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CENTRAL MORPHISMS AND ENVELOPES OF HOLOMORPHY

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Abstract: In this paper we study a particular class of continuous algebra morphisms the so-called $\mathbf{C}$-central $\mathbf{A}$-morphisms; i.e. continuous $\mathbf{A}$-morphisms between topological $\mathbf{A}$-algebras (viz. we take coefficients from a topological algebra $\mathbf{A}$) such that their images have $\mathbf{C}$-trivial center. In particular, we examine such morphisms for algebra-valued holomorphic functions on a complex manifold X, giving conditions that the set of the previous morphisms be the classical envelope of holomorphy of X.

1 – Introduction

The envelope of holomorphy of a domain $X \subseteq \mathbf{C}^n$, $\mathrm{Env}_{O(X)}(X)$, is classically defined as the maximal Riemann domain to which every holomorphic $\mathbf{C}$-valued function on X can be extended [8]. Moreover, $\mathrm{Env}_{O(X)}(X)$ is equal to the spectrum (Gel'fand space) of $O(X)$, denoted by $\mathrm{M}(O(X))$ [6], i.e. the set of continuous $\mathbf{C}$-characters of the algebra $O(X)$ of $\mathbf{C}$-valued holomorphic functions on X, the latter algebra being endowed with the compact open topology. In this context, one is interested in the following: Given a complex (analytic) manifold X and a locally m-convex algebra $\mathbf{A}$ (not necessarily commutative), which is the maximal complex manifold Y such that every $\mathbf{A}$-valued holomorphic function on X can be extended to Y? An answer was given by J.G. Craw [1] in the case of a Riemann domain X and a Banach algebra $\mathbf{A}$, considering as Y the set of $\mathbf{A}$-characters of $O(X,\mathbf{A})$ (viz. $\mathbf{A}$-morphisms of $O(X,\mathbf{A})$ into $\mathbf{A}$) which restrict to a character of $O(X)$. In [4] (see also [5]) we gave another interpretation of the previous result by realizing it as the set of $\mathbf{C}$-central $\mathbf{A}$-morphisms of $O(X,\mathbf{A})$ into a locally

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m-convex algebra (cf. §3 for definitions). This set is the $\mathrm{Env}_{O(X,\mathbf{A})}(X)$ (envelope of X with respect to $O(X,\mathbf{A})$), in analogy with the complex case; indeed, one can prove that it is a Riemann domain and $O(X,\mathbf{A})$-convex, when X is a Riemann domain and $\mathbf{A}$ a Fréchet locally m-convex algebra (ibid.).

We consider below, in an entirely general context, continuous $\mathbf{A}$-morphisms (§2) and $\mathbf{C}$-central $\mathbf{A}$-morphisms (§3) with respect to topological tensor product algebras. Precisely, using the above technique, we study these sets of morphisms defined on algebra-valued function algebras, since these algebras can be expressed as tensor product algebras under suitable conditions (cf. (2.5), (2.6)). Thus, we take analogous results to the classical case of the $\mathbf{C}$-valued algebras through the corresponding spectra (cf. Corollaries 3.2, 3.3). In particular, we examine in §4 the set of $\mathbf{C}$-central $\mathbf{A}$-morphisms on $O(X,\mathbf{A})$, and give conditions under which one identifies the set $\mathrm{Env}_{O(X,\mathbf{A})}(X)$ with the $\mathrm{Env}_{O(X)}(X)$ (cf. Corollary 4.1). To this end we use the notion of a Runge pair with respect to the sheaf $O \mathbin{\widehat{\otimes}} \mathbf{A}$ (cf. also [7]).

2 – The generalized A-spectrum of $E \mathbin{\widehat{\otimes}_\tau} F$

Given a topological algebra $\mathbf{A}$, an $\mathbf{A}$-algebra E is called a *topological* $\mathbf{A}$*-algebra* if E is a topological algebra and the "action" of $\mathbf{A}$ on E is a (jointly) continuous map.

If E, F are topological $\mathbf{A}$-algebras, the *generalized* $\mathbf{A}$*-spectrum* of E with respect to (w.r.t.) F is the set $\mathrm{M}_{\mathbf{A}}(E,F)$ of non-zero continuous $\mathbf{A}$-morphisms of E into F, equipped with the topology induced on it by $\mathcal{L}_{\mathbf{A}}(E,F)_s$ (the space of continuous $\mathbf{A}$-linear maps between the corresponding modules with the simple convergence topology on E; cf. [2: §3]). If the algebras involved have identities, the respective morphisms are assumed to be "identity preserving".

Now, given a ($\mathbf{C}$-) algebra E and an $\mathbf{A}$-algebra F, we consider the corresponding (algebraic) tensor product algebra $E \otimes F$, which is an $\mathbf{A}$-algebra such that

$$a \cdot (x \otimes y) := x \otimes a \cdot y \ , \tag{2.1}$$

for any $a \in A$ and $x \otimes y \in E \otimes F$. Analogously, if E is an $\mathbf{A}$-algebra and F a ($\mathbf{C}$-) algebra.

Definition 2.1. Let E, F be topological ($\mathbf{C}$-) algebras with F being a topological $\mathbf{A}$-algebra. By a *compatible topology* on the corresponding tensor product **A**-algebra $E \otimes F$ we mean a (Hausdorff) topology τ such that the pair $(E \otimes F, \tau) \equiv E \underset{\tau}{\otimes} F$ is a topological **A**-algebra of the same type as E, F.

This type of compatibility of a tensorial topology is analogous to that of [6: Chapter X, Definition 3.1] and [2: Definition 1.1] suitably modified in our case. In the sequel, we are interested in compatible topologies τ satisfying the following conditions

(2.2) *The canonical map of $E \times F$ into $E \underset{\tau}{\otimes} F$ is separately continuous.*

(2.3) *For every topological* **A***-algebra G, and for any pair $(f, g) \in \mathrm{M}_{\mathbf{C}}(E, G) \times \mathrm{M}_{\mathbf{A}}(F, G)$ one has $f \otimes g \in \mathcal{L}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)$.*

Here by $\mathrm{M}_{\mathbf{C}}(E, G)$ we consider the generalized **C**-spectrum of E (w.r.t.) G and moreover $f \otimes g$ is defined by $(f \otimes g)(x \otimes y) := f(x) \cdot g(y)$, $x \otimes y \in E \underset{\tau}{\otimes} F$.

A stronger version of (2.3) is applied when one has to consider completed tensor product **A**-algebras. That is, we assume that

(2.4) *For any equicontinuous subsets $M \subseteq \mathrm{M}_{\mathbf{C}}(E, G)$, $N \subseteq \mathrm{M}_{\mathbf{A}}(F, G)$, the set $M \otimes N \equiv \{f \otimes g : f \in M, g \in N\}$ is an equicontinuous subset of $\mathcal{L}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)$.*

Examples. In case of locally convex algebras the projective (**C**-)tensorial topology π is a compatible topology on $E \otimes F$ (Definition 2.1) satisfying (2.2) and (2.4) (and hence (2.3), cf. [6: Chapter X, Lemma 3.1]). The remark is still in force concerning locally convex **A**-algebras with continuous multiplication, or yet in case of locally m-convex ones. On the other hand, the preceding is still possible within the context of not necessarily locally convex **A**-algebras. Thus, if **A** is a locally bounded algebra with continuous multiplication [6], then every locally bounded **A**-algebra with continuous multiplication is a topological **A**-algebra (not necessarily locally convex). So for a pair (E, F) of locally bounded algebras with F being a locally bounded **A**-algebra, the corresponding compatible topology on $E \otimes F$ (cf. [6: Chapter VI, Theorem 3.1]) is, in fact, a (not necessarily locally convex) topology on $E \otimes F$, as above.

One has the above situation taking algebra-valued function algebras which may be considered as topological tensor product **A**-algebras in the sense of the previous remarks.

Thus, if X is a completely regular k-space and E a complete locally convex

$\mathbf{A}$-algebra, then one has

$$C_c(X,E) = C_c(X) \underset{\varepsilon}{\widehat{\otimes}} E \tag{2.5}$$

within an isomorphism of locally convex $\mathbf{A}$-algebras (cf. [6: Chapter XI, Theorem 1.1]). Here $C_c(X,E)$ (resp. $C_c(X)$) is the algebra of E- (resp. $\mathbf{C}$-) valued continuous functions on X, equipped with the compact-open topology, which is also a locally convex $\mathbf{A}$-algebra defining $(af)(x) := a \cdot f(x)$, for every $a \in \mathbf{A}$, $f \in C_c(X,E)$, $x \in E$. In the second member of (2.5) ε denotes the biprojective tensorial topology (ibid.), which is also a compatible topology as above through the isomorphism (2.5). One has analogous results by considering E-valued holomorphic (resp. C^∞-) functions. Precisely, if K is a compact subset of a second countable complex manifold X (resp. a paracompact C^∞-manifold X) and E a locally convex $\mathbf{A}$-algebra, then one gets the following isomorphisms of locally convex $\mathbf{A}$-algebras

$$O(K,E) = O(K) \underset{\varepsilon}{\widehat{\otimes}} E \quad (\text{resp. } C^\infty(X,E)) = C^\infty(X) \underset{\varepsilon}{\widehat{\otimes}} E \tag{2.6}$$

(cf. §4 and also [6: Chapter XI, Lemma 4.1, Theorem 2.1, (2.8)]). For the relevant definitions cf. [6].

Proposition 2.1. *Let E, F be unital topological algebras with F being also a topological $\mathbf{A}$-algebra and G a unital topological $\mathbf{A}$-algebra with continuous multiplication. Moreover, let τ be a compatible topology on $E \otimes F$ satisfying (2.2), (2.3) and the closed set*

$$Q = \Big\{(f,g)\colon\ f(x)\,g(y) = g(y)\,f(x);\ x \in E,\ y \in F\Big\} \subseteq \mathrm{M}_{\mathbf{C}}(E,G) \times \mathrm{M}_{\mathbf{A}}(F,G)\ . \tag{2.7}$$

Then one gets the next homeomorphism

$$\mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G) = Q \tag{2.8}$$

and for G commutative

$$\mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G) = \mathrm{M}_{\mathbf{C}}(E,G) \times \mathrm{M}_{\mathbf{A}}(F,G)\ . \tag{2.9}$$

Proof: For any $h \in \mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)$ we define

$$f(x) := h(x \otimes 1_F),\quad x \in E, \qquad \text{and} \qquad g(y) := h(1_E \otimes y),\quad y \in F\ , \tag{2.10}$$

with 1_E, 1_F the identities of E, F respectively, such that one has

$$h = f \otimes g \tag{2.11}$$

and moreover, $(f,g) \in \mathrm{M}_{\mathbf{C}}(E,G) \times \mathrm{M}_{\mathbf{A}}(F,G)$; hence one gets the map

$$\mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G) \to \mathrm{M}_{\mathbf{C}}(E,G) \times \mathrm{M}_{\mathbf{A}}(F,G)\ , \tag{2.12}$$

which is an injection (cf. (2.11)). Furthermore for a given $(f,g) \in Q$ one has $h := f \otimes g \in \mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)$ and every element thus defined yields according to (2.11) the initial pair (f,g), that is (2.12)is a bijection onto Q. The bicontinuity of (2.12) can be proved analogously to [6: Chapter XII, Lemma 3.1]. Concerning (2.9), this is immediate from (2.7), (2.8). ∎

For the study of the generalized spectrum of the (complete) topological $\mathbf{A}$-algebra $E \widehat{\underset{\tau}{\otimes}} F$ we need some more terminology.

Thus, let E, G be topological $\mathbf{A}$-algebras where E has continuous multiplication and G is complete. The continuous bijection

$$\mathrm{M}_{\mathbf{A}}(E,G) \to \mathrm{M}_{\mathbf{A}}(\widehat{E},G) \colon\ f \mapsto \overline{f}\ , \tag{2.13}$$

where $\overline{f}$ is the (continuous) extension of f to the completion $\widehat{E}$ of E, *is a homeomorphism iff either one of the sets* $\mathrm{M}_{\mathbf{A}}(E,G)$, $\mathrm{M}_{\mathbf{A}}(\widehat{E},G)$ *is locally equicontinuous* (cf. [2: (3.10), (3.11)]).

Proposition 2.1 and relation (2.13) yield the following lemma, whose proof is analogous to [6: Chapter VI, Lemma 6.2].

Lemma 2.1. *Let E, F, G be topological algebras as in Proposition 2.1 and τ a compatible topology satisfying (2.2), (2.3). Moreover, consider the following assertions*

i) $\mathrm{M}_{\mathbf{C}}(E,G)$, $\mathrm{M}_{\mathbf{A}}(F,G)$ *are both locally equicontinuous;*

ii) $\mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)$ *is locally equicontinuous.*

Then i)⇒ii). Besides, or every $(f,g) \in Q$, there exist an equicontinuous neighbourhood U of f in $\mathrm{M}_{\mathbf{C}}(E,G)$ and V of g in $\mathrm{M}_{\mathbf{A}}(F,G)$ such that $U \otimes V$ is an equicontinuous neighbourhood of $f \otimes g$ in $\mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)$. In particular ii)⇒i) as well, whenever G is commutative. ∎

Now, we have the main result of this section as follows.

Theorem 2.1. *Let E, F be unital topological algebras with continuous multiplication where F is, in particular, a topological $\mathbf{A}$-algebra. Moreover, suppose that $\mathrm{M}_{\mathbf{C}}(E,G)$, $\mathrm{M}_{\mathbf{A}}(F,G)$ are locally equicontinuous, where G is a unital complete topological $\mathbf{A}$-algebra with continuous multiplication and let τ be a compatible topology on $E\otimes F$ satisfying (2.2), (2.4). If Q is the set (2.7), then*

$$\begin{aligned}\mathrm{M}_{\mathbf{A}}(E \mathop{\widehat{\otimes}}_{\tau} F,G) = Q = \mathrm{M}_{\mathbf{A}}(E \mathop{\otimes}_{\tau} F,G) &\subsetneqq \mathrm{M}_{\mathbf{C}}(E,G)\times \mathrm{M}_{\mathbf{A}}(F,G)\\ &= \mathrm{M}_{\mathbf{C}}(\widehat{E},G)\times \mathrm{M}_{\mathbf{A}}(\widehat{F},G)\end{aligned} \tag{2.14}$$

within homeomorphisms. In case G is commutative, the "inclusion map" in (2.14) may be replaced by an equality.

Proof: Lemma 2.1 shows that $\mathrm{M}_{\mathbf{A}}(E \mathop{\otimes}_{\tau} F,G)$ is locally equicontinuous so that $\mathrm{M}_{\mathbf{A}}(E \mathop{\widehat{\otimes}}_{\tau} F,G) = \mathrm{M}_{\mathbf{A}}(E \mathop{\otimes}_{\tau} F,G)$ (cf. (2.13)). Thus, the assertion is immediate from Proposition 2.1. ∎

If E is a topological $\mathbf{A}$-algebra, a non-zero continuous $\mathbf{A}$-morphism $f\colon E\to\mathbf{A}$ is called a *continuous $\mathbf{A}$-character of E*.

Now, the set $\mathrm{M}_{\mathbf{A}}(\mathbf{A},\mathbf{A})$ is equal to $\{\mathrm{id}_{\mathbf{A}}\}$ (identity of $\mathbf{A}$) since for any $f\in \mathrm{M}_{\mathbf{A}}(\mathbf{A},\mathbf{A})$, $a\in\mathbf{A}$, one has $f(a)=f(a\cdot 1_{\mathbf{A}})=f(a\cdot 1_{\mathbf{A}})=a\cdot f(1_{\mathbf{A}})=a\cdot 1_{\mathbf{A}}=a$. Thus, by Proposition 2.1 there exists a map

$$\mathrm{M}_{\mathbf{A}}(E \mathop{\otimes}_{\tau} \mathbf{A},\mathbf{A}) \to \mathrm{M}_{\mathbf{C}}(E,\mathbf{A})\times\{\mathrm{id}_{\mathbf{A}}\}\colon\ h\mapsto (f,\mathrm{id}_{\mathbf{A}})\ , \tag{2.15}$$

such that

$$h = f\otimes \mathrm{id}_{\mathbf{A}}\ . \tag{2.16}$$

So, Proposition 2.1 and Theorem 2.1 yield the next

Corollary 2.1. *Let E, $\mathbf{A}$ be unital topological algebras with continuous multiplications and τ a compatible topology on $E\otimes\mathbf{A}$ satisfying (2.2), (2.3). Moreover, let S be the subset of $\mathrm{M}_{\mathbf{C}}(E,\mathbf{A})$ consisting of all $f\in\mathrm{M}_{\mathbf{C}}(E,\mathbf{A})$ such that $\mathbf{A}=(\mathrm{Im}\,f)' := \{a\in\mathbf{A}\colon\ a\cdot f(x)=f(x)\cdot a,\ x\in E\}$. Then*

$$\mathrm{M}_{\mathbf{A}}(E \mathop{\otimes}_{\tau} \mathbf{A},\mathbf{A}) = S\ , \tag{2.17}$$

within a homeomorphism. Furthermore, if $\mathbf{A}$ is complete, $\mathrm{M}_{\mathbf{A}}(E,\mathbf{A})$ is locally equicontinuous and τ satisfies (2.4), then

$$\mathrm{M}_{\mathbf{A}}(E \mathop{\widehat{\otimes}}_{\tau} \mathbf{A},\mathbf{A}) = \mathrm{M}_{\mathbf{A}}(E \mathop{\otimes}_{\tau} \mathbf{A},\mathbf{A}) = S\ . \tag{2.18}$$

In case $\mathbf{A}$ *is commutative, the relations (2.17), (2.18) take the form*

$$\mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} \mathbf{A}, \mathbf{A}) = \mathrm{M}_{\mathbf{C}}(E, \mathbf{A})\,, \qquad \mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\widehat{\otimes}} \mathbf{A}, \mathbf{A}) = \mathrm{M}_{\mathbf{C}}(\widehat{E}, \mathbf{A})\ , \tag{2.19}$$

respectively. ∎

Corolary 2.1 and relation (2.5) gives the following.

Corollary 2.2. *Let* X *be a completely regular* k*-space and* $\mathbf{A}$ *a complete locally convex algebra with continuous multiplication. Then each* $h \in \mathrm{M}_{\mathbf{A}}(C_c(X, \mathbf{A}), \mathbf{A})$ *is of the form*

$$h = f \otimes \mathrm{id}_{\mathbf{A}}\ , \tag{2.20}$$

with $f \in \mathrm{M}_{\mathbf{C}}(C_c(X), \mathbf{A})$. *In particular, if* $\mathbf{A}$ *is commutative, then*

$$\mathrm{M}_{\mathbf{A}}(C_c(X, \mathbf{A}), \mathbf{A}) = \mathrm{M}_{\mathbf{C}}(C_c(X), \mathbf{A}) \tag{2.21}$$

within a homeomorphism. ∎

One has analogous results to Corollary 2.2 in the case of vector-valued holomorphic (resp. C^∞-) functions (cf. (2.6) and also §4 below).

Under suitable conditions analogous to [2: §3] the results of this section are also in force if the involved algebras have bounded approximate identities instead of identities.

3 – C-central A-morphisms

In this section we examine a new class of central morphisms, as the title indicates which is different from that of central morphisms defined in [2,3].

So a continuous $\mathbf{A}$-morphism $h\colon E \to F$ between two topological $\mathbf{A}$-algebras E, F with identities 1_E, 1_F is said to be $\mathbf{C}$*-central* if the center of $\overline{\mathrm{Im}\, h}$ (closure of $\mathrm{Im}\, h$ in F) is $\mathbf{C}$-trivial, in the sense that

$$\mathcal{G}(\overline{\mathrm{Im}\, h}) \equiv \overline{\mathrm{Im}\, h} \cap (\overline{\mathrm{Im}\, h})' = \mathbf{C} \cdot 1_F \simeq \mathbf{C} \subsetneq \mathbf{A}\ . \tag{3.1}$$

Denoting by $\mathrm{M}^0_{\mathbf{A}}(E, F)_{\mathbf{C}}$ this set of morphisms, we endow it with the simple convergence topology in E, being thus a subset of $\mathrm{M}_{\mathbf{A}}(E, F) \subseteq \mathcal{L}_{\mathbf{A}}(E, F)_s$.

Each $\mathbf{C}$*-central* $\mathbf{A}$*-morphism is a central* $\mathbf{A}$*-morphism as in [3], while if, in addition,* $\mathrm{Im}\, h$ *is closed, is as in [2]. The converse is true in case* $\mathbf{A} = \mathbf{C}$.

Now, if E is a commutative topological $\mathbf{A}$-algebra, every $h \in \mathrm{M}^0_{\mathbf{A}}(E,F)_{\mathbf{C}}$ takes the form

$$h = \chi \otimes 1_F \ , \tag{3.2}$$

where $\chi \in \mathcal{M}(E)$ (spectrum of E) such that $(\chi \otimes 1_F)(x) := \chi(x) \cdot 1_F$, $x \in E$. Indeed, by (3.1) $\mathcal{G}(\overline{\mathrm{Im}\, h}) = \overline{\mathrm{Im}\, h} = \mathbf{C} \cdot 1_F$ such that $f(x) = \lambda_x \cdot 1_F$, $\lambda_x \in \mathbf{C}$, for every $x \in E$; hence one defines a map $\chi: E \to \mathbf{C}: x \mapsto \chi(x) := \lambda_x$ which is an element of $\mathcal{M}(E)$ such that $f(x) = \chi(x) \cdot 1_F = (\chi \otimes 1_F)(x)$. Thus, since $\mathbf{C} \simeq \mathbf{C} \cdot 1_F \subsetneq F$, one obtains the next homeomorphism

$$\mathrm{M}_{\mathbf{A}}(E,F)_{\mathbf{C}} = \mathcal{M}(E) \ . \tag{3.3}$$

The following theorem is analogous to [2: Propositions 4.1, 4.2] and [3: Lemma 1.1] in the present framework.

Theorem 3.1. *Let E, F be unital topological algebras with E commutative and F a topological $\mathbf{A}$-algebra. Moreover let G be a complete unital topological $\mathbf{A}$-algebra with continuous multiplication and τ a compatible topology on $E \otimes F$ satisfying (2.2), (2.3). Then,*

$$\mathrm{M}^0_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)_{\mathbf{C}} = \mathcal{M}(E) \times \mathrm{M}^0_{\mathbf{A}}(F,G)_{\mathbf{C}} \tag{3.4}$$

within a homeomorphism. Moreover, if E, F have continuous multiplications, τ satisfies (2.4) and $\mathcal{M}(E)$, $\mathrm{M}^0_{\mathbf{A}}(F,G)_{\mathbf{C}}$ are locally equicontinuous, then

$$\mathrm{M}^0_{\mathbf{A}}(E \underset{\tau}{\widehat{\otimes}} F, G)_{\mathbf{C}} = \mathcal{M}(\widehat{E}) \times \mathrm{M}^0_{\mathbf{A}}(\widehat{F},G)_{\mathbf{C}} \tag{3.5}$$

within a homeomorphism.

Proof: Each $h \in \mathrm{M}^0_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)_{\mathbf{C}}$ is of the form $h = f \otimes g$ with $(f,g) \in \mathrm{M}_{\mathbf{C}}(E,G) \times \mathrm{M}_{\mathbf{A}}(F,G)$ (cf. Proposition 2.1) such that the commutativity of E yields

$$\mathcal{G}(\overline{\mathrm{Im}\, g}) \subseteq \mathcal{G}(\overline{\mathrm{Im}\, h}) \, , \qquad \mathcal{G}(\overline{\mathrm{Im}\, f}) = \overline{\mathrm{Im}\, f} \subseteq \mathcal{G}(\overline{\mathrm{Im}\, h}) \ .$$

Thus, for h as above one has $(f,g) \in \mathcal{M}(E) \times \mathrm{M}^0_{\mathbf{A}}(F,G)_{\mathbf{C}}$ (cf. (3.6), (3.3)). Conversely, if $(f,g) \in \mathrm{M}^0_{\mathbf{C}}(E,G)_{\mathbf{C}} \times \mathrm{M}^0_{\mathbf{A}}(F,G)_{\mathbf{C}}$ then $f(x) = \lambda_x \cdot 1_G$, $\lambda_x \in \mathbf{C}$ (cf. (3.3), (3.6)) such that

$$f(x) \cdot g(y) = \lambda_x \cdot g(y) = g(y) \cdot f(x)$$

for all $(x,y) \in E \times F$, so that $h = f \otimes g \in \mathrm{M}_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)$ (cf. Proposition 2.1). Moreover,

$$\mathbf{C} \cdot 1_G = \mathcal{G}(\overline{\mathrm{Im}\, f}) \subseteq \mathcal{G}(\overline{\mathrm{Im}\, h}) \subseteq \mathcal{G}(\overline{\mathrm{Im}\, g}) = \mathbf{C} \cdot 1_G \ ,$$

i.e., $h \in \mathrm{M}^0_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)_{\mathbf{C}}$. That is, $h = f \otimes g$ is a $\mathbf{C}$-central $\mathbf{A}$-morphism iff this is true for f, g, so that (3.4) is imediate from Proposition 2.1, relation (3.3).

Now, one obtains that $\mathrm{M}^0_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)_{\mathbf{C}}$ is locally equicontinuous iff $\mathcal{M}(E)$, $\mathrm{M}^0_{\mathbf{A}}(F, G)_{\mathbf{C}}$ are locally equicontinuous (cf. also Lemma 2.1) such that $\mathrm{M}^0_{\mathbf{A}}(E \underset{\tau}{\widehat{\otimes}} F, G)_{\mathbf{C}} = \mathrm{M}^0_{\mathbf{A}}(E \underset{\tau}{\otimes} F, G)_{\mathbf{C}}$ (cf. also (3.1), (3.6)); hence (3.4) implies (3.5). ∎

Corollary 2.1 and Theorem 3.1 imply the following.

Corollary 3.1. *Let E, $\mathbf{A}$ be unital complete topological algebras with continuous multiplications and E commutative. Moreover, let τ be a compatible topology on $E \otimes \mathbf{A}$ satisfying (2.2), (2.3) and let $\mathcal{M}(E)$ be locally equicontinuous. Then one has*

$$\mathrm{M}^0_{\mathbf{A}}(E \underset{\tau}{\widehat{\otimes}} \mathbf{A}, \mathbf{A})_{\mathbf{C}} = \mathcal{M}(E) \tag{3.7}$$

within a homeomorphism. ∎

Corollary 3.1 and the relations (2.5), (2.6) (cf. also [6: Chapter VII, Theorems 1.2, 2.1]) prove the following corollaries.

Corollary 3.2. *Let X be a locally compact space and A a unital complete locally convex algebra with continuous multiplication. Then,*

$$\mathrm{M}^0_{\mathbf{A}}(C_c(X, \mathbf{A}), \mathbf{A})_{\mathbf{C}} = X \tag{3.8}$$

within a homeomorphism of the respective spaces. ∎

Corollary 3.3. *Let X be an n-dimensional compact C^∞-manifold and $\mathbf{A}$ a unital Fréchet locally convex algebra. Then,*

$$\mathrm{M}^0_{\mathbf{A}}(C^\infty(X, \mathbf{A}), \mathbf{A})_{\mathbf{C}} = X \tag{3.9}$$

within a homeomorphism of the respective spaces. ∎

4 – Envelopes of holomorphy

We use in this section the preceding by studying the algebra of vector-valued holomorphic functions.

Let X be a complex (analytic) manifold, K a compact subset of X and E a unital complete locally convex $\mathbf{A}$-algebra with continuous multiplication. Then, $O(K) \hat{\otimes} E$ is a locally convex $\mathbf{A}$-algebra (cf. (2.6) and also [5: §2]).

Now, if X is second countable, (U_n) a (denumerable) fundamental system of open neighbourhoods of K in X and E a complete locally m-convex $\mathbf{A}$-algebra, for which the respective topological vector space is a DF-space, then

$$O(K, E) = O(K) \hat{\otimes} E \tag{4.1}$$

within an isomorphism of locally convex $\mathbf{A}$-algebras (cf. [7] and relation (2.6)).

Theorem 3.1 and relation (4.1) prove the following.

Lemma 4.1. *Let X be a complex (analytic) manifold which is second countable and (U_n) a (denumerable) fundamental system of open neighbourhood of a compact subset K of X. Moreover, let E be a unital complete Fréchet localy m-convex $\mathbf{A}$-algebra and G a unital complete locally convex $\mathbf{A}$-algebra with continuous multiplication such that $\mathcal{M}(O(K))$, $\mathrm{M}^0_{\mathbf{A}}(E, G)_{\mathbf{C}}$ are locally equicontinuous. Then one has*

$$\mathrm{M}^0_{\mathbf{A}}(O(K, E), G)_{\mathbf{C}} = \mathcal{M}(O(K)) \times \mathrm{M}^0_{\mathbf{A}}(E, G)_{\mathbf{C}} \tag{4.2}$$

within a homeomorphism. ∎

In this concern, let K be a compact subset of a Stein manifold X and (U_n) an open basis of neighbourhoods of K in X. Considering the respective sequence $(\widetilde{U}_n)_{n \in \mathbb{N}}$ with $\widetilde{U}_n = \mathcal{M}(O(U_n))$, $n \in \mathbb{N}$, one gets a decreasing sequence of Stein manifolds containing K, such that

$$\mathcal{M}(O(K)) = \varprojlim \mathcal{M}(O(U_n)) = \varprojlim \widetilde{U}_n = \bigcap_n \widetilde{U}_n = K \tag{4.3}$$

within homeomorphisms (cf. also [6: p. 163]). Thus, Lemma 4.1 and (4.3) yield the following homeomorphism

$$\mathrm{M}^0_{\mathbf{A}}(O(K, E), G)_{\mathbf{C}} = K \times \mathrm{M}^0_{\mathbf{A}}(E, G)_{\mathbf{C}} \ . \tag{4.4}$$

In particular, if $E = \mathbf{A} = G$, the last relation gives the following homeomorphism

$$\mathrm{M}^0_{\mathbf{A}}(O(K, \mathbf{A}), \mathbf{A}) = K \ ; \tag{4.5}$$

i.e. the set of $\mathbf{C}$-central $\mathbf{A}$-morphisms of $O(K, \mathbf{A})$, with respect to $\mathbf{A}$, is a compact subset of a Stein manifold (cf. also [4, 5]).

On the other hand, if (X,O) is a complex space and A an open subset of X, we shall say that (X,A) is a *Runge pair, with respect to the (structure) sheaf* O, if the topological algebra $O(X) \varsubsetneq O(A)$ is dense in $O(A)$ (cf. [7]).

Let (X,O) be a complex space with X second countable and K a subset of X. Moreover, let (U_n) be a (denumerable) fundamental system of open neighbourhoods of K in X, such that (X,U_n), $n \in \mathbb{N}$, is a Runge pair with respect to O. If E is a unital Fréchet locally convex algebra, then (X,U_n) is a Runge pair with respect to $O \mathbin{\widehat{\otimes}} E$ (cf. [7: Corollary 3.1, p. 370]), such that (X,K) is also a Runge pair with respect to $O \mathbin{\widehat{\otimes}} E$. So we now have the following.

Theorem 4.1. *Let (X,O) be a complex space with X second countable and K a subset of X. Moreover, let (U_n) be a (denumerable) fundamental system of open neighbourhoods of K in X, such that (X,U_n), $n \in \mathbb{N}$, is a Runge pair with respect to O. Furthermore, let E be a unital Fréchet locally convex $\mathbf{A}$-algebra and G a unital complete locally convex $\mathbf{A}$-algebra with continuous multiplication such that $\mathrm{M}^0_{\mathbf{A}}(E,G)_{\mathbf{C}}$ is locally equicontinuous. Then*

$$\mathrm{M}^0_{\mathbf{A}}(O(X,E),G)_{\mathbf{C}} = \mathcal{M}(O(X)) \times \mathrm{M}^0_{\mathbf{A}}(E,G)_{\mathbf{C}} \tag{4.6}$$

within a homeomorphism.

Proof: The hypotheses and the above comments (cf. also Theorem 3.1) yield

$$\mathrm{M}^0_{\mathbf{A}}(O(X,E),G)_{\mathbf{C}} = \mathrm{M}^0_{\mathbf{A}}(\overline{O(K,E)},G)_{\mathbf{C}} = \mathrm{M}^0_{\mathbf{A}}(O(K,E),G)_{\mathbf{C}}$$

within homeomorphisms. Thus, Lemma 4.1 and the preceding definitions show the homeomorphism (4.6). ∎

As an application of the above we have the following.

Corollary 4.1. *Let the hypotheses of Theorem 4.1 be satisfied such that in particular $E = G = \mathbf{A}$. Then,*

$$\mathrm{Env}_{O(X,\mathbf{A})}(X) = \mathrm{Env}_{O(X)}(X) \ , \tag{4.7}$$

within a homeomorphism.

Proof: The hypotheses in connection with Theorem 4.1, Corollary 3.1 give the homeomorphism

$$\mathrm{M}^0_{\mathbf{A}}(O(X,\mathbf{A}),\mathbf{A})_{\mathbf{C}} = \mathcal{M}(O(X)) \ . \tag{4.8}$$

In [5] (cf. also [4]) the first member of (4.8) was proved to be a Riemann domain and moreover an $O(X,\mathbf{A})$-convex set in the case X is a Riemann domain; hence, it is, by definition, $\mathrm{Env}_{O(X,\mathbf{A})}(X)$. Moreover, the second member of (4.8) is, by definition the classical $\mathrm{Env}_{O(X)}(X)$ (cf. [6: Chapter V, Definition 4.1]). Thus (4.7) is an immediate consequence of (4.8) and the previous comments. ∎

The above Corollary 4.1 lead us to the following.

Theorem 4.2 (A. Mallios). *Let the hypotheses of Corollary 4.1 be satisfied. Then* $\mathrm{Env}_{O(X,\mathbf{A})}(X)$ *is independent of the algebra* $\mathbf{A}$*, range of the holomorphic functions considered.* ∎

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