A PROBLEM OF DIOPHANTOS–FERMAT AND CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

Gheorghe Udrea

Abstract: One shows that, if \((U_n)_{n \geq 0}\) is the sequence of Chebyshev polynomials of the second kind, then the product of any two distinct elements of the set
\[
\left\{ U_m, U_{m+2r}, U_{m+4r}; 4 \cdot U_{m+r}, U_{m+3r} \right\}, \quad m, r \in \mathbb{N},
\]
increased by \(U_a^2 \cdot U_b^2\), for suitable positive integers \(a\) and \(b\), is a perfect square.
This generalizes a result obtained by José Morgado in [4].

1 – Introduction

Chebyshev polynomials \((U_n)_{n \geq 0}\) are defined by the recurrence relation
\[
U_{n+1}(x) = 2 \cdot x \cdot U_n(x) - U_{n-1}(x), \quad (\forall) \; x \in \mathbb{C}, \quad (\forall) \; n \in \mathbb{N}^+, \tag{1.1}
\]
where \(U_0(x) = 1\) and \(U_1(x) = 2x\).

An important property of these polynomials is given by the formula
\[
U_{k-1}(\cos \varphi) = \frac{\sin k \varphi}{\sin \varphi}, \quad (\forall) \; \varphi \in \mathbb{C}, \quad \sin \varphi \neq 0, \quad (\forall) \; k \in \mathbb{N}^+. \tag{1.2}
\]

Also one has the relations
\[
U_k \left( \frac{i}{2} \right) = i^k \cdot F_{k+1}, \quad (\forall) \; k \in \mathbb{N}, \tag{1.3}
\]
where \(i^2 = -1\) and \((F_n)_{n \geq 0}\) is the sequence of so-called Fibonacci numbers:
\[
F_{n+1} = F_n + F_{n-1}, \quad (\forall) n \in \mathbb{N}^*, \quad F_0 = 0, \quad F_1 = 1. \tag{1.4}
\]

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We are going to prove the following

**Theorem.** If \((U_n)_{n \geq 0}\) is the sequence of Chebyshev polynomials of the second kind, then the product of any two distinct elements of the set

\[
\{ U_m, U_{m+2r}, U_{m+4r}; 4 \cdot U_{m+r} \cdot U_{m+2r} \cdot U_{m+3r} \}, \quad m, r \in \mathbb{N} ,
\]

increased by \(U_a^2 \cdot U_b^2\) for suitable positive integers \(a\) and \(b\), is a perfect square, \((\forall) m, r \in \mathbb{N}, (\forall) x \in \mathbb{C} \).

**Proof:** One has the identity

\[
U_m(x) \cdot U_{m+r+s}(x) + U_{r-1}(x) \cdot U_{s-1}(x) = U_{m+r} \cdot U_{m+s} , \quad (\forall) x \in \mathbb{C}, \ (\forall) m, r, s \in \mathbb{N}^* .
\]

Indeed, let \(x\) be an element of \(\mathbb{C}\); then \((\exists) \varphi \in \mathbb{C}\) such that \(x = \cos \varphi\). One has

\[
U_m(x) \cdot U_{m+r+s}(x) + U_{r-1}(x) \cdot U_{s-1}(x) =
\]

\[
= U_m(\cos \varphi) \cdot U_{m+r+s}(\cos \varphi) + U_{r-1}(\cos \varphi) \cdot U_{s-1}(\cos \varphi)
\]

\[
= \frac{\sin(m + 1) \varphi}{\sin \varphi} \cdot \frac{\sin(m + r + s + 1)}{\sin \varphi} + \frac{\sin r \varphi}{\sin \varphi} \cdot \frac{\sin s \varphi}{\sin \varphi}
\]

\[
= \frac{\cos(r + s) \varphi - \cos(2m + r + s + 2) \varphi}{2 \cdot \sin^2 \varphi} + \left[ \cos(r - s) \varphi - \cos(r + s) \varphi \right]
\]

\[
= \frac{\cos(r - s) \varphi - \cos(2m + r + s + 2) \varphi}{2 \cdot \sin^2 \varphi} = \frac{\sin(m + r + 1) \varphi}{\sin \varphi} \cdot \frac{\sin(m + s + 1) \varphi}{\sin \varphi}
\]

\[
= U_{m+r}(\cos \varphi) \cdot U_{m+s}(\cos \varphi) = U_{m+r}(x) \cdot U_{m+s}(x) , \text{ q.e.d.}
\]

By setting \(s = r\) in (2.1), one obtains

\[
U_m \cdot U_{m+2r} + U_{r-1}^2 = U_{m+r}^2 , \quad m, r \in \mathbb{N}^* ,
\]

which proves a part of the Theorem above, with \(a = r - 1, b = 0\).

Now, let us replace, in (2.2), \(m\) by \(m + 2r\); then

\[
U_{m+2r} \cdot U_{m+4r} + U_{r-1}^2 = U_{m+3r}^2 ,
\]

which proves also a part of the Theorem above, with \(a = r - 1, b = 0\).
By replacing, in (2.2), \( r \) by \( 2r \), it results

\[
U_m \cdot U_{m+4r} + U_{2r-1}^2 = U_{m+2r}^2 \quad (a = 2r - 1, \ b = 0).
\]

From the identity (2.1), it follows

\[
U_{r-1}^2 \cdot U_{s-1}^2 = (U_{m+r} \cdot U_{m+s} - U_m \cdot U_{m+r+s})^2
\]

and so

\[
4 \cdot U_m \cdot U_{m+r} \cdot U_{m+s} \cdot U_{m+r+s} + U_{r-1}^2 \cdot U_{s-1}^2 = (U_{m+r} \cdot U_{m+s} + U_m \cdot U_{m+r+s})^2, \quad m, r, s \in \mathbb{N}^+.
\]

If, in (2.5), one sets \( s = 2r \), one obtains

\[
4 \cdot U_m \cdot U_{m+r} \cdot U_{m+2r} \cdot U_{m+3r} + U_{r-1}^2 \cdot U_{s-1}^2 = (U_{m+r} \cdot U_{m+2r} + U_m \cdot U_{m+3r})^2 \quad (a = r - 1, \ b = 2r - 1).
\]

By replacing \( m \) by \( m + r \), in (2.6), it follows

\[
4 \cdot U_{m+r} \cdot U_{m+2r} \cdot U_{m+3r} \cdot U_{m+4r} + U_{r-1}^2 \cdot U_{s-1}^2 = (U_{m+2r} \cdot U_{m+3r} + U_{m+r} \cdot U_{m+4r})^2.
\]

Finally, from (2.5), it results, for \( s = r \) and replacing \( m \) by \( m + r \),

\[
4 \cdot U_{m+r} \cdot U_{m+2r} \cdot U_{m+3r} + U_{r-1}^4 = (U_{m+2r}^2 + U_{m+r} \cdot U_{m+3r})^2 \quad (a = b = r - 1).
\]

The relations (2.2)–(2.8) show that the Theorem holds.

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According to (1.3), from the relations (2.2)–(2.8) one obtains the following identities for Fibonacci numbers:

\[
\begin{align*}
(3.1) & \quad F_m \cdot F_{m+2r} + (-1)^m \cdot F_r^2 = F_{m+r}^2, \\
(3.2) & \quad F_{m+2r} \cdot F_{m+4r} + (-1)^m \cdot F_r^2 = F_{m+3r}^2, \\
(3.3) & \quad F_m \cdot F_{m+4r} + (-1)^m \cdot F_{2r}^2 = F_{m+2r}^2, \\
(3.4) & \quad 4 \cdot F_m \cdot F_{m+r} \cdot F_{m+2r} \cdot F_{m+3r} + F_r^2 \cdot F_{2r}^2 = (F_{m+r} \cdot F_{m+2r} + F_m \cdot F_{m+3r})^2.
\end{align*}
\]
These identities show that the product of any two distinct elements of the set
\[
\left\{ F_m, F_{m+2r}, F_{m+4r}; 4 \cdot F_{m+r} \cdot F_{m+2r} \cdot F_{m+3r} \right\}, \quad m, r \in \mathbb{N}
\]
increased by \( F_a^2 \cdot F_b^2 \) or \(-F_a^2 \cdot F_b^2\), where \( F_a \) and \( F_b \) are suitable elements of the Fibonacci sequence \( (F_n)_{n \geq 0} \), is a perfect square.

This remark is in fact the José Morgado’s result given in [4].

Moreover, the José Morgado’s result is also a generalization of some results of V.E. Hoggatt and E.G. Bergum, given in [1] and [2], about a problem of Diophantos–Fermat.

This problem claims to find four natural numbers such that the product of any two, added by unity, is a square.

REFERENCES


