PORTUGALIAE MATHEMATICA
Vol. 52 Fasc. 3 – 1995

NONEXISTENCE OF GLOBAL SOLUTIONS TO SOME DIFFERENTIAL INEQUALITIES OF THE SECOND ORDER AND APPLICATIONS

Philippe Souplet

Abstract: We establish the nonexistence of global nondecreasing positive solutions for a class of differential inequalities of the type $u'' + u'^p \geq u^q$. As a consequence, we prove the non global character of the solutions for nonlinear wave equations of the type $u_{tt} - \Delta u + u_t = \lambda u + |u|^q$, when the initial data have positive projections on the first eigenvector and a similar result for a heat equation with memory.

Résumé: Nous montrons qu'il n'existe pas de solutions globales positives et croissantes pour une classe d'inéquations différentielles du type $u'' + u'^p \geq u^q$. A titre d'application, nous prouvons le caractère non global des solutions pour des équations d'ondes non-linéaires du type $u_{tt} - \Delta u + u_t = \lambda u + |u|^q$, dès que les données initiales ont des projections positives sur la première fonction propre du Laplacien. Nous obtenons un résultat similaire pour une équation de la chaleur avec un terme de mémoire.

0 – Introduction

The existence of blowing-up solutions for hyperbolic nonlinear evolution equations:

$$u_{tt} - \Delta u = f(u) \tag{0.1}$$

or parabolic ones:

$$u_t - \Delta u = f(u) \tag{0.2}$$

has been extensively studied for more than thirty years.

Received: December 14, 1993.

The methods used were mainly of two types:

– the energy method, leading to the blow-up of solutions with large enough negative initial energy (cf. [1], [7], [8], [10]);

– the convexity method, where blow-up occurs for positive initial data (cf. [2], [3], [5], [6]).

Recently, the interaction between nonlinear source and damping terms in the problem

$$u_{tt} - \Delta u + u_t\Big(1 + |u_t|^{p-1}\Big) = \|u|^{q-1}\, u \ , \tag{0.3}$$

has been investigated ([4]). The authors, using the energy method, established the global existence of all solutions if $1 < q \leq p$ and the blow-up of "large" solutions if $1 < p < q$.

In the present paper, we shall prove the non global character of the solutions for nonlinear wave equations of the type:

$$u_{tt} - \Delta u + u_t = \lambda u \ + |u|^q \ , \tag{0.4}$$

when the initial data have positive projections on the first eigenvector.

In the first section, we provide a detailed study of a class of differential inequalities of the form

$$u'' + f(u') \geq g(u) \tag{0.5}$$

(typically:

$$u'' + u'^p \geq u^q \ , \tag{0.6}$$

with $q > p \geq 1$), for which we establish the non global character of the solutions when $u(0) > 0$ and $u'(0) \geq 0$. We here improve the results that we presented in [9], by allowing to handle generalized solutions of the inequality. In the second section, by combining the previous property and the convexity method, we deduce the result concerning the nonlinear wave equation and give a similar property for a heat equation with memory of the type:

$$u_t - \Delta u + a\, u = \int_0^t |u|^q(s)\, ds \ . \tag{0.7}$$

1 – The differential inequalities of the type $u'' + u'^p \geq u^q$

We wish to study the global nonexistence of nondecreasing positive solutions for the differential inequalities of the type:

$$u'' + u'^p \geq u^q \ . \tag{1.1}$$

In order to give results in a framework as general as possible, we shall investigate generalized solutions to some inequalities of the form:

$$u'' + f(u') \geq g(u) \ , \tag{1.2}$$

in the sense that u' is absolutely continuous and (1.2) is satisfied almost everywhere.

Let p, q be real numbers satisfying the condition:

$$1 \leq p < q \ . \tag{1.3}$$

Let $f, g \colon \mathbf{R} \to \mathbf{R}$, be such that:

$$\exists\, a, a' > 0, \quad \forall\, y \geq 0, \quad f(y) \leq a\, y^p + a' \ , \tag{1.4}$$

$$f(0) \leq 0 \quad \text{and} \quad \limsup_{s \to 0} f(s) \leq 0 \ , \tag{1.5}$$

$$\exists\, b, b' > 0, \quad \forall\, y > 0, \quad g(y) \geq b\, y^q - b' \ , \tag{1.6}$$

$$\forall\, y > 0, \quad g(y) > 0 \ \text{ and } \ \liminf_{s \to y} g(s) > 0 \ . \tag{1.7}$$

The main result of this section is:

Theorem 1.1. *Let $u \in W^{2,1}_{\rm loc}([0,T); \mathbf{R})$, with $0 < T \leq +\infty$, and suppose that (1.2) holds for a.e. $t \in (0,T)$. If*

$$u(0) > 0 \quad \textit{and} \quad u'(0) \geq 0 \ , \tag{1.8}$$

then necessarily $T < +\infty$ and besides we have:

$$u > 0 \ \textit{ and } \ u' > 0 \quad \textit{on} \ \ (0,T) \ . \tag{1.9}$$

Furthermore, if we assume f and g to be continuous on $\mathbf{R}^{+*}$, and denoting by T^* the maximal existence time, the solution blows-up in the sense that:

$$\|u\|_{W^{2,1}(0,T^*;\mathbf{R})} = +\infty \ . \tag{1.10}$$

Proof:

• First suppose that $u'(0) = 0$. Then by the continuity of u and u', and thanks to (1.5)(1.7), we have $g(u) \geq \delta$ and $f(u') \leq \delta/2$ on $(0,\varepsilon)$ for some ε, $\delta > 0$. Therefore, by (1.2), we get $u'' \geq \delta/2$ a.e. on $(0,\varepsilon)$ and $u' > 0$ on $(0,\varepsilon)$.

• Suppose that u' vanishes somewhere on $(0,T)$ and set:

$$t_0 = \min\{t > 0,\ u'(t) \leq 0\} > 0\ .$$

We have $u' > 0$ on $(0,t_0)$ so that $u(t_0) > 0$ and, by (1.7) and the continuity of u, $g(u) \geq \delta > 0$ on $(t_0-\varepsilon,t_0)$ for some $\delta,\varepsilon > 0$. On the other hand, using (1.5), the continuity of u' and $u'(t_0) = 0$, we get $f(u') \leq \delta/2$ on $(t_0-\varepsilon,t_0)$ for some possibly smaller ε. By (1.2), we then have

$$u''(t) \geq \delta/2, \qquad \text{a.e. on} \quad (t_0-\varepsilon,t_0)\ ,$$

and $u'(t_0-\varepsilon) < 0$, which contradicts the definition of t_0. Thus (1.9) is proved.

From now on, we suppose that $T = +\infty$.

• We first prove that:

$$\lim_{t\to+\infty} u(t) = +\infty\ . \tag{1.11}$$

Suppose the contrary; by (1.9) we have

$$\forall\, t \geq 0, \quad 0 < u(t) < \ell := \lim_{t\to+\infty} u(t) < +\infty\ .$$

From (1.2)(1.7), we deduce the existence of some $T,\delta > 0$ such that:

$$u''(t) + f(u'(t)) \geq \delta, \qquad \text{for a.e.} \quad t > T\ .$$

On the other hand, (1.5) implies:

$$\exists\, \varepsilon > 0, \quad \forall\, y \leq \varepsilon, \quad f(y) \leq \delta/2\ .$$

Therefore, there exists some $t_1 > T$, such that $u'(t_1) > \varepsilon$ (otherwise we would have $u'' \geq \delta/2$ for a.e. $t > T$), and since u is bounded and u' is continuous, we have:

$$\exists\, t_2 = \min\,\{t > t_1 \mid u'(t) \leq \varepsilon/2\} > t_1$$

and also

$$\exists\, t_3 \in (t_1,t_2), \quad \forall\, t \in (t_3,t_2), \quad u'(t) \leq \varepsilon\ .$$

But then $u'' \geq \delta/2$ a.e. on (t_3, t_2), hence $u' < u'(t_3) = \varepsilon/2$ on (t_3, t_2), which contradicts the definition of t_2.

- **Case** $p < 2q/(q+1)$.

Let us set:

$$\phi = \frac{1}{2} u^2 \ . \tag{1.12}$$

From (1.2)(1.9), we get, for a.e. $t > 0$:

$$\phi'' = u''u + u'^2 \geq g(u)\, u + u'^2 - f(u')\, u \ .$$

Using (1.4) and Young's inequality with exponents $2/p$ and $2/(2-p)$ yields:

$$f(u')\, u \leq a'u + u'^2 + C\, u^{2/(2-p)}$$

for some constant $C > 0$, and thus:

$$\phi'' \geq b\, u^{q+1} - b'u - a'u - C\, u^{2/(2-p)} \ .$$

Now, from the fact that $q + 1 > 2/(2-p)$ and thanks to (1.11), we can deduce the existence of some $\varepsilon > 0$ such that

$$\phi'' \geq \varepsilon\, u^{q+1} \geq \varepsilon\, \phi^{(q+1)/2} > 0, \quad \text{a.e. for } t \text{ large enough} \ . \tag{1.13}$$

Finally, (1.13) together with $\phi' = u\, u' > 0$ and $q > 1$ clearly imply the finite-time blow-up of ϕ and a contradiction.

- **Case** $p \geq 2q/(q+1)$.

By rescaling and using (1.11), we can reduce to the inequality:

$$u'' + u'^p \geq C\, u^q \ , \tag{1.14}$$

with $C > 0$. Setting $\alpha = q/p > 1$ and

$$\Psi(t) = \frac{u'}{u^\alpha} \ , \tag{1.15}$$

which belongs to $W^{1,1}_{\mathrm{loc}}(0, T; \mathbf{R})$, we note that $T = +\infty$ implies:

$$\liminf_{t \to +\infty} \Psi(t) = 0 \quad \text{and} \quad \limsup_{t \to +\infty} \Psi(t) > 0 \ . \tag{1.16}$$

(Otherwise, by (1.14), for some $\varepsilon > 0$, we would have either $u' \geq \varepsilon\, u^\alpha$, for t large enough, or $u'' \geq \varepsilon\, u^q$ a.e.). Therefore, if we fix a representant of Ψ', there exists some sequence $(t_n) \to +\infty$ such that:

$$\lim_{n \to \infty} \Psi(t_n) = 0 \tag{1.17}$$

and:

$$\forall\, n \in \mathbf{N}, \quad \Psi'(t_n) \leq 0 \ . \tag{1.18}$$

Indeed, if no such sequence did exist, there should be some $T, \varepsilon > 0$ such that:

$$\forall\, t \geq T, \quad |\Psi(t)| \leq \varepsilon \ \Rightarrow\ \Psi'(t) > 0 \ ,$$

which immediately contradicts (1.16). Condition (1.18) rewrites as $u''(t_n) \leq \alpha(u'^2/u)(t_n)$ and, by (1.14)(1.17), we have, for n large enough:

$$\frac{u'^2}{u^{q+1}}(t_n) \geq \frac{u''}{\alpha\, u^q}(t_n) \geq \frac{C}{2\alpha} > 0 \ . \tag{1.19}$$

Combining (1.17) and (1.19) yields:

$$u^{p(q+1)/2-q}(t_n) = \left[\frac{u'^p}{u^q}\ \frac{u^{p(q+1)/2}}{u'^p}\right](t_n) \ \to\ 0, \quad \text{as}\ \ n \to \infty \ ,$$

and since $\lim_{t\to+\infty} u(t) = +\infty$, we deduce $p(q+1) < 2q$: contradiction.

- We now just have to prove (1.10). Let us suppose that:

$$\int_0^{T^*} |u''(t)|\, dt < +\infty \ .$$

Therefore u and u' bear some limits, denoted by $u(T^*)$ and $u'(T^*)$, as t tends to T^{*-}. But then, since f and g are here assumed to be continuous, the solution of the inequality could be extended to the right by a *local* solution of the ODE:

$$\begin{cases} v'' = g(v) - f(v'), & t \geq T^* \ , \\ v(T^*) = u(T^*)\, , & v'(T^*) = u'(T^*) \ . \end{cases}$$

The proof is thus complete. ∎

Remark 1.1. If we require *classical* solutions of (1.2), i.e. such that u is twice differentiable and such that the inequality holds everywhere on $[0, T)$, it can occur that the solution ceases to exist without blowing-up, because of the coming out of a singularity. This stands in contrast with what happens for ordinary differential equations.

Let us for instance consider the following example:

Proposition 1.2. *Let p, q satisfy (1.3) and take $f(u') = \| u'|^p$ and $g(u) = \| u|^q$. Then there exist some $T > 0$ and some classical solution u of (1.2)(1.8)* *that cannot be extended to* $[0,T]$ *and such that* u, u' *and* u'' *remain bounded on* $[0,T)$. *More precisely, one such function is given by:*

$$u(t) = 3\,t^2 + 3\,T\,t + 2\,T^2 + (T-t)^2 \,\sin(\ln(T-t))\,, \qquad 0 \le t < T\ , \tag{1.20}$$

for any $T \le (1/3)2^{\frac{1}{2q}}$.

Proof: Computing:

$$u'(t) = 6\,t + 3\,T - 2(T-t)\,\sin(\ln(T-t)) - (T-t)\,\cos(\ln(T-t))$$

and

$$u''(t) = 6 + \sin(\ln(T-t)) + 3\cos(\ln(T-t))\ ,$$

one easily obtains the following bounds:

$$0 < T^2 \le u \le 9\,T^2\ ,$$
$$0 \le u' \le 12\,T\ ,$$
$$2 \le u'' \le 10\ .$$

Then we get:

$$u'' + u'^p \ge 2 \ge (3T)^{2q} \ge u^q \quad \text{on} \;\; [0,T)\ ,$$

so that u is a solution of (1.2)(1.8).

Last, u and u' can be continued until the point T but u' is not differentiable in T^-, so that the solution of (1.2)(1.8) is not extendable in the classical sense. ∎

Remark 1.2. The assumption $u'(0) \ge 0$ in Theorem 1.1 is essential. To see that, it suffices to take f and g as in Proposition 1.2 and to consider the function $u(t) = e^{-2t}$ which is a solution of (1.2) with $T = +\infty$ for any $p, q \ge 1$.

Remark 1.3. When $p = q \ge 1$, the function $u(t) = e^t$ is a global solution to (1.2)(1.8) (with f and g as in Proposition 1.2). This counter-example shows that the condition $1 \le p < q$ is optimal.

2 – Application: existence of nonglobal solutions for a nonlinear wave equation with damping and source terms and a heat equation with memory

The results of the previous section will allow us to provide conditions on the initial data ensuring global nonexistence of the solution for wave equations of the type:

$$(2.1) \qquad \begin{cases} u_{tt} - \Delta u + u_t = \lambda u + |u|^q & \text{in } \Omega \ , \\ u = 0 \quad \text{on } \partial\Omega \ . \end{cases}$$

As in section 1, we will consider more general terms $f(u_t)$ and $g(u)$.

Let Ω be a bounded open set of $\mathbf{R}^n$, λ_1 the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$, and Φ_1 the corresponding eigenfunction such that

$$\int_\Omega \Phi_1(x)\, dx = 1 \ .$$

Let $f, g, \tilde{f}, \tilde{g}$: $\mathbf{R} \to \mathbf{R}$, satisfying the following conditions:

$$(2.2) \qquad \forall\, y \in \mathbf{R}, \quad f(y) \le \tilde{f}(y) \ ,$$

$$(2.3) \qquad \tilde{f} \text{ is concave} \quad \text{and} \quad \tilde{f}(0) \le 0 \ ,$$

$$(2.4) \qquad \exists\, a, a' > 0, \quad \forall\, y \ge 0, \quad f(y) \le a\, y + a' \ ,$$

$$(2.5) \qquad \forall\, y \in \mathbf{R}, \quad g(y) \ge \tilde{g}(y) + \lambda_1\, y \ ,$$

$$(2.6) \qquad \tilde{g} \text{ is convex} \quad \text{and} \quad \forall\, y > 0, \quad \tilde{g}(y) > 0 \ ,$$

$$(2.7) \qquad \exists\, b, b' > 0, \quad \exists\, q > 1, \quad \forall\, y \ge 0, \quad \tilde{g}(y) \ge b\, y^q - b' \ .$$

We consider the problem:

$$(2.8) \quad u_{tt} - \Delta u + f(u_t) = g(u) \ ,$$

$$(2.9) \quad u \in L^1_{\text{loc}}([0,T); H_0^1(\Omega)) \cap W^{1,1}_{\text{loc}}([0,T); L^2(\Omega)) \cap W^{2,1}_{\text{loc}}([0,T); H^{-1}(\Omega)) \ ,$$

$$(2.10) \quad f(u_t),\, g(u) \in L^1_{\text{loc}}([0,T); H^{-1}(\Omega)) \ .$$

We then have the following result:

Theorem 2.1. *Let u be a solution of (2.8)–(2.10). As soon as*

$$(2.11) \qquad \int_\Omega u(0,x)\, \Phi_1(x)\, dx > 0 \quad \text{and} \quad \int_\Omega u_t(0,x)\, \Phi_1(x)\, dx \ge 0 \ ,$$

then u is nonglobal, i.e. $T < +\infty$.

Proof: We set:

$$w(t) = \int_\Omega u(t,x)\, \Phi_1(x)\, dx \ ,$$

which belongs to $W^{2,1}_{\rm loc}([0,T);\mathbf{R})$. Taking $\Phi_1 \in H^1_0(\Omega)\cap C^\infty(\Omega)$ as a test-function in (2.8) and using Green's formula, it comes, a.e. on $[0,T)$:

$$w'' + \int_\Omega f(u_t(t,x))\,\Phi_1(x)\,dx = \int_\Omega \Big[g(u(t,x)) - \lambda_1\,u(t,x)\Big]\,\Phi_1(x)\,dx\ .$$

Using (2.2)(2.5) and the fact that Φ_1 is nonnegative, we have:

$$w'' + \int_\Omega \tilde f(u_t(t,x))\,\Phi_1(x)\,dx \ge \int_\Omega \tilde g(u(t,x))\,\Phi_1(x)\,dx\ .$$

By Jensen's inequality, since $\tilde f$ is concave and $\tilde g$ convex, we get:

$$w'' + \tilde f(w') \ge \tilde g(w)\ .$$

Thanks to hypotheses (2.3)(2.4)(2.6)(2.7), we can then apply Theorem 1.1 that implies $T < +\infty$. ∎

Remark 2.1. The hypotheses on f and g in Theorem 2.1 were deliberately given as general as possible for the convexity method to operate. As a counterpart, we had to impose *a priori* a sufficient regularity class (2.9), (2.10) for the solutions so that the calculations be justified.

When the growth of f or g is too fast, it might happen that no local existence-uniqueness result is available. In the usual cases when such a result is known, for instance (2.1) with $q > 1$ and $(n-2)q \le n$, the result of Theorem 2.1 naturally becomes — via the continuation principle — a blow-up result in the norm of the local existence space.

Remark 2.2. In the case we deal with a solution u of (2.8)–(2.10) that would remain nonnegative a.e. in Ω as long as it exists, the result of Theorem 2.1 obviously holds without any restriction on $g(y)$ for $y > 0$, and applies in particular when $g(u) = \|u|^{q-1}\,u$.

By the method of Theorem 2.1, we can obtain a similar result for the following heat equation with memory:

$$\begin{cases} u_t - \Delta u + a\,u = \displaystyle\int_0^t |u|^q(s)\,ds & \text{in } \Omega\ , \\ u \ne 0 \quad \text{on } \partial\Omega\ . \end{cases} \tag{2.12}$$

Theorem 2.2. *Take $a\in\mathbf{R}$, $q>1$ and u a solution of (2.12) in $C(0,T;L^q(\Omega))\cap C^2(0,T;H^{-1}(\Omega))$. As soon as*

$$\int_\Omega u(0,x)\,\Phi_1(x)\,dx = 0 \tag{2.13}$$

and

$$u(0,\cdot) > 0, \quad \textit{on a subset of } \Omega \textit{ of positive measure}\ , \tag{2.14}$$

then u is nonglobal, i.e. $T < +\infty$.

Proof: Since u is in the regularity class $C(0,T;L^q(\Omega)) \cap C^2(0,T;H^{-1}(\Omega))$, we can differentiate the equation, which becomes:

$$u_{tt} - \Delta u_t + a u_t = \| u|^q\ . \tag{2.15}$$

Resuming the notations of the proof of Theorem 2.1, we have, by (2.12):

$$\begin{aligned} w'(0) &= \int_\Omega \{\Delta u - a\,u\}(0,x)\,\Phi_1(x)\,dx \\ &= -(\lambda_1 + a) \int_\Omega u(0,x)\,\Phi_1(x)\,dx = 0 \end{aligned}$$

and, by (2.14)(2.15):

$$\begin{aligned} w''(0) &= \int_\Omega \{\Delta u_t - a\,u_t + |u|^q\}(0,x)\,\Phi_1(x)\,dx \\ &> -(\lambda_1 + a) \int_\Omega u_t(0,x)\,\Phi_1(x)\,dx = 0\ , \end{aligned}$$

so that $w(\varepsilon) > 0$ and $w'(\varepsilon) > 0$ for $\varepsilon > 0$ small enough. Using the same technique as in Theorem 2.1, we are finally led to the inequality

$$w'' + (\lambda_1 + a)\,w' \geq w^q$$

and can conclude as above. ∎

Remark 2.3. If we impose the additional condition $a \leq -\lambda_1$, the result of Theorem 2.1 still holds when replacing (2.13)–(2.14) by $\int_\Omega u(0,x)\,\Phi_1(x)\,dx > 0$.

REFERENCES

[1] BALL, J.M. – Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford,* 28(2) (1977), 473–486.

[2] FUJITA, H. – On the blowing-up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo,* Sect. I, 13 (1966), 109–124.

[3] FUJITA, H. – On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations, *Proc. Symp. Pure Math.,* XVIII, Nonlinear Functional Analysis, *Amer. Math. Soc.*, 28 (1970), 105–113.

[4] GEORGIEV, V. and TODOROVA, G. – Existence of solution of the wave equation with nonlinear damping and source terms, *C.R. Acad. Sci. Paris,* Série I, 314 (1992), 205–209.

[5] GLASSEY, R.T. – Blow-up theorems for nonlinear wave equations, *Math. Z.,* 132 (1973), 183–203.

[6] KAPLAN, S. – On the growth of solutions of quasilinear parabolic equations, *Comm. Pure. Appl. Math.,* 16 (1963), 327–343.

[7] LEVINE, H. – Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = Au + F(u)$, *Trans. Amer. Math. Soc.,* 192 (1974), 1–21.

[8] LEVINE, H. – Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *S.I.A.M. J. Math. Anal.,* 5 (1974), 541–553.

[9] SOUPLET, PH. – *Propriétés globales de quelques équations d'évolution du second ordre non linéaires*, Part 3, Thèse, Université Pierre et Marie Curie, Paris, 1993, 116–127.

[10] TSUTSUMI, M. – On solutions of semilinear differential equations, *Math. Japonicae,* 17 (1972), 173–193.

Philippe Souplet,
Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie,
4, place Jussieu, 75252 Paris Cedex 05 – FRANCE