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NOTE ON THE CHEBYSHEV POLYNOMIALS AND APPLICATIONS TO THE FIBONACCI NUMBERS

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Abstract: In [12], Gheorghe Udrea generalizes a result obtained in [8], by showing that, if $(U_n)_{n\geq 0}$ is the sequence of Chebyshev polynomials of the second kind, then the product of any two distinct elements of the set

$$\Big\{U_n,\, U_{n+2r},\, U_{n+4r},\, 4U_{n+r}U_{n+2r}U_{n+3r}\Big\}\,, \qquad r,n\in\mathbb{N}\ ,$$

increased by $U_a^2 U_b^2$, for suitable nonnegative integers a and b, is a perfect square.

In this note, one obtains a similar result for the Chebyshev polynomials of the first kind and one states some generalizations of results contained in [12] and in [8].

1 – Preliminaries

Diophantus raised the following problem ([4], pp. 179–181):

"*To find four numbers such that the product of any two increased by unity is a square*" ,

for which he obtained the solution $\frac{1}{16}$, $\frac{33}{16}$, $\frac{68}{16}$, $\frac{105}{16}$.

Fermat ([3], p. 251) found the solution 1, 3, 8, 120.

In 1968, J.H. van Lint raised the problem whether the number 120 is the unique (positive) integer n for which the set $\{1, 3, 8, 120\}$ constitutes a solution for Diophantus' problem above; in the same year, A. Baker and H. Davenport [1] studied this question and concluded that, in fact, 120 is the unique integer satisfying the problem raised by J.H. van Lint.

In 1977, V.E. Hoggatt and G.E. Bergum [5] observed that 1, 3, 8 are, respectively, the terms F_2, F_4, F_6, of the Fibonacci sequence $(F_n)_{n\geq 0}$, defined by the conditions

$$F_0=0\,,\quad F_1=1\quad \text{and}\quad F_{n+2}=F_{n+1}+F_n,\ \ n\geq 0\ ,$$

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and formulated the problem of finding a positive integer n such that

$$F_{2t}n+1\,,\qquad F_{2t+2}n+1\,,\qquad F_{2t+4}n+1$$

be perfect squares.

Hoggatt and Bergum obtained the number

$$n = 4\,F_{2t+1}\,F_{2t+2}\,F_{2t+3}\ ,$$

which, for $t=1$, gives exactly $n=120$.

In 1984, this result was generalized ([8], p. 443), by showing that the product of any two distinct elements of the set

$$\Big\{F_n,\, F_{n+2r},\, F_{n+4r},\, 4F_{n+r}F_{n+2r}F_{n+3r}\Big\}\ ,$$

increased by $\pm F_a^2 F_b^2$ (for suitable integers a and b) is a perfect square, i.e., this set is a *Fibonacci quadruple*.

In 1987, this result was generalized by A.F. Horadam [6], who proved that the product of any two distinct elements of the set

$$\Big\{w_n,\, w_{n+2r},\, w_{n+4r},\, 4w_{n+r}w_{n+2r}w_{n+3r}\Big\}\ ,$$

increased by a suitable integer, is a perfect square, i.e., this set is a *Diophantine quadruple, not being necessarily a Fibonacci quadruple*.

The sequence $(w_n)_{n\geq 0}$ was introduced, in 1965, by A.F. Horadam [7]:

$$w_n = w_n(a,b;p,q),\quad w_0=a,\quad w_1=b \quad\text{and}\quad w_n = p\,w_{n-1} - q\,w_{n-2}\ ,$$

with a, b, p, q integers, and $n\geq 2$. This sequence generalizes the sequence $(F_n)_{n\geq 0}$, since one has $F_n = w_n(0,1;1,-1)$.

In the paper of Gheorghe Udrea [12], one obtains another generalization of the result contained in [8], by means of the Chebyshev polynomials of the second kind.

The sequence of Chebyshev polynomials of the first kind is the sequence $(T_n(x))_{n\geq 0}$, where $x\in\mathbb{C}$, defined by the recurrence relation

$$T_{n+2}(x) = 2\,x\,T_{n+1} - T_n(x)\ , \tag{1.1}$$

with $T_0(x)=1$ and $T_1(x)=x$. Thus, one has

$$T_2(x) = 2x^2-1,\quad T_3(x)=4x^3-3x,\quad T_4(x) = 8x^4-8x^2+1,\quad \ldots\ .$$

The sequence of Chebyshev polynomials of the second kind is the sequence $(U_n(x))_{n\geq 0}$, where $x \in \mathbb{C}$, defined by the same recurrence relation

$$U_{n+2}(x) = 2x\,U_{n+1}(x) - U_n(x) \;, \tag{1.2}$$

with $U_0(x) = 1$, and $U_1(x) = 2x$. Thus, one has

$$U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \quad U_4(x) = 16x^4 - 12x^2 + 1, \quad \dots \,.$$

The (ordinary) generating function of $(T_n(x))_{n\geq 0}$ is the formal series

$$g_1(y) = T_0(x) + T_1(x)\,y + T_2(x)\,y^2 + \dots + T_n(x)\,y^n + \dots \,. \tag{1.3}$$

By taking into account the recurrence relation (1.1), we are led to consider the reducing polynomial

$$k(y) = 1 - 2\,x\,y + y^2 \;.$$

One has clearly

$$\begin{aligned} g_1(y)\,k(y) &= \Big[T_0(x) + T_1(x)\,y + T_2(x)\,y^2 + \dots + T_n(x)\,y^n + \dots\Big]\,(1 - 2xy + y^2) \\ &= T_0(x) + \Big[T_1(x) - 2x\,T_0(x)\Big]y + \dots \\ &\quad + \dots + \Big[T_n(x) - 2x\,T_{n-1}(x) + T_{n-2}(x)\Big]y^n + \dots = 1 - xy \;, \end{aligned}$$

since, by (1.1), $T_n(x) - 2x\,T_{n-1}(x) + T_{n-2}(x)$ is the zero polynomial for $n \geq 2$. Thus, one obtains the generating function, $g_1(y)$, under a finite form,

$$g_1(y) = \frac{1 - xy}{1 - 2xy + y^2} \;,$$

which can be written as

$$\begin{aligned} g_1(y) &= \frac{1 - xy}{\Big[y - (x + \sqrt{x^2 - 1})\Big]\,\Big[y - (x - \sqrt{x^2 - 1})\Big]} \\ &= \frac{A}{y - (x + \sqrt{x^2 - 1})} + \frac{B}{y - (x - \sqrt{x^2 - 1})} \;, \end{aligned}$$

where

$$\begin{cases} A + B = -x, \\ A(x - \sqrt{x^2 - 1}) + B(x + \sqrt{x^2 - 1}) = -1 \;. \end{cases}$$

From this, it follows (with $x \neq \pm 1$) that

$$A = \frac{1 - x^2 - x\,\sqrt{x^2 - 1}}{2\sqrt{x^2 - 1}} \quad \text{and} \quad B = \frac{1 - x^2 + x\,\sqrt{x^2 - 1}}{2\sqrt{x^2 - 1}} \;,$$

and, consequently,

$$\begin{aligned}g_1(y) &= \frac{1}{2\sqrt{x^2-1}}\left[\frac{\sqrt{x^2-1}}{1-(x+\sqrt{x^2-1})\,y}+\frac{\sqrt{x^2-1}}{1-(x-\sqrt{x^2-1})\,y}\right]\\ &= \frac{1}{2}\Big[1+(x+\sqrt{x^2-1})\,y+(x+\sqrt{x^2-1})^2\,y^2+...+(x+\sqrt{x^2-1})^n\,y^n+...\Big]\\ &\quad+\frac{1}{2}\Big[1+(x-\sqrt{x^2-1})\,y+(x-\sqrt{x^2-1})^2\,y^2+...+(x-\sqrt{x^2-1})^n\,y^n+...\Big]\,.\end{aligned}$$

Since, by (1.3) $T_n(x)$ is the coefficient of y^n, one concludes that

$$T_n(x) = \frac{1}{2}\Big[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\Big]\,. \tag{1.4}$$

For the Chebyshev polynomials of the second kind, one finds, by a similar way, the corresponding generating function, under a finite form (with $x \neq \pm 1$):

$$\begin{aligned}g_2(y) &= \frac{1}{y^2-2xy+1}\\ &= \frac{1}{2\sqrt{x^2-1}}\left[\frac{x+\sqrt{x^2-1}}{1-(x+\sqrt{x^2-1})\,y}-\frac{x-\sqrt{x^2-1}}{1-(x-\sqrt{x^2-1})\,y}\right]\end{aligned}$$

and one obtains, after the developments in power series of

$$\frac{x+\sqrt{x^2-1}}{1-(x+\sqrt{x^2-1})\,y} \quad \text{and} \quad \frac{x-\sqrt{x^2-1}}{1-(x-\sqrt{x^2-1})\,y}\,,$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\Big[(x+\sqrt{x^2-1})_{n+1} - (x-\sqrt{x^2-1})_{n+1}\Big]\,. \tag{1.5}$$

Since, for each $x \in \mathbb{C}$, there is some $\theta \in \mathbb{C}$ such that $x = \cos\theta$, one can write

$$T_n(\cos\theta) = \frac{1}{2}\Big[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\Big] = \cos n\,\theta\,, \tag{1.6}$$

$$\begin{aligned}U_n(\cos\theta) &= \frac{1}{2i\sin\theta}\Big[(\cos\theta+i\sin\theta)^{n+1} - (\cos\theta-i\sin\theta)^{n+1}\Big]\\ &= \frac{\sin(n+1)\,\theta}{\sin\theta}\,.\end{aligned} \tag{1.7}$$

By means of the relations (1.6) and (1.7), it is easy to see that the following connections, between the two kinds of Chebyshev polynomials, hold:

$$T_n(x) = U_n(x) - xU_{n-1}(x)\,, \qquad n \geq 1\,, \tag{1.8}$$

$$(1-x^2)\,U_n(x) = x\,T_{n+1}(x) - T_{n+2}(x)\,, \qquad n \geq 0\,, \tag{1.9}$$

$$T^2_{n+1}(x) = 1 + (x^2-1)\,U^2_n(x)\,, \qquad n \geq 0\,. \tag{1.10}$$

The Chebyshev polynomials, $T_n(x)$ and $U_n(x)$, are special ultraspherical (or Gegenbauer) polynomials. The ultraspherical polynomials are special cases of the Jacobi polynomials, i.e., of the polynomials $P_n^{(\alpha,\beta)}(x)$ such that ([11], pp. 71–73),

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n\,(1-x)^{-\alpha}\,(1+x)^{-\beta}}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}\Big[(1-x)^{\alpha+n}\cdot(1+x)^{\beta+n}\Big]\ .$$

The ultraspherical polynomials are the Jacobi polynomials, for which one has $\alpha=\beta$; for the Chebyshev polynomials of the first kind, one has $\alpha=\beta=-\frac{1}{2}$ and, for the Chebyshev polynomials of the second kind, one has $\alpha=\beta=\frac{1}{2}$.

By taking into account (1.6), it is natural to extend the meaning of T_n for $n<0$: one puts

$$T_{-r}(x) = T_{-r}(\cos\theta) = \cos(-r)\,\theta = \cos r\,\theta = T_r(\cos\theta) = T_r(x)\ .$$

2 – Some properties of the Chebyshev polynomials of the first kind

In order to obtain, for the Chebyshev polynomials of the first kind, a result analogous to that obtained by Gheorghe Udrea for the Chebyshev polynomials of the second kind, we need to prove the following lemma:

Lemma 1. *If $(T_n(x))_{n\geq 0}$ is the sequence of Chebyshev polynomials of the first kind, then one has:*

$$T_n(x)\,T_{n+r+s}(x) + \frac{1}{2}\Big[T_{r-s}(x) - T_{r+s}(x)\Big] = T_{n+r}(x)\,T_{n+s}(x)\ . \tag{2.1}$$

$$\begin{aligned}4T_n(x)\,T_{n+r}(x)\,T_{n+s}(x)\,T_{n+r+s}(x) + \frac{1}{4}\Big[T_{r-s}(x) - T_{r+s}(x)\Big]^2 &= \\ = \Big[T_n(x)\,T_{n+r+s}(x) + T_{n+r}\,T_{n+s}(x)\Big]^2 &\ .\end{aligned} \tag{2.2}$$

Proof: (Sometimes, instead of $T_n(x)$, we shall write plainly T_n).

By setting $x=\cos\theta$ (and so $T_n=\cos n\,\theta$), one has

$$T_n\,T_{n+r+s} = \cos n\theta\,\cos(n+r+s)\,\theta = \frac{1}{2}\Big[\cos(2n+r+s)\,\theta + \cos(r+s)\,\theta\Big]$$

and

$$T_{n+r}\,T_{n+s} = \cos(n+r)\theta\,\cos(n+s)\theta = \frac{1}{2}\Big[\cos(2n+r+s)\,\theta + \cos(r-s)\,\theta\Big]$$

and, consequently,

$$T_n\,T_{n+r+s} - T_{n+r}\,T_{n+s} = \frac{1}{2}\Big[\cos(r+s)\,\theta - \cos(r-s)\,\theta\Big]\ .$$

Hence,

$$T_n\,T_{n+r+s} + \frac{1}{2}(T_{r-s} - T_{r+s}) = T_{n+r}\,T_{n+s}\ ,$$

which proves (2.1).

One has clearly

$$\frac{1}{4}(T_{r-s} - T_{r+s})^2 = T_{n+r}^2\,T_{n+s}^2 + T_n^2\,T_{n+r+s}^2 - 2T_n\,T_{n+r}\,T_{n+s}\,T_{n+r+s}$$

and so

$$4T_n\,T_{n+r}\,T_{n+s}\,T_{n+r+s} + \frac{1}{4}(T_{r-s} - T_{r+s})^2 = (T_n\,T_{n+r+s} + T_{n+r}\,T_{n+s})^2\ ,$$

which proves (2.2). ∎

Now, we are going to state the following

Theorem 1. *If $(T_n)_{n\geq 0}$ is the sequence of Chebyshev polynomials of the first kind, then the product of any two distinct elements of the set*

$$\Big\{T_n,\, T_{n+2r},\, T_{n+4r},\, 4T_{n+r}T_{n+2r}T_{n+3r}\Big\}\,, \qquad n, r \in \mathbb{N}\ ,$$

increased by $[\frac{1}{2}(T_h - T_k)]^t$, where T_h and T_k, with $k > h \geq 0$, are suitable terms of the sequence $(T_n)_{n\geq 0}$, and t is 1 or 2, according to the number of factors T, in that product, is 2 or 4, is a perfect square.

Proof: Indeed, if one sets $s = r$, in (2.1), one obtains

$$T_n\,T_{n+2r} + \frac{1}{2}(T_0 - T_{2r}) = T_{n+r}^2\ . \tag{2.3}$$

If r is replaced by $2r$, in (2.3), one gets

$$T_n\,T_{n+4r} + \frac{1}{2}(T_0 - T_{4r}) = T_{n+2r}^2\ . \tag{2.4}$$

By replacing, in (2.3) n by $n + 2r$, one obtains

$$T_{n+2r}\,T_{n+4r} + \frac{1}{2}(T_0 - T_{2r}) = T_{n+3r}^2\ . \tag{2.5}$$

If one puts $s = 2r$, in (2.2), one gets

$$4T_n\,T_{n+r}\,T_{n+2r}\,T_{n+3r} + \Big[\frac{1}{2}(T_r - T_{3r})\Big]^2 = (T_n\,T_{n+3r} + T_{n+r}\,T_{n+2r})^2 \ . \tag{2.6}$$

Now, by changing n into $n+r$, in (2.6), it comes

$$\begin{aligned} 4T_{n+r}\,T_{n+2r}\,T_{n+3r}\,T_{n+4r} + \Big[\frac{1}{2}(T_r - T_{3r})\Big]^2 &= \\ &= (T_{n+r}\,T_{n+4r} + T_{n+2r}\,T_{n+3r})^2 \ . \end{aligned} \tag{2.7}$$

If one replaces n by $n+r$, in (2.2), and, furthermore, one puts $s = r$, one obtains

$$4T_{n+r}\,T_{n+2r}^2\,T_{n+3r} + \Big[\frac{1}{2}(T_0 - T_{2r})\Big]^2 = (T_{n+r}\,T_{n+3r} + T_{n+2r}^2)^2 \ , \tag{2.8}$$

which completes the proof of the theorem above. ∎

3 – Applications to the Fibonacci numbers

There is a connection between, on the one hand, the sequence of Fibonacci numbers, $(F_n)_{n\geq 0}$, with

$$F_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big], \tag{3.1}$$

and, on the other hand, the sequences $(U_n)_{n\geq 0}$ and $(T_n)_{n\geq 0}$.

Indeed, from (1.5), it results

$$U_n\Big(\frac{i}{2}\Big) = \frac{1}{i\sqrt{5}}\Big[\Big(\frac{i}{2} + \frac{i}{2}\sqrt{5}\Big)^{n+1} - \Big(\frac{i}{2} - \frac{i}{2}\sqrt{5}\Big)^{n+1}\Big] = i^n\,F_{n+1} \ . \tag{3.2}$$

Now, from (1.8) and (3.2), one finds

$$T_n\Big(\frac{i}{2}\Big) = U_n\Big(\frac{i}{2}\Big) - \frac{i}{2}\,U_{n-1}\Big(\frac{i}{2}\Big) = \frac{i^n}{2}(2F_{n+1} - F_n)$$

and, since $F_{n+1} = F_n + F_{n-1}$, one has

$$T_n\Big(\frac{i}{2}\Big) = \frac{i^n}{2}(F_n + 2F_{n-1}) \ . \tag{3.3}$$

Thus, from (2.3) and (3.3), it follows

$$\begin{aligned} \frac{i^n}{2}(F_n + 2F_{n-1}) \cdot \frac{i^{n+2r}}{2}(F_{n+2r} + 2F_{n+2r-1}) + \frac{1}{2}\Big[1 - \frac{i^{2r}}{2}(F_{2r} + 2F_{2r-1})\Big] &= \\ &= \Big[\frac{i^{n+r}}{2}(F_{n+r} + 2F_{n+r-1})\Big]^2 \ , \end{aligned}$$

that is to say,

$$(-1)^{n+r}\Big(\frac{1}{4}\,F_n\,F_{n+2r}+\frac{1}{2}\,F_n\,F_{n+2r-1}+\frac{1}{2}\,F_{n-1}\,F_{n+2r}+F_{n-1}\,F_{n+2r-1}\Big)-$$
$$-\frac{1}{2}\Big[(-1)^r\Big(\frac{1}{2}\,F_{2r}+F_{2r-1}\Big)-1\Big]=(-1)^{n+r}\Big(\frac{1}{2}F_{n+1}+F_{n+r-1}\Big)^2\;,$$

and hence,

$$\begin{aligned}(3.4)\qquad & F_n\,F_{n+2r}+2F_{n+1}\,F_{n+2r-1}+2F_{n-1}\,F_{n+2r+1}+\\ & +2(-1)^{n+r}-(-1)^n\,(F_{2r}+2F_{2r-1})=F_{n+r}^2+4F_{n+r-1}\,F_{n+r+1}\;,\end{aligned}$$

and, analogously from the relations (2.4)–(2.8) and (3.3) other equalities can be obtained.

Other more interesting results can be obtained by making use of another connection between T_n and F_n. In fact, from (1.4), it results

$$\begin{aligned}T_n\Big(\frac{i}{2}\Big)&=\frac{i^n}{2}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n+\Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big]\\ &=\frac{i^n}{2}\Big\{\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{2n}-\Big(\frac{1-\sqrt{5}}{2}\Big)^{2n}\Big]\Big/\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n-\Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big]\Big\}\;,\end{aligned}$$

and hence, for $n>0$,

$$(3.5)\qquad T_n\Big(\frac{i}{2}\Big)=\frac{i^n}{2}\cdot\frac{F_{2n}}{F_n}\;.$$

Thus, from (2.3) and (3.5), one obtains

$$\frac{i^{2n+2r}}{4}\,\frac{F_{2n}}{F_n}\cdot\frac{F_{2n+4r}}{F_{n+2r}}+\frac{1}{2}\Big(1-\frac{i^{2r}}{2}\,\frac{F_{4r}}{F_{2r}}\Big)=\frac{i^{2n+2r}}{4}\Big(\frac{F_{2n+2r}}{F_{n+r}}\Big)^2\;,$$

whence,

$$(3.6)\qquad \frac{F_{2n}}{F_n}\cdot\frac{F_{2n+4r}}{F_{n+2r}}+(-1)^n\Big[2(-1)^r-\frac{F_{4r}}{F_{2r}}\Big]=\Big(\frac{F_{2n+2r}}{F_{n+r}}\Big)^2\;.$$

Analogously, from (2.4) and (3.5), it follows that

$$\frac{i^{2n+4r}}{4}\cdot\frac{F_{2n}}{F_n}\cdot\frac{F_{2n+8r}}{F_{n+4r}}+\frac{1}{4}\Big(2-i^{4r}\,\frac{F_{8r}}{F_{4r}}\Big)=\Big(\frac{i^{n+r}}{2}\cdot\frac{F_{2n+4r}}{F_{n+2r}}\Big)^2\;,$$

and, consequently, one has

$$(3.7)\qquad \frac{F_{2n}}{F_n}\cdot\frac{F_{2n+8r}}{F_{n+4r}}+(-1)^n\Big(2-\frac{F_{8r}}{F_{4r}}\Big)=\Big(\frac{F_{2n+4r}}{F_{n+2r}}\Big)^2\;.$$

By using (2.5) and (3.5), one obtains

$$(3.8)\qquad \frac{F_{2n+4r}}{F_{n+2r}}\cdot\frac{F_{2n+8r}}{F_{n+4r}}+(-1)^n\left(2(-1)^r-\frac{F_{4r}}{F_{2r}}\right)=\left(\frac{F_{2n+6r}}{F_{n+3r}}\right)^2 .$$

from (2.6) and (3.5), it results

$$(3.9)\qquad 4\cdot\frac{F_{2n}}{F_n}\cdot\frac{F_{2n+2r}}{F_{n+r}}\cdot\frac{F_{2n+4r}}{F_{n+2r}}\cdot\frac{F_{2n+6r}}{F_{n+3r}}+\left(\frac{F_{2r}}{F_r}-(-1)^r\,\frac{F_{6r}}{F_{3r}}\right)^2= \\ =\left(\frac{F_{2n}}{F_n}\cdot\frac{F_{2n+6r}}{F_{n+3r}}+\frac{F_{2n+2r}}{F_{n+r}}\cdot\frac{F_{2n+4r}}{F_{n+2r}}\right)^2.$$

from (2.7) and (3.5), one obtains

$$(3.10)\qquad 4\cdot\frac{F_{2n+2r}}{F_{n+r}}\cdot\frac{F_{2n+4r}}{F_{n+2r}}\cdot\frac{F_{2n+6r}}{F_{n+3r}}\cdot\frac{F_{2n+8r}}{F_{n+4r}}+\left(\frac{F_{2r}}{F_r}-(-1)^r\,\frac{F_{6r}}{F_{3r}}\right)^2= \\ =\left(\frac{F_{2n+2r}}{F_{n+r}}\cdot\frac{F_{2n+8r}}{F_{n+4r}}+\frac{F_{2n+4r}}{F_{n+2r}}\cdot\frac{F_{2n+6r}}{F_{n+3r}}\right)^2.$$

Finally, from (2.8) and (3.5), it results

$$(3.11)\qquad 4\cdot\frac{F_{2n+2r}}{F_{n+r}}\cdot\left(\frac{F_{2n+4r}}{F_{n+2r}}\right)^2\cdot\frac{F_{2n+6r}}{F_{n+3r}}+\left(2-(-1)^r\,\frac{F_{4r}}{F_{2r}}\right)^2= \\ =\left[\frac{F_{2n+2r}}{F_{n+r}}\cdot\frac{F_{2n+6r}}{F_{n+3r}}+\left(\frac{F_{2n+4r}}{F_{n+2r}}\right)^2\right]^2.$$

This means that the following holds:

Theorem 2. *If $(F_n)_{n\geq 0}$ is the sequence of Fibonacci numbers, then the product of any two distinct elements of the set*

$$(3.12)\qquad \left\{\frac{F_{2n}}{F_n},\ \frac{F_{2n+4r}}{F_{n+2r}},\ \frac{F_{2n+8r}}{F_{n+4r}},\ 4\cdot\frac{F_{2n+2r}}{F_{n+r}}\cdot\frac{F_{2n+4r}}{F_{n+2r}}\cdot\frac{F_{2n+6r}}{F_{n+3r}}\right\},\qquad \textit{with}\ \ n>0\ ,$$

increased by $\pm(\pm 2-\frac{F_{2h}}{F_h})$, if only 2 factors occur in that product; increased by $(\frac{F_{2l}}{F_l}-\frac{F_{2h}}{F_h})^2$, if 4 different factors occur in that product, and increased by $(2\pm\frac{F_{2h}}{F_h})^2$, if 4 factors occur in that product, but only three are different; h is the difference between the greatest and the least subscripts of F in the denominators of the factors and l is the difference between the subscripts of F in the denominators of the intermediate factors.

It is clear that the four integers belonging to the set (3.12) are not necessarily Fibonacci numbers and so the set (3.12) is a Diophantine quadruple, but, in general, it is not a Fibonacci quadruple.

In pursuance of a suggestion of the referee, we are going to present the results contained in Theorem 2, under another form, through the introduction of the Lucas numbers.

In [2], p. 395, L.E. Dickson says that E. Lucas

> "employed the roots a, b of $x^2 = x + 1$ and set
> $$u_n = \frac{a^n - b^n}{a - b}\,, \qquad v_n = a^n + b^n = \frac{u_{2n}}{u_n} = u_{n-1} + u_{n+1}\ .$$
> The u's form the series of Pisano [Fibonacci] with terms 0, 1 prefixed, so that $u_0 = 0$, $u_1 = u_2 = 1$, $u_3 = 2$."

The v's are the Lucas numbers.

One has $v_n = u_{n-1} + u_{n+1}$. In fact, the equality

$$a^n + b^n = \frac{a^{n-1} - b^{n-1}}{a - b} + \frac{a^{n+1} - b^{n+1}}{a - b}$$

is equivalent to

$$a^{n+1} - b^{n+1} + a\,b^n - a^n\,b = a^{n-1} - b^{n-1} + a^{n+1} - b^{n+1}\ ,$$

and this is equivalent to

$$a\,b(b^{n-1} - a^{n-1}) = a^{n-1} - b^{n-1}\ ,$$

which is true, since $a\,b = -1$.

One has also

$$v_n = a^n + b^n = \frac{(a^{2n} - b^{2n})/(a - b)}{(a^n - b^n)/(a - b)} = \frac{u_{2n}}{u_n} = \frac{F_{2n}}{F_n} = L_n\ ,$$

with $n > 0$.

Thus, by taking into account Theorem 2, one concludes that the following holds:

Theorem 2′. *If $(L_n)_{n>0}$ is the sequence of the Lucas numbers, then the product of any two distinct elements of the set*

$$\text{(3.12)}' \qquad \Big\{L_n,\, L_{n+2r},\, L_{n+4r},\, 4L_{n+r}L_{n+2r}L_{n+3r}\Big\}, \quad \textit{with}\ \ n > 0\ ,$$

increased by $\pm(\pm 2 - L_h)$, if only 2 factors L occur in that product; increased by $(L_k - L_h)^2$, if 4 different factors L occur in that product, and increased by $(2 \pm L_h)^2$, if 4 factors L occur in that products, but only three are different; h is the difference between the greatest and the least subscripts of L, and k is the difference between the subscripts of L in the intermediate factors.

4 – A generalization of the Chebyshev polynomials of the first and the second kind

Let us consider the sequence of polynomials $(S_n(x))_{n\geq 0}$ defined by the recurrence relation

$$S_{n+2}(x) = 2\,x\,S_{n+1}(x) - S_n(x)\,, \qquad n \geq 0\;, \tag{4.1}$$

with $S_0(x) = a$ and $S_1(x) = b$, being $a, b \in \mathbf{Z}[x]$.

Let $g(y) = S_0(x) + S_1(x)\,y + ... + S_n(x)\,y^n + ...$, be the generating function of the sequence $(S_n(x))_{n\geq 0}$. By making use of the reducing polynomial, $k(y) = 1 - 2xy + y^2$, one obtains the following finite form for $g(y)$:

$$g(y) = \frac{a + (b - 2\,a\,x)\,y}{1 - 2\,x\,y + y^2}\;,$$

which can be written as

$$g(y) = \frac{A}{y - (x + \sqrt{x^2-1})} + \frac{B}{y - (x - \sqrt{x^2-1})}\;,$$

with

$$A = \frac{(b - 2\,a\,x)\,\sqrt{x^2-1} + a + (b - 2\,a\,x)\,x}{2\sqrt{x^2-1}}\;,$$

$$B = \frac{(b - 2\,a\,x)\,\sqrt{x^2-1} - \Big[a + (b - 2\,a\,x)\,x\Big]}{2\sqrt{x^2-1}}\;,$$

where $x \neq \pm 1$.

By operating as in §1 in order to get the formula (1.4), one obtains

$$\begin{aligned} S_n(x) = {} & \left(\frac{a}{2} + \frac{a\,x - b}{2\sqrt{x^2-1}}\right)\left(x - \sqrt{x^2-1}\right)^n \\ & + \left(\frac{a}{2} - \frac{a\,x - b}{2\sqrt{x^2-1}}\right)\left(x + \sqrt{x^2-1}\right)^n\;. \end{aligned} \tag{4.2}$$

One sees that, for $a = 1$ and $b = x$, one has $S_n(x) = T_n(x)$ and, for $a = 1$ and $b = 2x$, one has $S_n(x) = U_n(x)$.

It follows also immediately that, if one sets $x = \cos\theta$ in (4.2), then

$$S_n(\cos\theta) = a\,\cos n\,\theta - \frac{(a\,\cos\theta - b)\,\sin n\,\theta}{\sin\theta}\;. \tag{4.3}$$

If one puts $a = 1$ and $b = x = \cos\theta$, one obtains

$$S_n(\cos\theta) = \cos n\,\theta = T_n(\cos\theta)\;,$$

and, for $a = 1$ and $b = 2x = 2\cos\theta$, one obtains

$$S_n(\cos\theta) = \cos n\,\theta - \frac{-\cos\theta\,\sin n\,\theta}{\sin\theta} = \frac{\sin(n+1)\,\theta}{\sin\theta} = U_n(\cos\theta)\ ,$$

as was to be expected.

If $a = T_j(x)$ and $b = T_{j+1}(x)$, then one has:

$$\begin{aligned}
S_n(x) = S_n(\cos\theta) &= \cos j\,\theta\,\cos n\,\theta - \frac{\Big[\cos j\,\theta\,\cos\theta - \cos(j+1)\theta\Big]\sin n\,\theta}{\sin\theta} \\
&= \cos(j+n)\theta + \sin j\,\theta\,\sin n\,\theta - \frac{\sin j\,\theta\,\sin\theta\,\sin n\,\theta}{\sin\theta} \\
&= \cos(j+n)\theta = T_{j+n}(x)\ .
\end{aligned}$$

If $a = U_j(x)$ and $b = U_{j+1}(x)$, then one has, by (4.3),

$$\begin{aligned}
S_n(x) = S_n(\cos\theta) &= \frac{\sin(j+1)\theta}{\sin\theta}\,\cos n\theta - \frac{\Big[\sin(j+1)\theta\,\cos\theta - \sin(j+2)\theta\Big]\,\sin n\theta}{\sin^2\theta} \\
&= \frac{\sin(j+1)\theta\,\cos n\theta}{\sin\theta} + \frac{\cos(j+1)\theta\,\sin n\theta}{\sin\theta} = \frac{\sin(j+n+1)\theta}{\sin\theta} \\
&= U_{j+n}(\cos\theta) = U_{j+n}(x)\ .
\end{aligned}$$

Now, we are going to prove, for S_n $(= S_n(x) = S_n(\cos\theta))$, a result analogous to Lemma 1.

Lemma 2. *If $(S_n)_{n\ge 0}$ is the sequence of polynomials defined by (4.1), then one has:*

$$S_n\,S_{n+r+s} + \frac{1}{2}\cdot\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,(T_{r-s} - T_{r+s}) = S_{n+r}\,S_{n+s} \tag{4.4}$$

and

$$4\,S_n\,S_{n+r}\,S_{n+s}\,S_{n+r+s} + \left[\frac{1}{2}\cdot\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,(T_{r-s} - T_{r+s})\right]^2 = (S_n\,S_{n+r+s} + S_{n+r}\,S_{n+s})^2\ . \tag{4.5}$$

Proof: Indeed, by taking into account the relation (4.3), one has

$$S_n(\cos\theta)\, S_{n+r+s}(\cos\theta) - S_{n+r}(\cos\theta)\, S_{n+s}(\cos\theta) =$$

$$= \left(a \cos n\theta - \frac{a \cos\theta - b}{\sin\theta} \sin n\theta\right) \left[a \cos(n+r+s)\theta - \frac{a \cos\theta - b}{\sin\theta} \sin(n+r+s)\,\theta\right] -$$

$$- \left[a \cos(n+r)\theta - \frac{a \cos\theta - b}{\sin\theta} \sin(n+r)\theta\right] \left[a \cos(n+s)\theta - \frac{a \cos\theta - b}{\sin\theta} \sin(n+s)\theta\right] =$$

$$= a^2 \Big[\cos n\theta \cos(n+r+s)\theta - \cos(n+r)\theta \cos(n+s)\theta\Big]$$

$$+ \left(\frac{a \cos\theta - b}{\sin\theta}\right)^2 \Big[\sin n\theta \sin(n+r+s)\theta - \sin(n+r)\theta \sin(n+s)\theta\Big]$$

$$+ a \cdot \frac{a \cos\theta - b}{\sin\theta} \Big\{ \Big[\sin(n+r)\theta \cos(n+s)\theta + \cos(n+r)\theta \sin(n+s)\theta\Big]$$

$$- \Big[\cos n\theta \sin(n+r+s)\theta + \sin n\theta \cos(n+r+s)\theta\Big] \Big\} =$$

$$= a^2 \left[\frac{\cos(2n+r+s)\theta + \cos(r+s)\theta}{2} - \frac{\cos(2n+r+s)\theta + \cos(r-s)\theta}{2}\right] +$$

$$+ \left(\frac{a \cos\theta - b}{\sin\theta}\right)^2 \left[\frac{\cos(r+s)\theta - \cos(2n+r+s)\theta}{2} - \frac{\cos(r-s)\theta - \cos(2n+r+s)\theta}{2}\right] =$$

$$= \frac{1}{2}\left[a^2 + \left(\frac{a \cos\theta - b}{\sin\theta}\right)^2\right] \Big[\cos(r+s)\theta - \cos(r-s)\theta\Big]$$

$$= \frac{1}{2} \cdot \frac{a^2 + b^2 - 2\,a \cos\theta}{\sin^2\theta} \cdot \Big[\cos(r+s)\theta - \cos(r-s)\theta\Big] \;,$$

and, consequently, since $T_j(x) = \cos jx$, one has (4.4).

Now, from (4.4), one obtains

$$4\,S_n\, S_{n+r}\, S_{n+s}\, S_{n+r+s} + \left\{\frac{1}{2} \cdot \frac{a^2 + b^2 - 2\,a\,b \cos\theta}{\sin^2\theta} \Big[\cos(r-s)\theta - \cos(r+s)\theta\Big]\right\}^2 =$$

$$= 4\,S_n\, S_{n+r}\, S_{n+s}\, S_{n+r+s} + (S_n\, S_{n+r+s} - S_{n+r}\, S_{n+s})^2$$

$$= (S_n\, S_{n+r+s} + S_{n+r}\, S_{n+s})^2 \;,$$

as desired. ∎

It is clear that (4.4) generalizes (2.1); in fact, for $a = 1$ and $b = x$, one obtains $S_0(x) = T_0(x) = 1$ and $S_1(x) = x = \cos\theta = T_1(x)$, and (4.4) becomes (2.1). One sees also that (4.4) generalizes the identity

$$U_n(x)\, U_{n+r+s}(x) + U_{r-1}(x)\, U_{s-1}(x) = U_{n+r}(x)\, U_{n+s}(x) \;,$$

obtained by Gheorghe Udrea for the Chebyshev polynomials of the second kind, in [12]; in fact, for $a = 1$ and $b = 2x = 2\cos\theta$, the identity (4.4) becomes the identity above, obtained in [12].

Now, one can state the following

Theorem 3. *Let $(S_n(x))_{n\geq 0}$ be the sequence of polynomials defined by the recurrence relation*

$$S_{n+2}(x) = 2\,x\,S_{n+1}(x) - S_n(x)\,, \qquad n \geq 0\;,$$

with $S_0(x) = a$, $S_1(x) = b$, $x \in \mathbb{C}$ and $x \neq \pm 1$, where $a, b \in \mathbf{Z}[x]$. Then, the product of any two distinct elements of the set

$$\Big\{S_n(x),\, S_{n+2r}(x),\, S_{n+4r}(x),\, 4S_{n+r}(x)S_{n+2r}(x)S_{n+3r}(x)\Big\}\;, \tag{4.6}$$

increased by

$$\left[\frac{1}{2}\cdot\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,(T_h(x) - T_k(x))\right]^t\;,$$

where T_h and T_k are suitable terms of the sequence $(T_n)_{n\geq 0}$, independent of n, with $h < k$, and $t = 1$ or $t = 2$, according to the number of factors S, in that product, is 2 or 4, is a perfect square.

Proof: One proceeds as in the proof of Theorem 1.

Thus, by setting $s = r$, in (4.4), one obtains

$$S_n\cos(\theta)\,S_{n+2r}(\cos\theta) + \frac{1}{2}\cdot\frac{a^2+b^2-2\,a\,b\,\cos\theta}{\sin^2\theta}\,(1-\cos 2r\theta) = \Big(S_{n+r}(\cos\theta)\Big)^2\;,$$

that is to say,

$$S_n(x)\,S_{n+2r}(x) + \frac{1}{2}\cdot\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,(T_0(x) - T_{2r}(x)) = \Big(S_{n+r}(x)\Big)^2\;. \tag{4.7}$$

By replacing r by $2r$, in (4.7), one gets

$$S_n(x)\,S_{n+4r}(x) + \frac{1}{2}\cdot\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,(T_0(x) - T_{4r}(x)) = (S_{n+r}(x))^2\;. \tag{4.8}$$

By replacing n by $n+2r$, in (4.7), it results in

$$S_{n+2r}(x)\,S_{n+4r}(x) + \frac{1}{2}\,\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,(T_0(x) - T_{2r}(x)) = (S_{n+3r}(x))^2\;. \tag{4.9}$$

By setting $s = 2r$, in (4.5), one gets

$$
\begin{aligned}
(4.10)\qquad & 4\,S_n(x)\,S_{n+r}(x)\,S_{n+2r}(x)\,S_{n+3r}(x)\,+\\
&\qquad + \left\{\frac{1}{2}\,\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,\Big[T_r(x)-T_{3r}(x)\Big]\right\}^2 =\\
&\qquad\qquad = \Big[S_n(x)\,S_{n+3r}(x)+S_{n+r}(x)\,S_{n+2r}(x)\Big]^2 .
\end{aligned}
$$

If one replaces n by $n + r$, in (4.10) one obtains

$$
\begin{aligned}
(4.11)\qquad & 4\,S_{n+r}(x)\,S_{n+2r}(x)\,S_{n+3r}(x)\,S_{n+4r}(x)\,+\\
&\qquad + \left\{\frac{1}{2}\,\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,[T_r(x)-T_{3r}(x)]\right\}^2 =\\
&\qquad\qquad = \Big[S_{n+r}(x)\,S_{n+4r}(x)+S_{n+2r}(x)\,S_{n+3r}(x)\Big]^2 .
\end{aligned}
$$

Finally, by setting $s = r$, in (4.5), it results

$$
\begin{aligned}
(4.12)\qquad & 4\,S_n(x)\,(S_{n+r}(x))^2\,S_{n+2r}(x) + \left\{\frac{1}{2}\,\frac{a^2+b^2-2\,a\,b\,x}{1-x^2}\,[T_0(x)-T_{2r}(x)]\right\}^2 =\\
&\qquad\qquad = \Big[S_n(x)\,S_{n+2r}(x)+(S_{n+r}(x))^2\Big]^2 ,
\end{aligned}
$$

thus completing the proof. ∎

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