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A TOPOLOGY WITH ORDER, GRAPH AND AN ENUMERATION PROBLEM

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A possible topologies on a two-point set $S = \{x, y\}$ are, indiscreet, discrete and two copies of $\{\phi, \{x\}, S\}$, a topology on S, called the *Sierpinski topology*.

Craveiro de Carvalho and d'Azevedo Breda [1] call a topological space X *locally Sierpinski space*, if every point $x \in X$ has a neighbourhood homeomorphic to the Sierpinski space. (Remember the definition of a topological manifold). The authors then have established there some properties of locally Sierpinski spaces, but left the following enumeration problem open: Let X be a finite set with n elements. Find (up to homeomorphism) the number of different topologies making X into a Locally Sierpinski space. In this note we use a well-known ordering of a topological space and its corresponding diagram to solve this enumeration problem.

1 – Definition, order and representation

Definition 1 ([1]). A topological space X is called *locally Sierpinski space*, l.S. space for short, if every point has a neighbourhood homeomorphic to a *Sierpinski space*; this means that for $x \in X$ there exists $S_x = \{x, y\}$, $x \neq y$, called *Sierpinski set*, such that the relative topology on S_x is a copy of Sierpinski topology. Note that, *Sierpinski sets* are open and together with isolated points (called centers in [1]) they form a base for open subsets of X.

Example 1: Let X be a set with order $|X| \geq 2$.

Fix $x_0 \in X$. Take as an open basis the set formed by $\{x_0\}$ and the subsets $\{x_0, y\}$ with $x_0 \neq y$, $y \in X$.

Example 2: If X is a l.S. space and Y discrete space then $X \times Y$ is a l.S. space.

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A topological space X is *Alexandroff space* [3] iff, every point has a smallest neighborhood or equivalently arbitrary intersection of open sets is open. It is evident that, a l.S. space is T_0-Alexandroff space.

Let X be a l.S. space and $\overline{\{y\}}$ denote the closure of $\{y\}$ in X.

The *Alexandroff specialization order*: $x \leq y$ if $x \in \overline{\{y\}}$, $x, y \in X$, makes X a partially ordered set (poset). For $A \subseteq X$, define $\downarrow A = \{x : \exists y \in A$ such that $x \leq y\}$. Then $\downarrow A = A$ if A is closed therefore the topology of a l.S. space is completely determined by its specialization order. Note that, specialization order $\leq$ on l.S. space X is partial order on X, precisely because X satisfies T_0-axiom.

Proposition 1. *Let X be a l.S. space equipped with specialization order. Then for $x \in X$, $\{x\}$ is either open or closed. Furthermore with respect to this order, isolated points are maximal and closed points are minimal.*

Proof: If $\{x\}$ is not open, then there exists $y \in X$ such that $S_x = \{x, y\}$ is open. Now $y \in S_x$ implies that $x \in \overline{\{y\}}$ so $x \leq y$ and there exists $z \neq y$ such that $\{x, z\}$ is open. Hence if $z \in \overline{\{x\}}$ and $z \neq y$ then $z = x$, this means that $\overline{\{x\}} = \{x\}$ and also x is minimal. If $x \leq y$, $\{y\}$ intersects every open set containing x, so if $\{x\}$ is open then $x = y$ and x is maximal. ∎

We need the following Corollary later.

Corollary 1. *If X is a finite l.S. space of size n, then the number of isolated points can not exceed $[\frac{n}{2}]$, the integral part of $\frac{n}{2}$.*

Proof: Since every point is open or closed (maximal or minimal) but not both, the proof is clear.

Posets traditionally represented in more economical way by an *acyclic* directed graph, called Hasse diagram.

By Proposition 1, the Hasse diagram of a l.S. space as poset is bipartite upward directed graph, with maximal (top) elements as isolated points and minimal elements as closed points. ∎

Hence:

Each l.S. space with respect to its natural order determines a unique bipartite graph, with directed edges representing Sierpinski sets and they intersect (if any) only on centres (maximal points). Clearly two such spaces are homeomorphic if, their corresponding graphs are isomorphic. The above discussion leads to the following representation theorem.

Theorem 1. *Let X be a l.S. space. Then X can be represented as an ordered bipartite space, having as a diagram a certain transitive digraph with directed* *edges representing Sierpinski sets. Conversely, any such bipartite order (poset) determines a unique l.S. space.*

Proof: Suppose X is a l.S. space. Then $X = A \cup B$ where A and B are disjoint subsets of open (isolated) points and closed points of X respectively. Now the order of specialization makes X an ordered bipartite space.

Since isolated points and *Sierpinski sets* are basic open sets in X, all upper sets (i.e. sets U such that $x \in U$ and $x \leq y$ implies $y \in U$) are open in X.

Conversely, let $(X, \leq)$ be a poset, $X = A \cup B$, where A, B are disjoint subsets of maximal and minimal elements of X respectively. Suppose for each $b \in B$ there exists a unique element $a \in A$ such that $b \leq a$. Topologise X by

$$\mathcal{B} = \Big\{\{a\} \colon \ a \in A\Big\} \cup \Big\{(a, b) \colon \ a \in A, \ b \in B, \ b \leq a\Big\} ,$$

as a base for open sets. The ordered bipartite space X now is clearly a l.S. space in which the order of specialization agrees with defining order on X. ∎

The author is indebted to the referee for suggesting the above theorem.

Example 3: Let $X = \{a, b, c, d\}$ with topologies

$$\tau_1 = \Big\{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\Big\}$$

and

$$\tau_2 = \Big\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\Big\} .$$

The bipartite graphs representing these topologies are G_1 and G_2 respectively. See Fig. 1

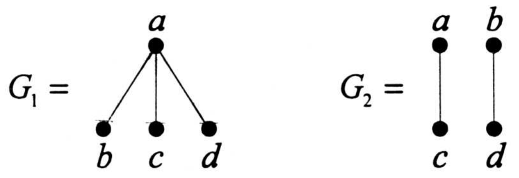


Fig. 1

Example 4: The possible diagrams for a l.S. space X of order $|X| = 7$ are as Fig. 2

Fig. 2

So we have only four topologies each one making X, $|X| = 7$ a locally Sierpinski space.

2 – Partitions of integers and counting topologies

A partition of a number n is a representation of n as the sum of any number of integral parts.

The number of partitions of n is usually denoted by $p(n)$. For example we have

$$p(1) = 1 \ ,$$

$$p(4) = 3 = 2 + 1 = 1 + 1 + 1 \ ,$$

$$p(4) = 5 : 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \ .$$

We also define $p(0) = 1$.

The ordinary generating functions provide a simple way for counting these numbers. In case of finitely many $p(0)$, $p(1)$, ..., $p(n)$, the generating function is as follows

$$G(x) = \frac{1}{(1-x)\,(1-x^2)\cdots(1-x^n)} \ ,$$

see [3].

Note that the coefficient of x^j is the number of partitions of j into summands that do not *exceed* n.

For example in

$$G(x) = \frac{1}{(1-x)\,(1-x^2)\,(1-x^3)} = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + ...$$

there are *three* ways to partition 3 and *seven* ways to partition the integer 6 with no parts exceeding 3.

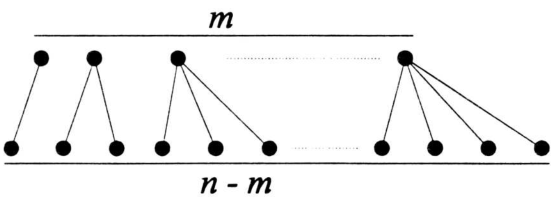
We need the following proposition in the sequel. For the proof, see [3].

Proposition 2. *The ordinary generation function of the number of partitions of positive integer n into exactly m parts is*

$$G(x) = \frac{x^m}{(1-x)(1-x^2)\cdots(1-x^m)} .$$

Let X be a finite set of size $n \geq 2$. Determine the number of different topologies (up to homeomorphism) making X into a locally Sierpinski space.

Note that a typical representation of a l.S. space X, with $|X| = n$, as bipartite graph might looks like (see Fig. 3).



where $1 \leq m \leq \left[\frac{n}{2}\right]$.

Fig. 3

Now, our main conclusion in this section is as follows.

Given positive integers n and m, $1 \leq m \leq [\frac{n}{2}]$, the number of l.S. topological structures on the set X with size $|X| = n$ and the number of isolated points m, is equal to the number of partitions of n into m parts. Therefore, if we denote by p_m^n the number of such partitions, then the number of l.S. spaces on X is equal to

$$p_1^{n-1} + p_2^{n-2} + \ldots + p_m^{n-m} \quad \text{where} \quad 1 \leq m \leq \left[\frac{n}{2}\right] .$$

For example if $n = 7$ (see Example 4) since $p_1^{n-1} = 1$ and generating functions for $n = 2$ and $n = 3$ are

$$\frac{x^2}{(1-x)(1-x^2)} = x^2 + x^3 + 2x^4 + 3x^5 + 3x^6 + \ldots$$

and

$$\frac{x^3}{(1-x)(1-x^2)(1-x^3)} = x^3 + x^4 + 2x^5 + 3x^6 + 7x^7 + \ldots$$

respectively, we have

$$p_1^6 + p_2^5 + p_3^4 = 1 + 2 + 1 = 4 ,$$

as we were expecting.

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