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CATALAN'S IDENTITY AND CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

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Abstract: Let $(U_n)_{n\geq 0}$ be the sequence of Chebyshev polynomials of the second kind and $(F_n)_{n\geq 0}$ the sequence of Fibonacci numbers. In the present work we prove some analogous of the Catalan's identity and generalize some identities for Chebyshev polynomials.

1. The Catalan's identity for Fibonacci numbers is

$$F_{n-r}\cdot F_{n+r}+(-1)^{n-r}\cdot F_r^2=F_n^2\ , \tag{1.1}$$

where $F_0=0$, $F_1=1$, $F_n=F_{n-1}+F_{n-2}$, $n\geq 2$, $n,r\in\mathbb{N}$, $n>r$.

The analogous of this identity for Chebyshev polynomials of the second kind can be considered the identity

$$U_{n-r-1}\cdot U_{n+r-1}+U^2_{r-1}=U^2_{n-1}\ ,\qquad n>r,\quad n,r\in\mathbb{N}^*\ . \tag{1.1$'$}$$

Proof: One has

$$U_0(x)=1,\ \ U_1(x)=2x,\ \ U_{n+1}=2x\cdot U_n(x)-U_{n-1}(x)\ ,\quad (\forall)\,n\in\mathbb{N}^*,\ \ (\forall)\,x\in\mathbb{C}\ .$$

On the other hand,

$$U_{n-1}(\cos\varphi)=\frac{\sin n\varphi}{\sin\varphi},\quad (\forall)\,\varphi\in\mathbb{C}\,,\qquad \sin\varphi\neq 0,\quad (\forall)\,n\in\mathbb{N}^*\ .$$

Since for any $x\in\mathbb{C}$ $(\exists)\varphi\in\mathbb{C}$ such that $x=\cos\varphi$, one has

$$U_{n-r-1}(x)\cdot U_{n+r-1}(x)+U^2_{r-1}(x)=U_{n-r-1}(\cos\varphi)\cdot U_{n+r-1}(\cos\varphi)+U^2_{r-1}(\cos\varphi)=$$

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$$= \frac{\sin(n+r)\,\varphi}{\sin\varphi} \cdot \frac{\sin(n-r)\,\varphi}{\sin\varphi} + \frac{\sin^2 r\varphi}{\sin^2\varphi} = \frac{\sin(n-r)\varphi \cdot \sin(n+r)\varphi + \sin^2 r\varphi}{\sin^2\varphi} =$$

$$= \frac{\dfrac{\cos 2r\varphi - \cos 2n\varphi}{2} + \sin^2 r\varphi}{\sin^2\varphi} = \frac{(1-2\sin^2 r\varphi) - (1-2\sin^2 n\varphi) + 2\sin^2 r\varphi}{2\sin^2\varphi} =$$

$$= \left(\frac{\sin n\varphi}{\sin\varphi}\right)^2 = U_{n-1}^2(\cos\varphi) = U_{n-1}^2(x)\ . \ \blacksquare$$

Let us remark that, from $(1.1')$, one obtains the identity (1.1), since $U_k(\frac{i}{2}) = i^k \cdot F_{k+1}$, $k \in \mathbb{N}$, where $i^2 = -1$.

If, in $(1.1')$, one puts $r+1$ instead of r, one obtains

$$U_{n-r-2} \cdot U_{n+r} + U_r^2 = U_{n-1}^2\ , \tag{1.2}$$

for $n, r \in \mathbb{N}^*$ and $n > r+1$.

By adding $(1.1')$ with (1.2), it results

$$U_{n-r-1} \cdot U_{n+r-1} + U_{n-r-2} \cdot U_{n+r} + U_r^2 + U_{r-1}^2 = 2 \cdot U_{n-1}^2\ , \tag{1.3}$$

$n, r \in \mathbb{N}^*$ and $n > r+1$.

If one subtracts (1.2) from $(1.1')$, one has

$$U_{n+r-1} \cdot U_{n-r-1} - U_{n-r-2} \cdot U_{n+r} = U_{2r}\ , \quad n, r \in \mathbb{N}^* \ \text{ and } \ n > r+1\ , \tag{1.4}$$

since $U_r^2 - U_{r-1}^2 = U_{2r}$, $(\forall)\, r \in \mathbb{N}^*$, as one concludes by taking into account that for every $x \in \mathbb{C}$, there is some $\varphi \in \mathbb{C}$ such that $U_{k-1}(x) = U_{k-1}(\cos\varphi) = \frac{\sin k\varphi}{\sin\varphi}$.

From (1.3) and (1.4), it results, for $r = 1$,

$$U_{n-2} \cdot U_n + U_{n-3} \cdot U_{n+1} + 4x^2 + 1 = 2 \cdot U_{n-1}^2\ , \quad n \in \mathbb{N},\ \ n \geq 3\ , \tag{1.3'}$$

$$U_{n-2} \cdot U_n - U_{n-3} \cdot U_{n+1} = 4x^2 - 1\ , \quad (\forall)\, x \in \mathbb{C},\ \ (\forall)\, n \in \mathbb{N},\ \ n \geq 3\ . \tag{1.4'}$$

If we put $x = \frac{i}{2}$ in $(1.3')$ and $(1.4')$, we obtain

$$F_{n-1} \cdot F_{n+1} + F_{n-2} \cdot F_{n+2} = 2 \cdot F_n^2\ , \quad n \in \mathbb{N},\ \ n \geq 2\ , \tag{1.3''}$$

and

$$F_{n-1} \cdot F_{n+1} - F_{n-2} \cdot F_{n+2} = 2 \cdot (-1)^n\ , \quad n \in \mathbb{N},\ \ n \geq 2\ . \tag{1.4''}$$

These results are also obtained in [2].

On the other hand, from $(1.3')$ and $(1.4')$, one deduces

$$\left[2 \cdot U_{n-1}^2 - (4x^2+1)\right]^2 - (4x^2-1)^2 = 4 \cdot U_{n-3} \cdot U_{n-2} \cdot U_n \cdot U_{n+1}\ ,$$

so that

$$(1.5)\qquad U_{n-1}^4 - (4x^2+1)\cdot U_{n-1}^2 + 4x^2 = \\ = U_{n-3}\cdot U_{n-2}\cdot U_n\cdot U_{n+1}\,, \qquad x\in\mathbb{C},\ \ n\in\mathbb{N},\ \ n\geq 3\ .$$

For $x=\frac{i}{2}$, from (1.5), it results

$$(1.5')\qquad F_n^4 - 1 = F_{n-2}\cdot F_{n-1}\cdot F_{n+1}\cdot F_{n+2}\,, \qquad n\in\mathbb{N},\ \ n\geq 2\ .$$

This result was stated by E. Gelin (Nouv. Corresp. Math., 6 (1880), p. 384) and proved by E. Cesàro (ibid., pp. 423–424; see [1], p. 401):

"The fourth power of the middle term of five consecutive terms of the sequence $(F_k)_{k\geq 0}$ differs from the product of the other four terms by unity".

2. Let us denote $m=n-r\in\mathbb{N}^*$. The identity (1.1′) can be written as follows

$$(2.1)\qquad U_{m-1}\cdot U_{m+2r-1} + U_{r-1}^2 = U_{m+r-1}^2\,, \qquad m,r\in\mathbb{N}^*\ .$$

From (2.1), we obtain

$$U_{r-1}^4 = (U_{m+r-1}^2 - U_{m-1}\cdot U_{m+2r-1})^2 \\ = (U_{m+r-1}^2 + U_{m-1}\cdot U_{m+2r-1})^2 - 4\cdot U_{m+r-1}^2\cdot U_{m-1}\cdot U_{m+2r-1}\ ,$$

i.e.,

$$(2.2)\qquad 4\cdot U_{m-1}\cdot U_{m+2r-1}\cdot U_{m+r-1}^2 + U_{r-1}^4 = \\ = (U_{m+r-1}^2 + U_{m-1}\cdot U_{m+2r-1})^2\,, \qquad m,r\in\mathbb{N}^*\ .$$

For $x=\frac{i}{2}$ and $r\in\{1,2\}$, from (2.2), it results that the integers $P_m = 4\cdot F_m\cdot F_{m+1}^2\cdot F_{m+2}+1$ and $Q_m = 4\cdot F_m\cdot F_{m+2}^2\cdot F_{m+4}+1$ are squares for any $m\in\mathbb{N}$.

Also, we have the identity:

$$(2.3)\qquad U_{n-1}\cdot U_{n+2r} + U_{r-1}\cdot U_r = U_{n+r-1}\cdot U_{n+r}\,, \qquad r,n\in\mathbb{N}^*\ .$$

For $x=\frac{i}{2}$, one has

$$(2.3')\qquad F_n\cdot F_{n+2r+1} + (-1)^n\cdot F_r\cdot F_{r+1} = F_{n+r}\cdot F_{n+r+1}\,, \qquad n,r\in\mathbb{N}\ .$$

This generalizes a result stated by E. Gelin and proved by E. Cesàro, mentioned in [1], p. 401:

"The product of the means of four consecutive terms differs from the product of the extremes by ± 1"

(in (2.3′) one puts $r = 1$).

By (2.3), it results

$$U_r^2 \cdot U_{r-1}^2 = (U_{n+r-1} \cdot U_{n+r} + U_{n-1} \cdot U_{n+2r})^2 - 4 \cdot U_{n+r-1} \cdot U_{n+r} \cdot U_{n-1} \cdot U_{n+2r} \ ,$$

or

$$\begin{aligned}(2.4) \quad & 4 \cdot U_{n-1} \cdot U_{n+r-1} \cdot U_{n+r} \cdot U_{n+2r} + U_{r-1}^2 \cdot U_r^2 = \\ & = (U_{n+r-1} \cdot U_{n+r} + U_{n-1} \cdot U_{n+r})^2 \, , \quad n, r \in \mathbb{N}^* \ .\end{aligned}$$

From (2.4), one has, for $x = \frac{i}{2}$,

$$\begin{aligned}(2.4') \quad & 4 \cdot F_n \cdot F_{n+r} \cdot F_{n+r+1} \cdot F_{n+2r+1} + F_r^2 \cdot F_{r+1}^2 = \\ & = (F_{n+r} \cdot F_{n+r+1} + F_n \cdot F_{n+2r+1})^2 \, , \quad n, r \in \mathbb{N} \ .\end{aligned}$$

The identity (2.4′) is a generalization of the two results obtained in [3], p. 344, namely, the numbers $H_n = 1 + 4 \cdot F_n \cdot F_{n+1} \cdot F_{n+2} \cdot F_{n+3}$ and $L_n = 1 + F_n \cdot F_{n+2} F_{n+3} \cdot F_{n+5}$ are squares $(\forall)\, n \in \mathbb{N}$.

3. In [5] D. Everman, A.E. Danese and K. Venkannayah give the following generalization of the Catalan's identity:

$$(3.1) \qquad F_n \cdot F_{n+r+s} + (-1)^n \cdot F_r \cdot F_s = F_{n+r} \cdot F_{n+s} \, , \quad (\forall)\, n, r, s \in \mathbb{N} \ .$$

Now, the identity (3.1) follows from the identity

$$(3.1') \qquad U_{n-1} \cdot U_{n+r+s-1} + U_{r-1} \cdot U_{s-1} = U_{n+r-1} \cdot U_{n+s-1} \, , \quad n, r, s \in \mathbb{N}^*, \ x \in \mathbb{C} \ .$$

Proof: Let be $x = \cos\varphi$, $\varphi \in \mathbb{C}$; one has

$$U_{n-1}(x) \cdot U_{n+r+s-1}(x) + U_{r-1}(x) \cdot U_{s-1}(x) =$$

$$
\begin{aligned}
&= U_{n-1}(\cos\varphi)\cdot U_{n+r+s-1}(\cos\varphi) + U_{r-1}(\cos\varphi)\cdot U_{s-1}(\cos\varphi) \\
&= \frac{\sin n\varphi \cdot \sin(n+r+s)\varphi + \sin r\varphi\cdot \sin s\varphi}{\sin^2\varphi} \\
&= \frac{\cos(r+s)\varphi - \cos(2n+r+s)\varphi}{2\sin^2\varphi} + \frac{\cos(s-r)\varphi - \cos(s+r)\varphi}{2\sin^2\varphi} \\
&= \frac{\cos(s-r)\varphi - \cos(2n+r+s)\varphi}{2\sin^2\varphi} = \frac{\sin(n+r)\varphi\cdot\sin(n+s)\varphi}{\sin^2\varphi} \\
&= \frac{\sin(n+r)\varphi}{\sin\varphi}\cdot\frac{\sin(n+s)\varphi}{\sin\varphi} \\
&= U_{n+r-1}(\cos\varphi)\cdot U_{n+s-1}(\cos\varphi) = U_{n+r-1}(x)\cdot U_{n+s-1}(x) \quad . \blacksquare
\end{aligned}
$$

From (3.1′), one obtains

$$
\begin{aligned}
U_{r-1}^2 \cdot U_{s-1}^2 = &(U_{n+r-1}\cdot U_{n+s-1} + U_{n-1}\cdot U_{n+r+s-1})^2 \\
&- 4\cdot U_{n+r-1}\cdot U_{n+s-1}\cdot U_{n+s+r-1}\cdot U_{n-1} \ ,
\end{aligned}
$$

and further, it results

$$
\begin{aligned}
\text{(3.2)}\quad &4\cdot U_{n-1}\cdot U_{n+r-1}\cdot U_{n+s-1}\cdot U_{n+r+s-1} + U_{r-1}^2\cdot U_{s-1}^2 = \\
&\quad = (U_{n+r-1}\cdot U_{n+s-1} + U_{n-1}\cdot U_{n+r+s-1})^2 , \qquad n,r,s \in \mathbb{N}^* \ .
\end{aligned}
$$

Now, for $x = \frac{i}{2}$, from (3.2), it results

$$
\begin{aligned}
\text{(3.2′)}\quad &4\cdot F_n\cdot F_{n+r}\cdot F_{n+s}\cdot F_{n+r+s} + F_r^2\cdot F_s^2 = \\
&\quad = (F_n\cdot F_{n+r+s} + F_{n+r}\cdot F_{n+s})^2 , \qquad n,r,s\in\mathbb{N} \ .
\end{aligned}
$$

This result is a generalization of the identities that can be obtained from (2.2) and (2.4). From (3.2), it results also that the natural numbers

$$
\begin{aligned}
T_{n,r,s}(x) = &4\cdot U_{n+r-1}(x)\cdot U_{n+s-1}(x)\cdot U_{n-1}(x)\cdot U_{n+r+s-1}(x) \\
&+ U_{r-1}^2(x)\cdot U_{s-1}^2(x) , \qquad x\in\mathbb{N},\ \ n,r,s\in\mathbb{N}^* \ ,
\end{aligned}
$$

are squares.

4. If, in the identity (3.1′), we have change s by $s+t$ and r by $r+t$, we obtain

$$
\text{(4.1)}\qquad U_{n-1}\cdot U_{n+r+s+t-1} + U_{r-1}\cdot U_{s+t-1} = U_{n+r-1}\cdot U_{n+s+t-1}
$$

and, respectively,

$$(4.2)\quad U_{n-1}\cdot U_{n+r+s+t-1}+U_{s-1}\cdot U_{r+t-1}=U_{n+s-1}\cdot U_{n+r+t-1}\,,\qquad n,r,s,t\in\mathbb{N}^*\,.$$

Also, one has

$$(4.3)\quad U_{n-1}\cdot U_{n+r+s+t-1}+U_{t-1}\cdot U_{r+s-1}=U_{n+t-1}\cdot U_{n+r+s-1}\,,\qquad n,r,s,t\in\mathbb{N}^*\,.$$

By adding (4.1), (4.2) and (4.3), it results

$$(4.4)\quad \begin{aligned}3\cdot U_{n-1}\cdot U_{n+r+s+t-1}+U_{r-1}\cdot U_{s+t-1}+U_{s-1}\cdot U_{r+t-1}+U_{t-1}\cdot U_{r+s-1}=\\=U_{n+r-1}\cdot U_{n+s+t-1}+U_{n+s-1}\cdot U_{n+r+t-1}+U_{n+t-1}\cdot U_{n+r+s-1}\,.\end{aligned}$$

By multiplying the identity (4.4) by U_{n-1}, it follows

$$(4.5)\quad 3\cdot U_{n-1}^2\cdot U_{n+r+s+t-1}=3\cdot U_{n+r-1}\cdot U_{n+s-1}\cdot U_{n+t-1}-(A+B\cdot U_{n-1})\,,$$

where

$$A=U_{r-1}\cdot U_{s-1}\cdot U_{n+t-1}+U_{s-1}\cdot U_{t-1}\cdot U_{n+r-1}+U_{t-1}\cdot U_{r-1}\cdot U_{n+s-1}\,,$$
$$B=U_{r-1}\cdot U_{s+t-1}+U_{s-1}\cdot U_{r+t-1}+U_{r+s-1}\cdot U_{t-1}\,,\qquad n,,s,t\in\mathbb{N}^*\,.$$

For $x=\frac{i}{2}$, in (4.5), it results the identity (4.1) in [3], p. 347.
On the other hand, from (4.5), it follows the identities:

$$(4.6)\qquad (U_{n-1}^2\cdot U_{n+r+s+t-1}-U_{n+r-1}\cdot U_{n+s-1}\cdot U_{n+t-1})^2=\frac{1}{9}\cdot(A+B\cdot U_{n-1})^2$$

and

$$(4.7)\quad \begin{aligned}4\cdot U_{n-1}^2\cdot U_{n+r-1}\cdot U_{n+s-1}\cdot U_{n+t-1}\cdot U_{n+r+s+t-1}+\frac{1}{9}\cdot(A+B\cdot U_{n-1})^2=\\=(U_{n-1}^2\cdot U_{n+r+s+t-1}+U_{n+s-1}\cdot U_{n+r-1}\cdot U_{n+t-1})^2\,,\qquad n,r,s,t\in\mathbb{N}^*\,.\end{aligned}$$

For $t=0$, one has

$$(4.8)\quad \begin{aligned}4\cdot U_{n-1}^3\cdot U_{n+r-1}\cdot U_{n+s-1}\cdot U_{n+r+s-1}+\frac{1}{9}\cdot(A+B\cdot U_{n-1})^2=\\=(U_{n-1}^2\cdot U_{n+r+s-1}+U_{n-1}\cdot U_{n+s-1}\cdot U_{n+r-1})^2\,,\end{aligned}$$

where

$$A=U_{n-1}\cdot U_{r-1}\cdot U_{s-1}\qquad (U_{-1}(x)=0=F_0,\quad x\in\mathbb{C})\,,$$
$$B=2\cdot U_{r-1}\cdot U_{s-1}\,,$$

or, equivalently,

$$4\cdot U_{n-1}^3\cdot U_{n+r-1}\cdot U_{n+s-1}\cdot U_{n+r+s-1}+\frac{1}{9}\cdot(3\cdot U_{n-1}\cdot U_{r-1}\cdot U_{s-1})^2=$$
$$=(U_{n-1}^2\cdot U_{n+r+s-1}+U_{n-1}\cdot U_{n+r-1}\cdot U_{n+s-1})^2\ ,$$

For $x=\frac{i}{2}$ in (4.9), it results

$$(4.9)\qquad 4\cdot U_{n-1}^3\cdot U_{n+r-1}\cdot U_{n+s-1}\cdot U_{n+r+s-1}+U_{n-1}^2\cdot U_{r-1}^2\cdot U_{s-1}^2=$$
$$=(U_{n-1}^2\cdot U_{n+r+s-1}+U_{n-1}\cdot U_{n+r+s-1}\cdot U_{n+s-1})^2\,,\qquad n,r,s\in\mathbb{N}^*\ .$$

following directly from (4.7), one has the

Theorem. *If $(U_m)_{m\geq 0}$ denotes the sequence of Chebyshev polynomials of the second kind, then the integers*

$$(4.9')\qquad 4\cdot F_n^3\cdot F_{n+r}\cdot F_{n+s}\cdot F_{n+r+s}+F_n^2\cdot F_r^2\cdot F_s=$$
$$=(F_n^2\cdot F_{n+r+s}+F_n\cdot F_{n+r}\cdot F_{n+s})^2\,,\qquad n,r,s\in\mathbb{N}\ .$$

where A and B are defined by the relations

$$A=U_{n+r-1}\cdot U_{s-1}\cdot U_{t-1}+U_{n+s-1}\cdot U_{r-1}\cdot U_{t-1}+U_{n+t-1}\cdot U_{r-1}\cdot U_{s-1}$$

and

$$B=U_{r-1}\cdot U_{s+t-1}+U_{s-1}\cdot U_{t+r-1}+U_{t-1}\cdot U_{r+s-1}\ ,$$

are perfect squares, for $(\forall)\,n,r,s,t,\,x\in\mathbb{N}^*$.

That is in fact a generalization of Theorem 3, given in [3], p. 348.

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