) = I_F(-Sy) + I_G(y) + \langle Sy, y\rangle \ ,$$
$$J_2(y) = I_G(y) + (I_{G+K})^*(S^* y) \ .$$

- **a**) *J_1 and J_2 take both finite values. V is a solution to (Q_2) if and only if J_1 (resp. J_2) attains on V a minimum, with value zero.*
- **b**) *If there exists $\alpha \in Y$ such that $I_G(\alpha) \in \mathbb{R}$ and I_F is finite and continuous at $(-S\alpha)$ with respect to τ, then J_2 admits at least a zero minimum on Y.*
- **c**) *Let c be an element of X. If there exists $\alpha \in Y$ such that $I_G(\alpha) \in \mathbb{R}$ and I_F is finite and continuous at $(c - S\alpha)$ relative to τ then there exists $u \in Y$ such that $u \in \partial I_F(c - Su)$.*

The aim of the following proposition is to write the second equation of (Q), which holds $d\theta$ a.e., under another equivalent form expressed relative to the duality $X\langle\ \rangle Y$, where now $X = L^1(I, \mathbb{R}^n, d\theta)$ and $Y = L^\infty(I, \mathbb{R}^n, d\theta)$.

Proposition 3.2. *Let*

$$F : I \times \mathbb{R}^n \to]-\infty, +\infty]$$
$$(t,x) \to F(t,x) = \psi^*_{\Gamma(t)}(x) \ ,$$

where $\Gamma : I \to \mathbb{R}^n$ *is a given measurable multifunction with nonempty convex and compact values. Then*

a) F *is a convex normal integrand.*

b) *If we take* $H = \{y \in L^\infty(I, \mathbb{R}^n, d\theta) \mid y(t) \in \Gamma(t)\, d\theta \text{ a.e.}\}$ *then we have*

$$y(t) \in \partial\psi^*_{\Gamma(t)}(x(t))\, d\theta \text{ a.e.} \quad \Leftrightarrow \quad y \in \partial\psi^*_H(x) \ .$$

Proof: a) It is easy to verify that $\psi^*_{\Gamma(t)}(x) \in \mathbb{R}$ for all $t \in I$ and $x \in \mathbb{R}^n$. By virtue of [1] (theorem III.15, p. 70), the support function of $\Gamma(t)$ is measurable and the function $F^*_t : \mathbb{R}^n \to \mathbb{R}$, $F^*_t(x) = \psi_{\Gamma(t)}(x)$ with t being fixed, is convex. Here $\psi_{\Gamma(t)}$ denotes the indicator function. Since Γ takes bounded values in $\mathbb{R}^n$, then for all $t \in I$, there exists $M(t) \geq 0$ such that for all $x, y \in \mathbb{R}^n$ we have

$$\left|\psi^*_{\Gamma(t)}(x) - \psi^*_{\Gamma(t)}(y)\right| \leq M(t)\, \|x - y\|_{\mathbb{R}^n} \ .$$

Accordingly, lemma III.14, p. 70 of [1] shows that F is globally measurable. Since F_t is convex, proper and continuous by virtue of [6], F is a convex normal integrand.

b) Let us take

$$I_F(x) = \int_I \psi^*_{\Gamma(t)}(x(t))\, d\theta(t) \quad \text{and} \quad I_G(y) = \int_I \psi_{\Gamma(t)}(y(t))\, d\theta(t) \ .$$

Both $I_F : L^1(I, \mathbb{R}^n, d\theta) \to \mathbb{R}$ and $I_G : L^\infty(I, \mathbb{R}^n, d\theta) \to \mathbb{R}$ take finite values and by virtue of [6], $(I_F)^* = I_G$ and $I_F = (I_G)^*$. Moreover, we have by the same result of [6]

$$y(t) \in \partial F_t(x(t))\, d\theta \text{ a.e.} \quad \Leftrightarrow \quad y \in \partial I_F(x) \ .$$

If we take H as indicated (the set of bounded selections of Γ) then we obtain $I_G(y) = \psi_H(y)$ and $I_F(x) = (I_G)^*(x) = \psi^*_H(x)$. Hence

$$y(t) \in \partial\psi^*_{\Gamma(t)}(x(t))\, d\theta \text{ a.e.} \quad \Leftrightarrow \quad y \in \partial\psi^*_H(x) \ . \ \blacksquare$$

The next corollary is formulated for a slightly more general ν than needed here.

Corollary 3.3. *We consider*

$$\begin{aligned}\Gamma : I &\to \mathbb{R}^n \\ t &\to \nu(t)\, C\ ,\end{aligned}$$

where C is a fixed nonempty compact interval of $\mathbb{R}$ and $\nu = (\nu_1, \nu_2, ..., \nu_n)$ defined from I into $\mathbb{R}^n$ is assumed measurable and bounded on I. Γ is a measurable multifunction and admits at least one selection belonging to $L^\infty(I, \mathbb{R}^n, d\theta)$. If we denote $H = \nu\, C$ then we have

$$y(t) \in \partial\psi^*_{\nu(t)C}(x(t))\, d\theta \text{ a.e.} \quad \Leftrightarrow \quad y \in \partial\psi^*_{\nu C}(x)\ .$$

Proof: Since Γ is a multifunction with nonempty convex compact values, then, by virtue of [1] (theorem III.9, p. 67), Γ is measurable iff there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of measurable selections of Γ such that

$$\mathrm{cl}\Big\{\sigma_n(t) \mid n \in \mathbb{N}\Big\} = \Gamma(t)$$

(cl means the topological closure). For that, take $B = \mathbb{Q} \cap C$ where $\mathbb{Q}$ is the set of rational numbers. B may be written as a sequence (z_n) since it is a countable set. Let us consider $\sigma_n = \nu(t)\, z_n$. Clearly (σ_n) is a countable family of measurable selections of Γ. For all $t \in I$ we have

$$\mathrm{cl}\Big\{\sigma_n(t) \mid n \in \mathbb{N}\Big\} = \mathrm{cl}\Big\{\nu(t)\, z_n \mid n \in \mathbb{N}\Big\} = \nu(t)\, C = \Gamma(t)\ . \blacksquare$$

If we take $Z = -v + e$ (here e is assumed to be a constant function), then the system (Q_1) will be transformed into an equivalent system in the interval $[a, b]$

$$\begin{cases} -\dfrac{dZ}{dt} \in \partial\psi^*_{\frac{1}{m}\nu(t)C}(Z(t))\, dt \quad \text{a.e.}, \\ Z(a) = -V_0 + e\ . \end{cases}$$

Let us define $\Gamma(t) = \frac{1}{m}\, \nu(t)\, C$, which is a measurable multifunction (by corollary 3.3). Thus, proposition 3.2 applies and so

$$F(t, x) = \psi^*_{\frac{1}{m}\nu(t)C}(x) \tag{3.1}$$

is a convex normal integrand. Moreover, (Q_1) takes the form of (Q_2). We solve (Q_2) by means of proposition 3.1. I_F is a functional defined on $L^1([a, b], \mathbb{R}, dt)$ with finite values. By the previous results, the polar function of I_F is exactly ψ_H, the indicator function of the set

$$H = \left\{\varphi \in L^\infty([a, b], \mathbb{R}, dt) \mid \varphi(t) \in \frac{1}{m}\, \nu(t)\, C\ dt \quad \text{a.e.}\right\}\ .$$

Notice that the functional $x \to I_F(c - Sx)$, defined on $L^\infty([a,b], \mathbb{R}, dt)$ and with $c = -V_0 + e$, is lipschitzian and that I_G is finite on H. Then by proposition 3.1 c) the problem (Q_1) admits at least one solution V which is an absolutely continuous function; in fact, its derivative is in $L^\infty([a,b], \mathbb{R}, dt)$, so that V is lipschitzian.

4 – The approximated problem (Q_n)

Now, we go back to problem (Q) and we apply to it an approximation method. For that purpose let us consider the approximated problem (Q_n) of (Q) defined by

$$\left.\begin{aligned} &m\, dU_n = f_n\, d\theta_n, && (4.1)\\ &f_n(t) \in \partial\psi^*_{\nu(t)C}(-U_n^+(t) + e)\ d\theta_n \ \text{ a.e.}, && (4.2)\\ &U_n(t_0) = 0\ , \end{aligned}\right\}\ (Q_n)$$

where

$$d\theta_n = d\mu_n + dt\ .$$

I_n, the support of the atomic measure $d\mu_n$, is a finite set. Its elements are $\lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_p$ with $\lambda_0 = t_0$ and $\lambda_p = t_0 + T$. Let $j \in \{0, 1, ..., p\}$. By integrating (4.1) on the singleton $\{\lambda_j\}$ we get

$$U_n^+(\lambda_j) = U_n^-(\lambda_j) + \frac{a_j}{m}\, f_n(\lambda_j)\ . \tag{4.3}$$

Here $t \to U_n^+(t)$ and $t \to U_n^-(t)$ denote respectively the right-continuous and left-continuous regularized functions of U_n, which are of bounded variation on I, if U_n is so. Here and below, we must treat slightly differently the special cases of the endpoints. For $j = 0$, we consider $U_n^-(t_0) := U_n(t_0) = 0$, so that $U_n^+(t_0) = \frac{a_0 f_n(t_0)}{m}$ where $f_n(t_0) = f_n(\lambda_0)$ is defined by (4.4) below. For $j = p$, it is enough to define $U_n(t_0 + T) := U_n^+(\lambda_p)$ as in (4.3) and $f_n(t_0 + T) = f_n(\lambda_p)$ as in (4.4) below.

The equation (4.2) is equivalent $d\theta_n$ a.e. to

$$f_n(t) = \mathrm{proj}_{\nu(t)C}\Big[f_n(t) - \rho(U_n^+(t) - e)\Big]\ ,$$

with $\rho > 0$ (ρ is arbitrary). If we take $t = \lambda_j$ and $\rho = \frac{m}{a_j}$ then we get

$$f_n(\lambda_j) = \mathrm{proj}_{\nu(\lambda_j)C}\Big[-\frac{m}{a_j}\, U_n^-(\lambda_j) + \frac{m}{a_j}\, e\Big]\ . \tag{4.4}$$

Let us consider the following system

$$(Q_{n0})\quad \begin{cases} m\,\dfrac{dV_{n0}}{dt} = f_{n0}, \\ f_{n0}(t) \in \partial\psi^*_{\nu(t)C}(-V_{n0}(t)+e)\ dt \ \text{ a.e.}, \\ V_{n0}(\lambda_0) = U_n^+(\lambda_0) = \dfrac{a_0\, f_n(t_0)}{m}\ , \end{cases}$$

where V_{n0} and f_{n0} are defined from $[\lambda_0,\lambda_1]$ into $\mathbb{R}$. This system has the same form as (Q_1), therefore it admits at least one solution (V_{n0}, f_{n0}) where V_{n0} is an absolutely continuous function and $f_{n0} \in L^\infty([\lambda_0,\lambda_1],\mathbb{R},dt)$. By replacing respectively into (4.3) and (4.4) $U_n^-(\lambda_1)$ by $V_{n0}(\lambda_1)$, then we get $f_n(\lambda_1)$ and $U_n^+(\lambda_1)$. In the same way as (Q_{n0}), the following system

$$(Q_{n1})\quad \begin{cases} m\,\dfrac{dV_{n1}}{dt} = f_{n1}, \\ f_{n1}(t) \in \partial\psi^*_{\nu(t)C}(-V_{n1}(t)+e)\ dt \ \text{ a.e.}, \\ V_{n1}(\lambda_1) = U_n^+(\lambda_1)\ , \end{cases}$$

admits also one solution (V_{n1}, f_{n1}), where V_{n1} is an absolutely continuous function and $f_{n1} \in L^\infty([\lambda_1,\lambda_2],\mathbb{R},dt)$. This ensures the existence of $U_n^+(\lambda_2)$ and $f_n(\lambda_2)$, as in (4.3) and (4.4). As I_n is a finite subset of D therefore the system (Q_{nj}) admits at least one solution (V_{nj}, f_{nj}) for all $j \in \{0,1,\ldots,p-1\}$ where $V_{nj} : [\lambda_j,\lambda_{j+1}] \to \mathbb{R}$ is an absolutely continuous function and $f_{nj} \in L^\infty([\lambda_j,\lambda_{j+1}],\mathbb{R},dt)$. If we take $U_n\colon I \to \mathbb{R}$ and $f_n\colon I \to \mathbb{R}$ defined by

$$\begin{cases} U_n(t_0) = 0, \\ U_n = V_{nj} \quad \text{on } \ [\lambda_j,\lambda_{j+1}[\setminus\{t_0\} \ \ (0 \le j \le p-1), \\ U_n(t_0+T) = V_{np-1}(t_0+T) + \dfrac{1}{m}\, a_p\, f_n(\lambda_p)\ , \end{cases}$$

and

$$\begin{cases} f_n = f_{nj} \quad \text{on } \ [\lambda_j,\lambda_{j+1}[\ \ (0 \le j \le p-1), \\ f_n(\lambda_p) = \operatorname{proj}_{\nu(\lambda_p)C}\left[-\dfrac{m}{a_p}\, V_{np-1}(t_0+T) + \dfrac{m}{a_p}\, e\right]\ , \end{cases}$$

we easily verify that (U_n, f_n) is a solution to the problem (Q_n).

Remark. (U_n) is constructed as a function of bounded variation which is right-continuous in $]t_0, t_0+T[$.

5 – Convergence of the problem (Q_n)

We recall that the second equation of (Q_n) is solved $d\theta_n$ a.e. We denote by E_n the $d\theta_n$-null subset of I in which the second equation of (Q_n) is not satisfied and we decompose it as $E_n = A_n \cup B_n$ where $A_n \subset I - D$ and $B_n \subset D$. Then we can extend f_n by taking

$$f_n(\alpha) = 0 \quad \text{ for all } \ \alpha \in B_n \ .$$

This enables us to write $f_n(t) \in \nu(t) C \, d\theta$ a.e. since A_n is a $d\theta$-null set. Moreover, since $\nu \colon \ I \to \mathbb{R}^n$ is measurable and bounded on I and $d\theta$ is a finite positive measure, then $f_n \in L^\infty(I, \mathbb{R}, d\theta)$ and

$$\|f_n\|_{L^\infty(I,\mathbb{R},d\theta)} \leq \gamma \, \|\nu\|_\infty \ .$$

As $L^1(I, \mathbb{R}, d\theta)$ is a separable space, then we may extract a subsequence from (f_n) which converges to $f \in L^\infty(I, \mathbb{R}, d\theta)$ with respect to the weak topology $\sigma(L^\infty, L^1, d\theta)$. We also notice that $f_n \, d\theta_n$ converges weakly to the measure $f \, d\theta$, in the following sense

$$\lim_{n \to +\infty} \int_I \varphi \, f_n \, d\theta_n = \int_I \varphi \, f \, d\theta$$

for all $\varphi \in C(I, \mathbb{R})$. In fact, we have

$$\Big| \int_I \varphi (f_n \, d\theta_n - f \, d\theta) \Big| \leq \|\varphi\|_\infty \, \|\nu\|_\infty \, \gamma \|\mu_n' - 1\|_{L^1(I,\mathbb{R},d\mu)} + \Big| \int_I \varphi (f_n - f) \, d\theta \Big|$$

with $f_n \to f \ \sigma(L^\infty, L^1, d\theta)$ and $\|\mu_n' - 1\|_{L^1(I,\mathbb{R},d\mu)} \to 0$ (by prop. 2.1) as $n \to +\infty$.

We define U as the function of bounded variation, right-continuous in $]t_0, t_0 + T[$ such that $m \, dU = f \, d\theta$ (i.e. its Stieltjes measure is $\frac{f}{m} \, d\theta$) with $U(t_0) = 0$. Now it remains to verify that the second equation of (Q) holds.

Proposition 5.1. *For all $\varphi \in \nu \, C$ where*

$$\nu \, C = \Big\{ \varphi \in L^\infty(I, \mathbb{R}, d\theta) \mid \varphi(t) \in \nu(t) \, C \ \ d\theta \ \text{ a.e.} \Big\}$$

we have

$$\int_I \Big(U(t) - e(t) \Big) \Big(\varphi(t) - f(t) \Big) \, d\theta \geq 0 \ . \tag{5.1}$$

Remark. Having shown that (5.1) holds for every $\varphi \in \nu \, C$, we conclude that

$$f \in \partial \psi^*_{\nu C}(-U + e) \ ,$$

with respect to the usual duality. If we recall proposition 3.2 we may now write that

$$f(t) \in \partial\psi^*_{\nu(t)C}\Big(-U(t)+e\Big)\; d\theta \quad \text{a.e.}\ ,$$

thus ending the proof that U solves problem (Q) (with $P=0$ and e constant).

Proof: We know that f_n, U_n and $d\theta_n$ are uniformly bounded:

$$\|d\theta_n\| \le \|d\theta\|\,, \qquad \|f_n\|_{L^\infty(I,\mathbb{R},d\theta)} \le \gamma\, \|\nu\|_\infty$$

and

$$\|U_n\|_\infty \le \frac{\gamma}{m}\, \|\nu\|_\infty\, \|d\theta\|\ .$$

Thus, there exist M and $M' \ge 0$ such that for all $\varphi \in \nu\, C$ we get

$$\begin{aligned}\bigg|\int_I \Big(U_n(t)-e\Big)\, \Big(\varphi(t)-f_n(t)\Big)\, d\theta_n - \int_I \Big(U(t)-e\Big)\, \Big(\varphi(t)-f(t)\Big)\, d\theta\bigg| &\le \\ \le M\|\mu'_n-1\|_{L^1(I,\mathbb{R},d\mu)} + M'\|U_n-U\|_{L^1(I,\mathbb{R},d\theta)}& \\ + \bigg|\int_I \Big(U(t)-e\Big)\, \Big(f_n(t)-f(t)\Big)\, d\theta\bigg|\ .& \end{aligned} \tag{5.2}$$

First we show that

$$\lim_{n\to+\infty} \|U_n-U\|_{L^1(I,\mathbb{R},d\theta)} = 0\ . \tag{5.3}$$

By integrating the measure $dU_n - dU = \frac{f_n}{m}\, d\theta_n - \frac{f}{m}\, d\theta$ on $[t_0,t]$ we obtain

$$U_n(t)-U(t) = \frac{1}{m}\int_I \kappa_{[t_0,t]}\, f_n(\mu'_n-1)\, d\mu + \frac{1}{m}\int_I \kappa_{[t_0,t]}\, (f_n-f)\, d\theta\ ,$$

where $\kappa_{[t_0,t]}(\cdot)$ is the characteristic function of $[t_0,t]$. Hence

$$\Big|U_n(t)-U(t)\Big| \le \frac{A}{m}\, \|\mu'_n-1\|_{L^1(I,\mathbb{R},d\mu)} + \frac{M}{m}\bigg|\int_I \kappa_{[t_0,t]}\, (f_n-f)\, d\theta\bigg|\ ,$$

where $A = \|\nu\|_\infty\, \gamma$. As $|U_n(t)| \le \frac{1}{m}\, \gamma\|\nu\|_\infty \int_I d\theta$ then $U_n \in L^1(I,\mathbb{R},d\theta)$. By using proposition 2.1 and the weak convergence of f_n to f we obtain that U_n converges pointwise to U on I. By the dominated convergence theorem we deduce that (5.3) holds. Using (5.3), proposition 2.1 and the fact that

$$\lim_{n\to\infty} \int_I \Big(U(t)-e\Big)\, \Big(f_n(t)-f(t)\Big)\, d\theta = 0$$

(since $t \to U(t)-e$ is in $L^1(I,\mathbb{R},d\theta)$), we now obtain from (5.2) that

$$\lim_{n\to\infty} \int_I \Big(U_n(t)-e\Big)\, \Big(\varphi(t)-f_n(t)\Big)\, d\theta_n = \int_I \Big(U(t)-e\Big)\, \Big(\varphi(t)-f(t)\Big)\, d\theta \ge 0\ .$$

In fact, the integrals depending on n are all nonnegative, since U_n solves (Q_n). ∎

6 – The general case

In this section we consider that, in addition, the particle is submitted to an external force which we denote by P. We assume that P is a function of time and tangential to the line of motion with $P \in L^\infty(I, \mathbb{R}, d\theta)$. We also allow the transport velocity e to vary and assume that e is lipschitzian.

If we take $W = -U + e$, then the restriction of system (Q) to a subinterval $[a,b]$ of I, where the motion of the particle is assumed smooth, is given by

$$(Q') \quad \begin{cases} -\dfrac{dW}{dt} \in \partial\psi^*_{\frac{1}{m}\nu(t)C - \dot{e}(t) + \frac{P(t)}{m}}(W(t))\ dt \quad \text{a.e.}, \\ W(a) = -U(a) + e(a)\ . \end{cases}$$

Replacing (3.1) by

$$F(t,x) = \psi^*_{\frac{1}{m}\nu(t)C - \dot{e}(t) + \frac{P(t)}{m}}(x)\ ,$$

it is easily seen, by applying the same technique used for solving the system (Q_1), that the system (Q') admits at least one lipschitzian solution. By using this result and the same procedure applied for solving (Q), we could show that system (Q) admits at least one solution (U, f) where $f \in L^\infty([a,b], \mathbb{R}, d\theta)$ and $U\colon I \to \mathbb{R}$ is a function of bounded variation satisfying the initial condition $U(t_0) = 0$, which is right-continuous in $]t_0, t_0 + T[$.

ACKNOWLEDGEMENT – This work was done while both authors stayed at the University of Montpellier II and they are indebted to Professors J.J. Moreau and M. Jean for many helpful discussions.

The second author gratefully acknowledges the support of Fundação C. Gulbenkian (Lisboa).

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M. Laghdir,
Faculté des Sciences de Rabat – MAROC

and

Manuel D.P. Monteiro Marques,
Faculdade de Ciências e C.M.A.F., Universidade de Lisboa,
Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex – PORTUGAL