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A WAVE EQUATION WITH A DIRAC DISTRIBUTION

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Abstract: The sine-Gordon equation including a Dirac distribution: $u_{tt} - u_{xx} + \sin u = \delta_a \sin u$ models the interaction between a soliton and a local impurity occuring during transmission through disordered media. Here, we study the Cauchy problem for this equation and we show the existence and uniqueness of a solution in the energy space.

1 – Introduction

In 1958, Skyrme introduced the well-known sine-Gordon equation in nonlinear field theory

$$u_{tt} - u_{xx} = m^2 \sin u .$$

Then, in 1962, Perring and Skryme discovered after numerical simulation that a solution of this equation represents the head-on collision of two kinks (see [3]). In order to understand soliton transmission through a disordered media, Kivshar et al. [1] [2] study the interaction between a soliton and a local impurity. They consider a sine-Gordon equation including a Dirac distribution

$$u_{tt} - u_{xx} + \sin u = \delta_a \sin u .$$

They study numerically the reflection of a soliton by an impurity when its initial velocity is in a certain resonance "window". Here we do not go into such questions, our main goal is to show that there is a unique solution of the Cauchy problem in the energy space.

The main result is the following

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Theorem 1.1. *If* $(u_0, v_0) \in H^1_0(I)\times L^2(I)$ *then there exists a unique solution* $u \in C(\mathbf{R}_+, H^1_0(I)) \cap C^1(\mathbf{R}_+, L^2(I)) \cap C^2(\mathbf{R}_+, H^{-1}(I))$ *of*

$$(1.1) \qquad \begin{cases} u_{tt} - u_{xx} + \sin u = \delta_a(x)\, \sin u, & \text{for } \ (t,x) \in [0,\infty)\times I \ , \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), & \text{for } \ x \in I \ , \end{cases}$$

where $I =]\alpha, \beta[$ *is a bounded, open interval of* $\mathbf{R}$ *and* $a \in I$. *In addition*

$$(1.2) \qquad \begin{aligned} E(u(t), u_t(t)) &= \int_I \Big\{ \frac{1}{2}\, |u_t(t,x)|^2 + \frac{1}{2}\, |u_x(t,x)|^2 - \cos u(t,x) \Big\}\, dx + \cos u(t,a) \\ &= E(u_0, v_0) \ . \end{aligned}$$

To prove this theorem we first approximate the Dirac distribution by a family of smooth bounded functions for which we can apply the classic results for a wave equation (see the following section); we obtain a sequence of approximate solutions.

Then we prove uniform estimates on these solutions thanks to the fact that I is bounded. Next we use these estimates to pass to the limit in the approximate equation and we show the conservation of energy.

In the last section, we present some extensions for more general Dirichlet conditions and for unbounded intervals.

2 – Preliminaries

2.1. The Cauchy problem for a wave equation

Let J be any interval of $\mathbf{R}$, bounded or not, and let $T > 0$. We consider a function $g \in C(J\times\mathbf{R}, \mathbf{R})$ and a real $0 \le \alpha < \infty$ such that for all $(x,y,z) \in J\times\mathbf{R}^2$

$$|g(x,y) - g(x,z)| \le C_1(|y|^\alpha + |z|^\alpha)\, |y-z| \ .$$

If J is not bounded we also assume that $g(x,0) = 0$ for all $x \in J$. We note

$$G(x,y) = \int_0^y g(x,z)\ dz \ ,$$

and we assume $G(x,y) \le C_2\, |y|^2$ for all $(x,y) \in J\times\mathbf{R}$. It is well known that for every $(\varphi,\psi) \in H^1_0(J)\times L^2(J)$, there exists a unique solution $u \in C(\mathbf{R}_+, H^1_0(J)) \cap$

$C^1(\mathbf{R}_+, L^2(J)) \cap C^2(\mathbf{R}_+, H^{-1}(J))$ of the following problem

$$
\begin{cases} u_{tt} - u_{xx} = g(x,u), & \text{for } \ (t,x) \in [0,\infty) \times J \ , \\ u(0,x) = \varphi(x), \ \ u_t(0,x) = \psi(x), & \text{for } \ x \in J \ , \end{cases} \tag{2.1}
$$

Moreover,

$$
\begin{aligned} \int_J \Big\{ |u_t(t,x)|^2 + |u_x(t,x)|^2 - 2\,G(x,u(t,x)) \Big\}\, dx = \\ = \int_J \Big\{ |\psi(x)|^2 + |\varphi_x(x)|^2 - 2\,G(x,\varphi(x)) \Big\}\, dx \ . \end{aligned} \tag{2.2}
$$

(See for example Cazenave and Haraux [4]).

2.2. Solution of a wave equation on $\mathbf{R}_+ \times \mathbf{R}$

For $a \in \mathbf{R}$ we set

$$
\mathcal{T}_a = \Big\{ (t,x) \in \mathbf{R}^2, \ \ t > 0, \ t - |x-a| > 0 \Big\} \ .
$$

We consider two functions f and h in $C(\mathbf{R}_+ \times \mathbf{R}, \mathbf{R})$.

Then the problem

$$
\begin{cases} w_{tt} - w_{xx} + h(t,x) = \delta_a(x)\, f(t,a), & \text{for } \ (t,x) \in \mathbf{R}_+ \times \mathbf{R} \ , \\ w(0,x) = 0, \ \ w_t(0,x) = 0, & \text{for } \ x \in \mathbf{R} \ , \end{cases} \tag{2.3}
$$

has one and only one solution in $\mathcal{D}'(\mathbf{R}_+ \times \mathbf{R})$ given by

$$
w(t,x) = \begin{cases} -\dfrac{1}{2} \displaystyle\int_0^t ds \int_{x-(t-s)}^{x+t-s} h(s,\sigma)\, d\sigma + \dfrac{1}{2} \int_0^{t-|x-a|} f(s,a)\, ds & \text{if } (t,x) \in \mathcal{T}_a, \\ -\dfrac{1}{2} \displaystyle\int_0^t ds \int_{x-(t-s)}^{x+t-s} h(s,\sigma)\, d\sigma & \text{otherwise} \ . \end{cases} \tag{2.4}
$$

(see [5], p. 204).

3 – Proof of Theorem 1.1.

3.1. Uniqueness

In this section, we show that if u_1, u_2 are two solutions of (1.1) in $L^\infty((0,T),H_0^1(I))\cap W^{1,\infty}((0,T),L^2(I))$, then $u_1=u_2$. We note $W=u_1-u_2$. We first show that W is zero in the open rectangular triangle $\mathcal{T}_r$ (for *right triangle*) defined by the three points $(0,a)$, $(0,\beta)$, $(\beta-a,\beta)$ in the (t,x) plane (we recall that $I=]\alpha,\beta[$).

For u defined on $[0,T)\times I$, we will denote by $\tilde u$ the extension of u in $[0,T)\times\,]\alpha,2\beta-\alpha[$ by symmetry about to the line $x=\beta$, and by $\widetilde{\mathcal{T}_r}$ the extension of $\mathcal{T}_r$; thanks to the properties of the wave equation we know that $\widetilde W$ is a solution of the problem

$$\begin{cases} \widetilde W_{tt}-\widetilde W_{xx}+\sin\tilde u_1-\sin\tilde u_2= \\ \qquad =\Big(\delta_a(x)+\delta_{2\beta-a}(x)\Big)\,(\sin\tilde u_1-\sin\tilde u_2), \quad \text{for } \ (t,x)\in[0,T)\times\,]\alpha,2\beta-\alpha[\ , \\ \widetilde W(0,x)=0, \ \ \widetilde W_t(0,x)=0, \quad \text{for } \ x\in\,]\alpha,2\beta-\alpha[\ , \end{cases}$$

In particular, in the triangle $\widetilde{\mathcal{T}_r}$, the function $\widetilde W$ is a solution of the wave equation

$$\widetilde W_{tt}-\widetilde W_{xx}+2\sin\Big(\frac{\widetilde W}{2}\Big)\cos\Big(\frac{\tilde u_1+\tilde u_2}{2}\Big)=0$$

with zero initial data; so $\widetilde W=0$ in the triangle $\widetilde{\mathcal{T}_r}$. This proves that W is zero on $\mathcal{T}_r$. Similarly W is zero on $\mathcal{T}_l$ defined by $(0,\alpha)$, $(0,a)$, $(a-\alpha,\alpha)$.

Next, the triangle $\mathcal{T}_m$ (for *middle triangle*) defined by $(0,a)$, $(t_0,a-t_0)$, $(t_0,a+t_0)$ is contained in $I\times\,]0,T[$ for $t_0<\frac12\min(a-\alpha,\beta-a)$ (indeed, we take t_0 small enough to avoid the influence of boundary conditions on $\mathcal{T}_m$). Since $\mathcal{T}_m$ is a characteristic triangle for the equation satisfied by W we may apply **2.2** to obtain an expression of W for all $(t,x)\in\mathcal{T}_m$,

$$\begin{aligned} W(t,x)=&-\frac12\int_0^t ds\int_{x-(t-s)}^{x+t-s}\Big(\sin u_1(s,\sigma)-\sin u_2(s,\sigma)\Big)\,d\sigma \\ &+\frac12\int_0^{t-|x-a|}\sin u_1(s,a)-\sin u_2(s,a)\,ds\ . \end{aligned}$$

Then,

$$\begin{aligned} W(t,x)=&-\int_0^t ds\int_{x-(t-s)}^{x+t-s}\cos\Big(\frac{u_1(s,\sigma)+u_2(s,\sigma)}{2}\Big)\sin\Big(\frac{W(s,\sigma)}{2}\Big)\,d\sigma \\ &+\int_0^{t-|x-a|}\cos\Big(\frac{u_1(s,a)+u_2(s,a)}{2}\Big)\sin\Big(\frac{W(s,a)}{2}\Big)\,ds\ , \end{aligned}$$

and

$$\|W(t)\|_{L^\infty(I)} \leq C \int_0^t \|W(s)\|_{L^\infty(I)}\, ds \ ,$$

where C depends only on t_0.

Since $\|W(0)\|_{L^\infty(I)} = 0$ we obtain $\|W(t)\|_{L^\infty(I)} = 0$ (a.e.) by Gronwall's lemma. This shows uniqueness on $]0, t_0[\times I$. Uniqueness being a local property we obtain uniqueness on $]0, T[\times I$ in the space $L^\infty((0,T), H_0^1(I)) \cap W^{1,\infty}((0,T), L^2(I))$ for every $0 < T \leq \infty$.

3.2. Solution of the regularised equation

For every (u_0, v_0) in $H_0^1(I) \times L^2(I)$, and for every $n > 0$, $T > 0$, there exists a unique solution $u^n \in C([0,T], H_0^1(I)) \cap C^1([0,T], L^2(I))$ of the problem

$$\text{(3.1)} \qquad \begin{cases} u^n_{tt} - u^n_{xx} + \sin u^n = \rho^n_a(x)\, \sin u^n, & \textit{for } \ (t,x) \in [0, T[\, \times I \ , \\ u^n(0,x) = u_0(x), \quad u^n_t(0,x) = v_0(x), & \textit{for } \ x \in I \ , \end{cases}$$

moreover

$$\int_I \left\{ \frac{1}{2}\, |u^n_t(t,x)|^2 + \frac{1}{2}\, |u^n_x(t,x)|^2 - (1 - \rho^n_a(x))\, \cos u^n(t,x) \right\} dx = \\ = \int_I \left\{ \frac{1}{2}\, |v_0(x)|^2 + \frac{1}{2}\, |u_{0x}(x)|^2 - (1 - \rho^n_a(x))\, \cos u_0(x) \right\} dx \ ,$$

where $\rho^n_a \in \mathcal{D}(\mathbf{R})$, defined by

$$\begin{cases} \rho^n_a(x) = n \exp\left(\dfrac{1}{n(x-a)^2 - 1} \right), & \text{for } x \in \,]-1/n + a, a + 1/n[, \\ \rho^n_a(x) = 0 & \text{otherwise} \ , \end{cases}$$

is an approximation of δ_a.

This result follows immediately from **2.1** with $g(x,y) = -\sin(y)\,(1 - \rho^n_a(x))$ which verifies the right condition.

3.3. Existence

We are going to build a solution of the problem (1.1) as the limit of the sequence of functions (u^n).

A priori estimates. *For every $n > 0$ the solution u^n of the problem (3.1) verifies the following estimates:*

$$\|u^n\|_{L^\infty((0,T),H_0^1(I))} \leq C \ ,$$
$$\|u_t^n\|_{L^\infty((0,T),L^2(I))} \leq C' \ ,$$

where C and C' do not depend on n.

As a matter of fact,

$$\frac{1}{2}\int_I \Big\{ |u_t^n(t,x)|^2 + |u_x^n(t,x)|^2 \Big\}\, dx =$$
$$= \int_I (1-\rho_a^n(x))\ \cos u^n(t,x)\, dx$$
$$+ \int_I \Big\{ \frac{1}{2}\, |v_0(x)|^2 + \frac{1}{2}\, |u_{0x}(x)|^2 - (1-\rho_a^n(x))\ \cos u_0(x) \Big\}\, dx \ ,$$

Hence

$$\int_I |u_t^n(t,x)|^2 + |u_x^n(t,x)|^2\, dx \leq C_1 + 4|I| + 4 \ ,$$

where C_1 depends only on $|I|$ and initial data.

We derive

$$\|u^n(t)\|_{H_0^1(I)} \leq C_2 \|u_x^n(t)\|_{L^2(I)} \leq C \ ,$$

and

$$\|u_t^n(t)\|_{L^2(I)} \leq C' \ ,$$

for every $t \in [0,T)$.

Convergence of the sequence. *There exists $u \in L^\infty((0,T),H_0^1(I))$ such that:*

i) $u_t \in L^\infty((0,T),L^2(I))$,

ii) *u weakly continuous on $H_0^1(I)$ and u_t weakly continuous on $L^2(I)$,*

iii) *u is solution of (1.1).*

Indeed, according to the above a priori estimates, there exists a subsequence which we will denote by $(u^\mu)_{\mu\in\mathbf{N}}$ and a function $u \in L^\infty((0,T),H_0^1(I))$, weakly continuous on $H_0^1(I)$, such that u_t belongs to $L^\infty((0,T),L^2(I))$ and

- $u^\mu \rightharpoonup u$ on $L^\infty((0,T),H_0^1(I))$ weak $*$,
- $u_t^\mu \rightharpoonup u_t$ on $L^\infty((0,T),L^2(I))$ weak $*$,

- $u^\mu \to u$ on $L^2((0,T)\times I)$ and a.e. on $(0,T)\times I$.

Now we prove that u is a solution of the problem (1.1) by passing to the limit in the approximate equation

$$u^n_{tt} - u^n_{xx} + \sin u^n = \rho^n_a(x)\,\sin u^n, \qquad \text{for} \quad (t,x) \in [0,T)\times I\ ,$$

According to the above remarks we have

$$u^\mu_{tt} - u^\mu_{xx} + \sin u^\mu \underset{\mu\to\infty}{\longrightarrow} u_{tt} - u_{xx} + \sin u, \qquad \text{in} \quad \mathcal{D}'((0,T)\times I)\ .$$

Next we show that $\rho^a_\mu \sin u^\mu \to \delta_a \sin u$ in $\mathcal{D}'((0,T)\times I)$ by using the following lemma.

Lemma 3.1. *Let $(x_n)_{n\in\mathbf{N}}$ be a bounded sequence of functions of $L^\infty((0,T),H^1_0(I)) \cap W^{1,\infty}((0,T),L^2(I))$. Then there exists a subsequence of $(x_n)_{n\in\mathbf{N}}$ which converges in $L^\infty((0,T),L^\infty(I))$.*

(See [6] Corollary 4, p. 85.)

Using this lemma we can assume that the subsequence $(u_\mu)_{\mu\in\mathbf{N}}$ also converges in $L^\infty((0,T),L^\infty(I))$.

Let $\varphi \in H^1_0(I)$,

$$\begin{aligned}\int_I \rho^\mu_a(x)\,\sin u^\mu(t,x)\,\varphi(x)\,dx &= \\ = \sin u(t,a)\,\varphi(a) &+ \int_I \rho^\mu_a(x)\Big(\sin u^\mu(t,x)\,\varphi(x) - \sin u(t,x)\,\varphi(x)\Big)\,dx \\ &+ \int_I \rho^\mu_a(x)\Big(\sin u(t,x)\,\varphi(x) - \sin u(t,a)\,\varphi(a)\Big)\,dx\ .\end{aligned}$$

Besides,

$$\Big|\int_I \rho^\mu_a(x)\Big(\sin u^\mu(t,x)\,\varphi(x) - \sin u(t,x)\,\varphi(x)\Big)\,dx\Big| \le \\ \le \|\varphi\|_{L^\infty(I)}\,\|u^\mu - u\|_{L^\infty((0,T),L^\infty(I))}\ ,$$

and

$$\Big|\int_I \rho^\mu_a(x)\Big(\sin u(t,x)\,\varphi(x) - \sin u(t,a)\,\varphi(a)\Big)\,dx\Big| \le \\ \le \Big\|u(t,x)\,\varphi(x) - u(t,a)\varphi(a)\Big\|_{L^\infty((0,T),L^\infty(]-1/\mu+a,\,a+1/\mu[))}\ ,$$

By uniform convergence of (u^μ) and by continuity of u on $[0,T]\times I$, we obtain

$$\rho_a^\mu \sin u^\mu \underset{\mu\to\infty}{\longrightarrow} \delta_a \sin u \quad \text{in} \;\; L^\infty((0,T),H^{-1}(I)) \ ,$$

and u verifies the equation (1.1) in the sense of distributions.

We conclude this section, remarking that $u_{tt} = u_{xx} - (1-\delta_a(x))\sin u$, so $u_{tt} \in L^\infty((0,T),H^{-1}(I))$ and there exists a subsequence $(u^{\mu_k})_{\mu_k\in\mathbf{N}}$ such that $u_{tt}^{\mu_k} \to u_{tt}$ in $L^\infty((0,T),H^{-1}(I))$. It follows that $u_t(0)=v_0$. We conclude that u is the unique solution of (1.1).

3.4. Regularity

We show that $E(u(t),u_t(t)) = E(u_0,v_0)$ for every $t\in\,]0,T[$, then we derive the announced regularity result from this conservation law.

As a matter of fact we have conservation of energy for every function u^μ, id est:

$$\int_I \Big\{ |u_t^\mu(t,x)|^2 + |u_x^\mu(t,x)|^2 - 2(1-\rho_a^\mu(x))\,\cos u^\mu(t,x) \Big\}\,dx =$$

$$= \int_I \Big\{ |v_0(x)|^2 + |u_{0x}(x)|^2 - 2(1-\rho_a^\mu(x))\,\cos u_0(x) \Big\}\,dx \ .$$

The right term tends to $E(u_0,v_0)$ when $\mu\uparrow\infty$. Moreover from what precedes it is clear that

$$\int_I (1-\rho_a^\mu(x))\,\cos u^\mu(t,x)\,dx \underset{\mu\to\infty}{\longrightarrow} \int_I \cos u(t,x)\,dx - \cos u(t,a) \ .$$

Besides, it follows from Fatou's lemma that

$$\int_I \Big\{ |u_t(t,x)|^2 + |u_x(t,x)|^2 \Big\}\,dx \;\le\; \liminf_{\mu\to\infty} \int_I \Big\{ |u_t^\mu(t,x)|^2 + |u_x^\mu(t,x)|^2 \Big\}\,dx \ .$$

So that $E(u(t),u_t(t)) \le E(u_0,v_0)$ for every $t\in[0,T)$.

If we solve the backward problem with $(u(t),u_t(t))$ as initial data and taking into account the uniqueness of the solution on $[0,t]$ we conclude $E(u_0,v_0)\le E(u(t),u_t(t))$. This gives the conservation of energy.

The regularity of u is a consequence of the following lemma.

Lemma 3.2. *If $(f_k)_{1\le k\le n}$ are lower semicontinuous functions on $[0,T]$ and if $\sum_{1\le k\le n} f_k$ is continuous, then each f_k is continuous.*

We know that:

$$\int_I \Big\{ |u_t(t,x)|^2 + |u_x(t,x)|^2 - 2\cos u(t,x) \Big\}\,dx + 2\cos u(t,a) = 2E(u_0,v_0) \ .$$

Each term of the left member is lower semicontinuous in time, according to the above lemma each term is continuous. So the functions $\int_I |u_x(t,x)|^2\,dx$ and $\int |u_t(t,x)|^2\,dx$ are continuous in time; taking into account the weak continuity of the functions u and u_t we have shown that for an arbitrary $T>0$, u belongs to $C((0,T),H_0^1(I))\cap C^1((0,T),L^2(I))$. This completes the proof of Theorem 1.1. ∎

4 – Extensions

4.1. More general nonlinearities

We consider $h,f\in C(I\times\mathbf{R},\mathbf{R})$. We note $H(x,y)=\int_0^y h(x,z)\,dz$, $F(x,y)=\int_0^y f(x,z)\,dz$ and we suppose

- $|h(x,y)-h(x,z)|\le C_1|y-z|$, $|f(x,y)-f(x,z)|\le C_2|y-z|$ for all $(x,y,z)\in I\times\mathbf{R}^2$,
- $f(x,0)=h(x,0)=0$ for all $x\in J$.

Then, we have the following result

Theorem 4.1. *If* $(u_0,v_0)\in H_0^1(I)\times L^2(I)$ *then there exists a unique solution* $u\in C(\mathbf{R}_+,H_0^1(I))\cap C^1(\mathbf{R}_+,L^2(I))\cap C^2(\mathbf{R}_+,H^{-1}(I))$ *of*

$$(4.1)\qquad \begin{cases} u_{tt}-u_{xx}+h(x,u)=\delta_a(x)\,f(x,u), & \text{for } (t,x)\in[0,\infty)\times I\ , \\ u(0,x)=u_0(x),\quad u_t(0,x)=v_0(x), & \text{for } x\in I\ , \end{cases}$$

where $I=\,]\alpha,\beta[$ *is a bounded, open interval of* $\mathbf{R}$ *and* $a\in I$. *In addition*

$$(4.2)\qquad \begin{aligned} E(u(t),u_t(t)) &= \int_I\left\{\frac{1}{2}\,|u_t(t,x)|^2+\frac{1}{2}\,|u_x(t,x)|^2+H(x,u(t,x))\right\}dx-F(a,u(t,a)) \\ &= E(u_0,v_0)\ . \end{aligned}$$

Proof: The proof is an adaptation of the proof of Theorem 1.1. Note that the conditions on h and f are not optimum but it is all what we need for the two next sections. ∎

4.2. More general boundary conditions

We consider $I=\,]\alpha,\beta[$ a bounded, open interval of $\mathbf{R}$ and $a\in I$. We show the following result

Theorem 4.2. *If* $(u_0,v_0)\in H^1(I)\times L^2(I)$ *and* $u_0(\alpha)=K$, $u_0(\beta)=L$ *then there exists a unique solution* $u\in C(\mathbf{R}_+,H^1(I))\cap C^1(\mathbf{R}_+,L^2(I))\cap C^2(\mathbf{R}_+,H^{-1}(I))$ *of*

$$\begin{cases} u_{tt}-u_{xx}+\sin u=\delta_a(x)\,\sin u, & \text{for } (t,x)\in[0,\infty)\times I\ ,\\ u(0,x)=u_0(x),\quad u_t(0,x)=v_0(x), & \text{for } x\in I\ ,\\ u(t,\alpha)=K,\quad u(t,\beta)=L, & \text{for } t\geq 0\ . \end{cases} \tag{4.3}$$

In addition

$$\begin{aligned} E(u(t),u_t(t)) &= \int_I \left\{\frac{1}{2}\,|u_t(t,x)|^2+\frac{1}{2}\,|u_x(t,x)|^2-\cos u(t,x)\right\}dx+\cos u(t,a)\\ &= E(u_0,v_0)\ . \end{aligned}$$

Proof: Let A and B be such that $A\alpha+B=K$ and $A\beta+B=L$. From Theorem 4.1 we know that there exists a unique solution $v\in C(\mathbf{R}_+,H_0^1(I))\cap C^1(\mathbf{R}_+,L^2(I))$ of

$$\begin{cases} v_{tt}-v_{xx}+\sin(v+Ax+B)=\delta_a(x)\,\sin(v+Ax+B), & \text{for } (t,x)\in[0,\infty)\times I\ ,\\ v(0,x)=u_0(x)-Ax-B,\quad v_t(0,x)=v_0(x), & \text{for } x\in I\ . \end{cases}$$

Set $u=v+Ax+B$, then u is a solution of (4.3) and u belongs to $C(\mathbf{R}_+,H^1(I))\cap C^1(\mathbf{R}_+,L^2(I))\cap C^2(\mathbf{R}_+,H^{-1}(I))$. ∎

4.3. Unbounded domain

For $I=\mathbf{R}$ we have the following theorem

Theorem 4.3. *If* $(u_0,v_0)\in H^1(\mathbf{R})\times L^2(\mathbf{R})$ *then there exists a unique solution* $u\in C(\mathbf{R}_+,H^1(\mathbf{R}))\cap C^1(\mathbf{R}_+,L^2(\mathbf{R}))\cap C^2(\mathbf{R}_+,H^{-1}(\mathbf{R}))$ *of*

$$\begin{cases} u_{tt}-u_{xx}+\sin u=\delta_a(x)\,\sin u, & \text{for } (t,x)\in[0,\infty)\times\mathbf{R}\ ,\\ u(0,x)=u_0(x),\quad u_t(0,x)=v_0(x), & \text{for } x\in\mathbf{R}\ , \end{cases} \tag{4.4}$$

where $a\in\mathbf{R}$. *In addition*

$$\begin{aligned} E(u(t),u_t(t)) &= \int_{\mathbf{R}} \left\{\frac{1}{2}\,|u_t(t,x)|^2+\frac{1}{2}\,|u_x(t,x)|^2+1-\cos(u(t,x))\right\}dx+\cos(u(t,a))\\ &= E(u_0,v_0)\ . \end{aligned}$$

Proof: We proceed in three steps.

Step 1. We first consider $w \in C(\mathbf{R}_+, H^1(\mathbf{R})) \cap C^1(\mathbf{R}_+, L^2(\mathbf{R}))$ solution of

$$\text{(4.5)} \qquad \begin{cases} w_{tt} - w_{xx} + \sin w = 0, & \text{for } \ (t,x) \in [0,\infty) \times \mathbf{R}\ , \\ w(0,x) = u_0(x), \quad w_t(0,x) = v_0(x), & \text{for } \ x \in \mathbf{R}\ . \end{cases}$$

We use **2.1** with $g(x,y) = \sin y$, which verifies the additional property $g(x,0) = 0$.

Step 2. We consider $T > 0$ and we define $I =]-4T+a, a+4T[$. Let $(w_0, z_0) \in H_0^1(I) \times L^2(I)$ be such that $w_0 = u_0$ and $z_0 = v_0$ on $]-3T+a, a+3T[$. According to Theorem 1.1 there exits a unique solution $v \in C([0,T], H_0^1(I)) \cap C^1([0,T], L^2(I))$ of

$$\begin{cases} v_{tt} - v_{xx} + \sin v = \delta_a(x)\, \sin v, & \text{for } \ (t,x) \in [0,T] \times I\ , \\ v(0,x) = w_0(x), \quad v_t(0,x) = z_0(x), & \text{for } \ x \in I\ . \end{cases}$$

We will note $\mathcal{T}_1$ the triangle defined by $(0, a-T)$, $(0, a-3T)$, $(T, a-2T)$ and $\mathcal{T}_2$ the triangle defined by $(0, a+T)$, $(0, a+3T)$, $(T, a+2T)$. We claim that $w = v$ on $\mathcal{T}_1 \cup \mathcal{T}_2$. Indeed, w and v are solutions of the same wave equation with the same initial data in these two characteristic triangles.

Step 3. We note $\mathcal{M}$ the trapezium defined by $(0, a-T)$, $(0, a+T)$, $(T, a+2T)$, $(T, a-2T)$ and $\mathcal{N} = (]0,T[\, \times \mathbf{R}) \backslash (\mathcal{T}_1 \cup \mathcal{M} \cup \mathcal{T}_2)$. According to step 2 we may build the solution u as follows

- $u = v$ on $\mathcal{M}$,
- $u = v = w$ on $\mathcal{T}_1 \cup \mathcal{T}_2$,
- $u = w$ on $\mathcal{N}$.

Since T is arbitrary, we obtain a solution of the problem (4.4) in $\mathbf{R}_+ \times \mathbf{R}$. We know that the solution u is unique by the argument of **3.1**. This completes the proof of Theorem 4.3. ∎

We consider a function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(x) = 2\pi$ for $x > 1$ and $\varphi(x) = 0$ for $x < -1$. We give a last result.

Theorem 4.4. *If $(u_0, v_0) \in H^1_{loc}(\mathbf{R}) \times L^2(\mathbf{R})$ and $u_0 - \varphi \in H^1(\mathbf{R})$ then there exists a unique solution $u \in C(\mathbf{R}_+, H^1_{loc}(\mathbf{R})) \cap C^1(\mathbf{R}_+, L^2(\mathbf{R})) \cap C^2(\mathbf{R}_+, H^{-1}(\mathbf{R}))$ such that $u - \varphi \in C(\mathbf{R}_+, H^1(\mathbf{R}))$ of*

$$\text{(4.6)} \qquad \begin{cases} u_{tt} - u_{xx} + \sin u = \delta_a(x)\, \sin u, & \text{for } \ (t,x) \in [0,\infty) \times \mathbf{R}\ , \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), & \text{for } \ x \in \mathbf{R}\ , \end{cases}$$

where $a \in \mathbf{R}$. In addition

$$\begin{aligned} E(u(t), u_t(t)) &= \int_{\mathbf{R}} \left\{ \frac{1}{2} |u_t(t,x)|^2 + \frac{1}{2} |u_x(t,x)|^2 + 1 - \cos u(t,x) \right\} dx + \cos u(t,a) \\ &= E(u_0, v_0) \ . \end{aligned}$$

Remark 4.5. Note that the solution obtained by the above theorem does not depend on the choice of φ. Indeed the introduction of the function φ is only a way to formulate the boundary conditions at $+\infty$ and $-\infty$.

Proof of Theorem 4.4: One can easily verify that Theorem 4.1 is also available for $I = \mathbf{R}$ following the proof of Theorem 4.3. So there exists a unique solution $v \in C(\mathbf{R}_+, H^1(\mathbf{R})) \cap C^1(\mathbf{R}_+, L^2(\mathbf{R})) \cap C^2(\mathbf{R}_+, H^{-1}(\mathbf{R}))$ of

$$\begin{cases} v_{tt} - v_{xx} + \sin(v + \varphi(x)) = \delta_a(x) \, \sin(v + \varphi(x)) + \varphi''(x), & \text{for} \quad (t,x) \in [0,\infty) \times \mathbf{R} \ , \\ v(0,x) = u_0(x) - \varphi(x), \quad v_t(0,x) = v_0(x), & \text{for} \quad x \in \mathbf{R} \ . \end{cases}$$

Set $u = v + \varphi$, then u is the unique solution of (4.6) in $C(\mathbf{R}_+, H^1_{loc}(\mathbf{R})) \cap C^1(\mathbf{R}_+, L^2(\mathbf{R})) \cap C^2(\mathbf{R}_+, H^{-1}(\mathbf{R}))$ and $v = u - \varphi \in C(\mathbf{R}_+, H^1(\mathbf{R}))$. ∎

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