PORTUGALIAE MATHEMATICA
Vol. 52 Fasc. 4 – 1995

AZUMAYA ALGEBRAS OF A RING WITH A FINITE AUTOMORPHISM GROUP

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Abstract: Let R be a ring with 1, C the center of R, and G a finite automorphism group of R. Denote the subring of the elements of R fixed under each element in G by R^G and the commutator subring of R^G in R by V. Conditions are given such that two Azumaya algebras of the rings R, R^G and V imply that the third one is also an Azumaya algebra.

1 – Introduction

Let R be a ring with 1, C the center of R and G a group. In [5], F.R. DeMeyer and G.J. Janusz proved that the group ring RG is an Azumaya algebra if and only if the group algebra CG and R are Azumaya algebras. In [7], this fact was proved for a projective group ring RG_f where G is a finite automorphism group of R and f is a factor set induced by the standard basis $\{U_g \,/\, g \text{ in } G\}$ of RG_f. That is, RG_f is an Azumaya algebra if and only if the group algebra CG_f and R are Azumaya algebras where RG_f and CG_f are projective group ring and algebra respectively as defined by F.R. DeMeyer and the present authors in [3] and [7, 8]. We note that CG_f and CG are the commutator subrings of R in RG_f and in RG respectively. The purpose of the present paper is to prove the above fact that for the rings, R, R^G and V where R^G is the subring of the elements fixed under each element in G and V is the commutator subring of R^G in R. We shall give conditions under which any two of R, R^G and V Azumaya implies that the third one is also an Azumaya algebra.

Received: November 25, 1994.
1991 AMS Mathematics Subject Classification: Primary 16H05; Secondary 16W20.
Keywords and Phrases: Azumaya algebras, Galois extensions, skew group rings.

2 – Notations and definitions

Throughout, R denotes a ring with 1, G a finite automorphism group of R, $R^G = \{r \text{ in } R \,/\, g(r) = r \text{ for all } g \text{ in } G\}$, C the center of R, $V = \{r \text{ in } R \,/\, tr = rt$ for all t in $R^G\}$ and R^*G the skew group ring. That is, the ring with the elements of G as a free basis over R such that $gr = g(r)\, g$ for all g in G and r in R. Let $S \subset T$ be a ring extension (that is, S is a subring of T with the same 1). Then T is called a separable extension of S if there exist elements $\{a_i,\, b_i$ in T such that $\Sigma\, ta_i \otimes b_i = \Sigma\, a_i \otimes b_i t$ for all t in T where $i = 1, 2, ..., n$ for some integer n and $\otimes$ is over $S\}$, and $\Sigma\, a_i b_i = 1$. If S is contained in the center of T, then the separable extension T of S is called a separable algebra; and in particular, the separable algebra T over S is called an Azumaya algebra if S is the center of T. Let T be a ring extension of S with a finite automorphism group G. Then T is called a G-Galois extension of S if $T^G = S$ and there exist elements $\{c_i,\, d_i$ in $T\,/$ $\Sigma\, c_i\, g(d_i) = 0$ for each $g \neq 1$ in G and $\Sigma\, c_i d_i = 1$ where $i = 1, 2, ..., k$ for some integer $k\}$. The set $\{c_i, d_i\}$ is called a G-Galois set of T. It is well known that T is a G-Galois extension of S if and only if T is finitely generated and projective as a right S-module and $T^*G \cong \mathrm{Hom}_S(T, T)$ ([2, Theorem 1]).

3 – Azumaya subalgebras in a ring

In this section, we shall give conditions under which any two rings of R, R^G and V are Azumaya algebras imply that the third one is also an Azumaya algebra.

Theorem 3.1. *Let R be a G-Galois extension of R^G and R^G is an Azumaya algebra over C^G. Then, R is an Azumaya algebra if and only if V is an Azumaya algebra.*

Proof: Since R is a G-Galois extension of R^G, $R^*G \cong \mathrm{Hom}_{R^G}(R, R) =$ the commutator subring of R^G in $\mathrm{Hom}_{C^G}(R, R)$ and R is finitely generated and projective over R^G. Since R^G is an Azumaya algebra over C^G, R^G is finitely generated and projective over C^G. Hence R is finitely generated and projective over C^G. Thus R is a progenerator over C^G (for 1 is in C^G). Therefore $\mathrm{Hom}_{C^G}(R, R)$ is an Azumaya C^G-algebra with an Azumaya subalgebra R^G. As a commutator subalgebra of R^G in $\mathrm{Hom}_{C^G}(R, R)$, R^*G is also an Azumaya algebra over C^G ([4, Theorem 4.3, p. 57]). But then $R^*G \cong R^G \otimes_{C^G} V'$ where V' is the commutator subring of R^G in R^*G ([4, Theorem 4.3]). Clearly, $V' = V^*G$ so $R \cong R^G \otimes_{C^G} V = (R^G \otimes_{C^G} C) \otimes_C V$. Therefore, R is an Azumaya algebra if and only if so is V (for $R^G \otimes_{C^G} C$ is an Azumaya algebra over C by Lemma 5.1 in [4]). ∎

Theorem 3.2. *Let $R = R^G \otimes_{C^G} V$ and assume that there exists an element c in C such that $\mathrm{tr}_G(c) = 1$ where $\mathrm{tr}_G(\ )$ is the trace of G. Then, R is an Azumaya algebra if and only if R^G and V are Azumaya algebras.*

Proof: Since $R = R^G \otimes_{C^G} V$, C is contained in the center of V. Hence $R = (R^G \otimes_{C^G} C) \otimes_C V$. Thus R is an Azumaya algebra if and only if so are $R^G \otimes_{C^G} C$ and V. By hypothesis, there exists a c in C such that $\mathrm{tr}_G(c) = 1$, so C^G is a C^G-direct summand of C. But then $R^G \otimes_{C^G} C$ is an Azumaya algebra if and only if so are R^G and V. ∎

Corollary 3.3. *Let R^*G be a separable extension of R and $R = R^G \otimes_{C^G} V$. Then, R is an Azumaya algebra if and only if so are R^G and V.*

Proof: It is well known that R^*G is a separable extension of R if and only if there exists a c in C such that $\mathrm{tr}_G(c) = 1$, so the corollary is immediate. ∎

Theorem 3.4. *Let R be a G-Galois extension of R^G and assume that there exists a c in C such that $\mathrm{tr}_G(c) = 1$. Then, R is separable over C^G if and only if R^G is separable over C^G and V is separable over C.*

Proof: Since R is a G-Galois extension of R^G, it is separable over R^G. For the sufficiency, since R^G is separable over C^G, R is separable over C^G by the transitivity of separable extensions. Conversely, since $\mathrm{tr}_G(c) = 1$ for some c in C, the trace map $\mathrm{tr}_G(\ )$ implies that R^G is a direct summand of R as a bimodule over R^G. By hypothesis, R is a separable C^G-algebra, so R is left-$R \otimes_{C^G} R^\circ$-projective, where R° is the opposite algebra of R over C^G. But R is G-Galois over R^G, so it is finitely generated and projective over R^G ([2]). Hence $R \otimes_{C^G} R^\circ$ is finitely generated and projective over $R^G \otimes_{C^G} (R^G)^\circ$. Then, by the transitivity of projective modules, R is a left $R^G \otimes_{C^G} (R^G)^\circ$-projective module. Thus, as a direct summand of R, R^G is also left $R^G \otimes_{C^G} (R^G)^\circ$-projective. This implies that R^G is a separable C^G-algebra. Thus $R^G C$ is a separable subalgebra of the Azumaya algebra R over C. Therefore, V is also a separable algebra over C for it is the commutator subalgebra of $R^G C$ in R ([4, Theorem 4.3]). ∎

Theorem 3.5. *Let V be a G-Galois extension of C^G. Then R is an Azumaya algebra of C if and only if R^G is an Azumaya algebra over C^G and V is an Azumaya algebra over C.*

Proof: Since V is a G-Galois extension of C^G, R and $R^G V$ are G-Galois extensions of R^G. Hence $R = R^G V$ ([8, Proof of Lemma 3.5]). Thus the center of V is C (for R has center C). Also, for any x in the center of R^G, it is in the center of V, so it is in C^G. Thus the center of R^G is C^G. Noting that V is a G-Galois extension of C^G, we have that V is separable over C^G and so V is an Azumaya C-algebra and C is separable over C^G. Since V is a G-Galois extension of C^G again, there exists an element c in C such that $\mathrm{tr}_G(c) = 1$ (for C^G is a commutative ring with 1). Thus R^G is a direct summand of R as a bimodule over R^G ([8, Proof of Theorem 3.6]). Since R is separable over C and C is separable over C^G, R is separable over C^G. But then, by Theorem 3.4, R^G is separable over C^G. Therefore, R^G is an Azumaya C^G-algebra. The necessity holds. Conversely, R^G is an Azumaya algebra of C^G, $R^G C$ is an Azumaya algebra over C. By the early result, $R = R^G V = R^G C V$, so R is an Azumaya algebra over C (for so is V). ∎

Next we give conditions under which that R and V are Azumaya algebras over C implies that R^G is an Azumaya algebra.

Theorem 3.6. *Assume that R and V are Azumaya algebras and that C^G is a C^G-direct summand of C. Then, $R \cong R^G \otimes_{C^G} V$ if and only if R^G is an Azumaya algebra over C^G.*

Proof: By hypothesis, $R \cong R^G \otimes_{C^G} V \cong (R^G \otimes_{C^G} C) \otimes_C V$ such that R is an Azumaya algebra over C, $R^G \otimes_{C^G} C$ is an Azumaya algebra over C ([4, Theorem 4.4, p. 58]). Since C^G is a C^G-direct summand of C, R^G is an Azumaya algebra over C^G ([4, Corollary 1.10, p. 45]).

Conversely, R^G is an Azumaya algebra, so $R^G \otimes_{C^G} C$ is an Azumaya algebra over C such that $R^G C \cong R^G \otimes_{C^G} C$. But then $R \cong R^G C \otimes_C V \cong (R^G \otimes_{C^G} C) \otimes_C V \cong R^G \otimes_{C^G} V$. ∎

4 – Examples

We conclude the paper with three examples to demonstrate the results in Section 3.

Example 1: Let R be the real field, $Q = R[i,j,k]$ (= the real quaternion algebra), $A = Q \oplus Q$ (= direct sum of Q), and G is generated by α such that $\alpha(a,b) = (b,a)$ for (a,b) in A. Then we have the following properties:

(**1**) G is an automorphism group of A of order 2.

(**2**) $A^G = \{(a,a) \,/\, a \text{ in } Q\} \cong Q$.

(**3**) The center C of A is $R \oplus R$.

(**4**) Q is an Azumaya algebra over R $(= C^G)$.

(**5**) A^G is an Azumaya algebra over R (for $A^G \cong Q$ by (2) and (4)).

(**6**) V (= the commutator subring of A^G in A) $= R \oplus R = C$ by (3).

(**7**) A is an Azumaya algebra over its center $R \oplus R$ (for Q is an Azumaya algebra over R).

(**8**) $C^G = \{(b,b) \,/\, b \text{ in } R\} \cong R$.

(**9**) $A \cong A^G \otimes_{C^G} V \cong Q \otimes_R (R \oplus R) = Q \oplus Q$.

Example 2: Let Q be the ring as given in Example 1 and Q^*G (= the skew group ring). Then,

(**1**) The commutator V' of Q^G in Q^*G is $(R \oplus R)^*G$ which is V^*G by (6) of Example 1.

(**2**) $A^*G \cong A^G \otimes_{C^G} (V^*G) = A^G \otimes_{C^G} ((R \oplus R)^*G) = A^G \otimes_{C^G} (C^*G)$.

(**3**) The center of $A^*G = C^G$.

(**4**) A^*G is an Azumaya C^G-algebra.

(**5**) V' is an Azumaya C^G-algebra.

There exist rings R such that $R \neq R^G \otimes_{C^G} V$.

Example 3: Let Q be the ring of real quaternion $R[i,j,k]$ as given in Example 1, and $G = \{\alpha_1, \alpha_i \,/\, \alpha_1$ is the identity map of Q and α_i is the inner automorphism of Q induced by $i\}$.

Then we have:

(**1**) $Q^G = R[i]$ and V (= the commutator subring of Q^G in Q) $= R[i]$.

(**2**) $Q^G \otimes_{C^G} V = R[i] \otimes_R R[i]$ (hence $Q \neq Q^G \otimes_{C^G} V$).

(**3**) The center C of Q is R and $C^G = R$ (hence Q^G is not an Azumaya algebra over C^G for Q^G is commutative). Thus the sufficient condition of Theorem 3.1 does not hold.

(**4**) Q^*G is an Azumaya algebra.

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