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SINGULAR KLEIN MANIFOLDS

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Abstract: The aim of the present paper is an effort to make more exact some aspects of seven-parameter group of collineations in a five-dimensional Klein projective Space. Using the δ-variation of invariants in the first order contact elements, we derive several types of Klein manifolds on the Klein absolutum. Our study is carried out using Cartan's methods of moving frames [1], [2], [3].

1 – Introduction

The space $\overline{P}_3$ is defined as a homogeneous space $\overline{P}_3 \equiv (P_3, S)$, where S is a subgroup of the projective group $\mathrm{PG}(3, \mathbb{R})$. We will now assume that all transformations of S will be collineations of a 3-dimensional projective space P_3 that leave fixed two real points and a real plane through one of them [4]. The coordinate transformations in S are given by

$$\overline{x}_i = a_{ij}\, x_j\,, \qquad i,j = 1,2,3,4\ , \tag{1}$$

where (a_{ij}) is a non-singular matrix with the stationarity conditions

$$\begin{aligned} &a_{13} = a_{23} = a_{24} = a_{34} = a_{43} = 0\ , \\ &a_{14} = a_{41}, \quad a_{12} = a_{32}, \quad a_{11} = a_{33} + a_{31}\ . \end{aligned} \tag{2}$$

Hereafter, we assume that the Latin and Greek indices run over the ranges $\{1,2,3,4\}$ and $\{1,2,3\}$ except the indices μ, ν and η run over the ranges $\{1,2\}$ and $\{3,4\}$ respectively.

We introduce a special family of frames $\{A_i\}$ (A_i are linearly independent points) such that the vertices A_3 and A_4 coincide with the two fixed points,

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but the invariant plane is determined by means of the points A_2, A_4, $A_1 + A_3$. Therefore the fundamental equations of the frames are given by

$$dA_i = \omega_i^j \, A_j \ . \tag{3}$$

The one-forms ω_i^j satisfy the stationarity conditions

$$\begin{cases} \omega_3^1 = \omega_3^2 = \omega_3^4 = \omega_4^1 = \omega_4^2 = \omega_4^3 = 0 \ , \\ \omega_2^1 = \omega_2^3, \quad \omega_1^1 = \omega_1^3 + \omega_3^3, \quad \omega_1^3 + \omega_2^2 + 2\omega_3^3 + \omega_4^4 = 0 \ . \end{cases} \tag{4}$$

Thus, we have ω_i^j with the conditions (4) are the invariant one-forms of a seven-parameter group of collineations. The integrability conditions of the invariant group S are given by

$$\begin{cases} D\omega_2^2 = \omega_2^3 \wedge \omega_1^2, \quad D\omega_1^3 = -\omega_2^3 \wedge \omega_1^2, \quad D\omega_3^3 = 0 \ , \\ D\omega_2^3 = \omega_2^3 \wedge (\omega_1^3 + \omega_3^3 - \omega_2^2) \ , \\ D\omega_1^2 = -\omega_1^2 \wedge (\omega_1^3 + \omega_3^3 - \omega_2^2) \ , \\ D\omega_1^4 = -\omega_1^4 \wedge (2\omega_1^3 + 3\omega_3^3 + \omega_2^2) + \omega_1^2 \wedge \omega_2^4 \ , \\ D\omega_2^4 = -\omega_2^4 \wedge (\omega_1^3 + 2\omega_3^3 + 2\omega_2^2) + \omega_2^3 \wedge \omega_1^4 \ . \end{cases} \tag{5}$$

For a general discussion of Klein-representation (for brevity K-R) of line manifolds on the Klein-quadric (K-absolutum), the reader is referred to [5], [6], [7]. It is well-known that a line $\ell \subset \overline{P}_3$ is represented by a point ℓ^k of a Klein five dimensional projective space $\overline{P}{}_5^k$. The locus of ℓ^k as the line ℓ varies is the Grassmann manifold $\mathrm{Gr}(1,3)$ of all lines in $\overline{P}_3$. The manifold $\mathrm{Gr}(1,3)$ is equivalent to the K-R of the lines of $\overline{P}_3$ by K-points (P^{ij}) $(i<j,\ P^{ij} = -P^{ji})$ of the K-absolutum $\overline{Q}{}_4^2 \subset \overline{P}{}_5^k$.

We introduce the K-frames in $\overline{P}{}_5^k$ as a six-hedron moving K-frame $\{A_{ij}\}$ in which A_{ij} are the K-images of the edges (A_i, A_j) of the frame $\{A_i\} \subset \overline{P}_3$. The infinitesimal displacements of the K-frame $\{A_{ij}\}$ are given by

$$dA_{ij} = \omega_i^k \, A_{kj} + \omega_j^k \, A_{ik} \tag{6}$$

up to the stationarity conditions (4).

2 – Characterization of K-absolutum

From the displacements (6) with (4), it is easy to see that the space $\overline{P}{}_5^k$ contains a degenerate K-absolutum $\overline{Q}{}_4^2$. The absolutum $\overline{Q}{}_4^2$ consist of two invariant K-planes $P^1 \equiv (A_{13}, A_{23}, A_{34})$, $P^2 \equiv (A_{14}, A_{24}, A_{34})$, invariant K-point $A_{34} \equiv P^1 \cap P^2$ and invariant K-line $L_k \equiv (A_{24}, A_{14} + A_{34}) \subset P^2$.

Thus, we have the following:

Lemma 1. *The K-absolutum $\overline{Q}_4^2 \subset \overline{P}_5^k$ consist of two fixed planes P^1, P^2, fixed point $P^1 \cap P^2$ and fixed line $L_k \subset P^2$.*

From (6), (4) and (3), we have $\operatorname{Det}\Omega(dA_{12}, dA_{12}) = 1$, $\operatorname{trac}\Omega(dA_{12}, dA_{12}) = 0$ where Ω is a quadratic form defined as the following

$$\Omega(A_{ij}, A_{mn}) = \delta_{mn}^{ij} = \begin{cases} 1, & i \neq j \neq m \neq n, \\ 0, & \text{otherwise} . \end{cases}$$

Thus, we have proved the following:

Lemma 2. *The K-absolutum $\overline{Q}_4^2 \subset \overline{P}_5^k$ is a minimal hyper surface with Gaussian curvature equal one.*

The coordinates of the i-th vertex of the frame $\{A_i\}$ are δ_j^i. Thus the coordinates A_{mn}^{ij} of the K-R to the line A_{mn} are given by $A_{mn}^{ij} = \delta_m^i\, \delta_n^j - \delta_n^i\, \delta_m^j$ and so we have the following:

Lemma 3. *Each pair of the K-points P_{ij}, P_{mm}, whose index pairs contain at least one common number, satisfy*

$$\Omega(P_{ij}, P_{mn}) = 0 .$$

In the case of Lemma 3, the lines P_{ij}, P_{mn} are called in involution (projectively orthogonal).

3 – Three-dimensional K-manifolds

We establish the fundamental equations of a 3-dimensional K-manifold (line complex in $\overline{P}_3$) immersed in $\overline{Q}_4^2 \subset \overline{P}_5^k$, for brevity $M_3^k \subset \overline{Q}_4^2 \subset \overline{P}_5^k$. From the displacement dA_{12} in (6), it follows that the principal forms on the K-absolutum $\overline{Q}_4^2$ are ω_μ^η and from (5) we have $D\omega_2^3 \equiv 0 \pmod{\omega_2^3}$. Thus, the differential equation of the K-manifold M_3^k can be written as the following

$$\omega_2^3 = B_\alpha\, \theta^\alpha \, , \tag{7}$$

where $(\theta^\alpha) = (\omega_1^3, \omega_1^4, \omega_2^4)$ are the principal forms on M_3^k and B_α are real valued functions on the first order contact element U_1.

Exterior differentiation of (7) leads to the quadratic exterior equation

$$
\begin{aligned}
(8)\qquad & \Big\{dB_1 + B_1(B_1\theta_1 + \theta_2 - \theta_3) - 3B_2\theta^2\Big\} \wedge \theta^1 + \\
& \quad + \Big\{dB_2 + B_2(B_1\theta_1 + 4\theta_2) + B_1B_3\theta^1\Big\} \wedge \theta^2 + \\
& \quad\quad + \Big\{dB_3 + B_3(\theta_2 + 3\theta_3 + \theta^1) + \Delta^+\theta_1\Big\} \wedge \theta^3 = 0 \ ,
\end{aligned}
$$

where $\Delta^+ = B_2 + B_1B_3$.

The variations δB_α of the invariants B_α are given by [8]

$$
(9)\qquad \delta B_\alpha = -M_{\alpha\beta}\,\pi_\beta \ ,
$$

where $\theta^\alpha(\delta) = 0$, $\theta_\beta(\delta) = \pi_\beta$, $(\theta_\beta) = (\omega_1^2, \omega_3^3, \omega_2^2)$ and δ is the differentiation with respect to the secondary parameters. The matrix $(M_{\alpha\beta})$ in (9) is called the attitude matrix and is defined in terms of the invariants B_α as the following

$$
(10)\qquad
\begin{aligned}
& M_{11} = B_1^2, \quad M_{12} = -M_{13} = -B_1, \quad M_{22} = 0, \quad M_{23} = 4B_1 \ , \\
& 4M_{21} = M_{13}M_{23}, \quad M_{33} = 3B_3 = 3M_{32} \ , \\
& 4M_{31} = 4M_{32}M_{13} + M_{23} \ .
\end{aligned}
$$

In general the matrix $(M_{\alpha\beta})$ has rank $h = 3$, that is $B_1B_2\,\Delta^+ \neq 0$. From (6) ad (7), we get

$$
dA_{12} \equiv \theta^1(B_1A_{13} - A_{23}) + \theta^2(B_2A_{13} - A_{24}) + \theta^3(B_3A_{13} + A_{14}) \pmod{A_{12}} \ .
$$

Using Lemma 3, we have the quadratic form

$$
(11)\qquad \Omega(dA_{12}, dA_{12}) \equiv a_{\alpha\beta}\,\theta^\alpha\,\theta^\beta \ ,
$$

defined on the K-manifold M_3^k, where $a_{11} = 0$, $a_{12} = -B_1$, $a_{13} = -1$, $a_{22} = -2B_2$, $a_{23} = -B_3$, $a_{33} = 0$ and its determinant is $\Delta^- = B_2 - B_1B_3$. In general $(\Delta^- \neq 0)$ the rank h' of the quadratic form (11) is three.

The following definitions are very important in the sequel [5].

Definition 1. The K-manifold M_3^k for which $h < 3$ is called singular of rank $3 - h$, $h \leq 2$.

Definition 2. The K-manifold M_3^k for which $h' < 3$ is called special of order $3 - h'$, $h' \leq 2$.

From (11), one can see that h' can not be equal to zero or one and hence $h' = 2$ in the case where $\Delta^- = 0$. Thus, we have

Lemma 4. *The K-manifold M_3^k characterized by $\Delta^- = 0$ is of type special of order one.*

The definitions (1) and (2) lead to the following

Lemma 5. *Singular K-manifolds M_3^k of rank one and non-special are divided into three subclasses given by*

$$\begin{aligned} \omega_2^3 &= B_2\theta^2 + B_3\theta^3, && (\textit{Type } T_1)\\ \omega_2^3 &= B_1\theta^1 + B_3\theta^3, && (\textit{Type } T_2)\\ \omega_2^3 &= B_1\theta^1 - B_1B_3\theta^2 + B_3\theta^3, && (\textit{Type } T_3) \end{aligned}$$

Lemma 6. *Singular K-manifolds M_3^k of rank two and special of order one are given by the following $\omega_2^3 = B_3\theta^3$. We denote its type by T_4.*

Lemma 7. *Singular K-manifolds M_3^k of rank three and special of order one are given by the holonomic equation $\omega_2^3 = 0$.*

In each of the above types, the existence theorem can be proved using Cartan's common methods. Thus, we have:

Theorem 1. *The range of existence of the K-manifolds of types $T_\alpha(T_4)$ comprises one arbitrary function of two arguments (one argument).*

For the general K-manifold (7), we may specialize the frames such that

$$\theta_\alpha = C_{\alpha\beta}\,\theta^\beta\ . \tag{12}$$

Using Cartan's lemma in (8), we have

$$dB_\alpha + M_{\alpha\beta}\,\theta_\beta = E_{\alpha\beta}\,\theta^\beta\ , \tag{13}$$

where $E_{\alpha\beta}$ are invariants defined in the 2nd order contact element $U_2 \subset U_1$ on the K-manifold M_3^k. The invariants $E_{\alpha\beta}$ satisfy the integrability conditions

$$E_{12} = E_{21} + 2B_2 + \Delta^+\,, \qquad E_{23} = E_{32}\,, \qquad E_{13} = E_{31} + B_3\ . \tag{14}$$

From (12), (13) and (14), we get

$$dB_\alpha = b_{\alpha\beta}\,\theta^\beta\ , \tag{15}$$

where

$$b_{1\alpha} = E_{1\alpha} - B_1(B_1 C_{1\alpha} - C_{2\alpha} + C_{3\alpha}) + 3B_2(\alpha-1)(3-\alpha) ,$$

$$b_{2\alpha} = E_{2\alpha} - B_2(B_1 C_{1\alpha} + 4C_{3\alpha}) - B_1 B_3(2-\alpha)(3-\alpha)/2 ,$$

$$b_{3\alpha} = E_{3\alpha} - \Big(\Delta^+ C_{1\alpha} + B_3(C_{2\alpha} + 3C_{3\alpha} + (2-\alpha)(3-\alpha)/2)\Big) .$$

The Gauss equation is given by

$$d^2 A_{12} \equiv b^2_{\alpha\beta}\, \theta^\alpha\, \theta^\beta A_{13} - a_{\alpha\beta}\, \theta^\alpha\, \theta^\beta A_{34} . \tag{16}$$

The quantities $b^2_{\alpha\beta}$ are the components of covariant quadratic symmetric tensor defined in terms of the quadratic tensors $a_{\alpha\beta}$, $E_{\alpha\beta}$, $C_{\alpha\beta}$ as the following:

$$b^2_{11} = E_{11} + B_3 - B_1(1 + C_{31} - C_{21}) ,$$

$$b^2_{22} = E_{22} + B_2(B_3 - 4C_{32}) ,$$

$$b^2_{33} = E_{33} - B_3(C_{23} + 3C_{33}) ,$$

$$b^2_{12} = 2E_{12} + B_1(2B_3 - C_{32} + C_{22}) - 4B_2(1 + C_{31}) ,$$

$$b^2_{13} = 2E_{13} - B_3(2 + C_{21} + 3C_{31}) - B_1(C_{33} - C_{23}) ,$$

$$b^2_{23} = 2E_{23} + B_3(B_3 - C_{22} - 3C_{32}) - 4B_2 C_{33} .$$

The Wiengarten equations are

$$dA_{34} = -\Big(\theta^1 + (C_{3\alpha} + C_{2\alpha})\,\theta^\alpha\Big)\, A_{34} ,$$

$$dA_{13} = (\theta^1 + 2C_{3\alpha}\,\theta^\alpha)\, A_{13} - \theta^2 A_{34} + C_{1\alpha}\theta^\alpha A_{23} .$$

In our present investigation we are again concerned with the K-manifold M^k_3 given by (7) and we continue to require that $h=3$, $h'=3$.

4 – Two-dimensional K-manifold

If there exists a relation between the forms θ^α on M^k_3, we have a 2-dimensional K-manifold. Without loss of generality, if we take

$$\omega^4_2 = E_\mu\, \psi^\mu , \tag{17}$$

where $(\psi^\mu) = (\theta^\eta)$, such that the equation (7) for (17) represents a two-dimensional K-manifold (line congruence in $\overline{P}_3$) immersed in the K-manifold M^k_3 and we denote it by $\widehat{M}^k_2$. This immersion is given by [9], [10]

$$\omega^4_2 = E_\mu\, \psi^\mu , \quad \omega^3_2 = \widehat{E}_\mu\, \psi^\mu , \tag{18}$$

where $\widehat{E}_\mu = B_\mu + B_3 E_\mu$ and E_μ are functions defined in the 1st order contact element $\widehat{U}_1$ of the K-point A_{12} on

$$\widehat{M}_2^k \subset M_3^k \subset \overline{Q}_4^2 .$$

Exterior differentiation of (18) and using Cartan's lemma, there exist the real valued functions $\widehat{E}_{i\mu}\colon \widehat{U}_2 \subset \widehat{U}_1 \to \mathbb{R}$ such that

$$dE_i + \widehat{M}_{i\alpha}\,\theta_\alpha = \widehat{E}_{i\mu}\,\psi^\mu \ , \tag{19}$$

where $(E_\eta) = (\widehat{E}_\mu)$ and the invariants $\widehat{E}_{i\mu}$ satisfy the integrability conditions

$$\widehat{E}_{12} - \widehat{E}_{21} = 3\widehat{E}_2\,, \qquad E_{32} - E_{41} = E_2 - \widehat{E}_1 + \widehat{M}_{21} \ .$$

The invariants $\widehat{M}_{i\alpha}$ are defined in terms of E_μ, $\widehat{E}_\mu$ as the following:

$$\begin{aligned}
&\widehat{M}_{11} = E_1(E_2 + \widehat{E}_1)\,, \qquad \widehat{M}_{12} = \widehat{M}_{13} = -2E_1 \ ,\\
&\widehat{M}_{21} = E_1\widehat{E}_2 + E_2^2\,, \qquad \widehat{M}_{22} = -\widehat{M}_{23} = -E_2 \ ,\\
&\widehat{M}_{31} = E_1\widehat{E}_2 - \widehat{E}_1^2\,, \qquad \widehat{M}_{32} = -\widehat{M}_{33} = -\widehat{E}_1 \ ,\\
&\widehat{M}_{41} = \widehat{E}_2(E_2 + \widehat{E}_1)\,, \qquad \widehat{M}_{42} = 0\,, \qquad \widehat{M}_{43} = -4\widehat{E}_2 \ .
\end{aligned}$$

Using (12), the forms θ_α on the K-manifold $\widehat{M}_2^k$ are given by

$$\theta_\alpha = \widehat{C}_{\alpha\mu}\,\psi^\mu\,, \qquad \widehat{C}_{\alpha\mu} = C_{\alpha\mu} + C_{\alpha 3}E_\mu \ .$$

The Gauss and Wiengarten equations of the immersion (18) are given as the following

$$\begin{aligned}
&dA_{12} \equiv \psi^\mu\, e_\mu \pmod{A_{12}} \ ,\\
&d^2A_{12} \equiv \phi^{14}A_{14} + \phi^{13}A_{13} + \phi^{34}A_{34} \pmod{A_{12}, dA_{12}} \ ,
\end{aligned} \tag{20}$$

where $e_\mu = E_\mu A_{14} + \widehat{E}_\mu A_{13} - N_\mu$, $N_1 = A_{23}$, $N_2 = A_{24}$, $\phi^{14} = F^1_{\mu\nu}\,\psi^\mu\,\psi^\nu$, $\phi^{13} = F^2_{\mu\nu}\,\psi^\mu\,\psi^\nu$, $\phi^{34} \equiv \widehat{a}_{\mu\nu}\,\psi^\mu\,\psi^\nu \equiv \Omega(dA_{12}, dA_{12})$, $A_{12} \in \widehat{M}_2^k$.

The invariants $F^\mu_{\mu\nu}$, $\widehat{a}_{\mu\nu}$ are symmetric in the indices μ, ν and are given by

$$F^1_{11} = \widehat{E}_{31} + 2E_1\zeta_1\,, \quad F^1_{22} = \widehat{E}_{42} - \widehat{E}_2 + E_2\zeta_{-1}\ ,$$
$$F^1_{12} = \widehat{E}_{32} + \frac{1}{2}E_2(1+\zeta_{-1}) + E_1\zeta_1 - \widehat{E}_1\ ,$$
$$F^2_{11} = \widehat{E}_{11} - 1 + \widehat{E}_1\zeta_{-1}\,, \quad F^2_{22} = \widehat{E}_{22} - e\widehat{E}_2(C_{32} + C_{33}E_2)\ ,$$
$$F^2_{12} = \widehat{E}_{12} + \frac{1}{2}\widehat{E}_2\Big(3 - 4(C_{31} + C_{33}E_1)\Big) + \frac{1}{2}\widehat{E}_1\zeta_{-1}\ ,$$
$$\widehat{a}_{11} = -2E_1\,, \quad \widehat{a}_{12} = -(\widehat{E}_1 + E_2)\,, \quad \widehat{a}_{22} = -2\widehat{E}_2\ ,$$
$$\xi_\varepsilon = C_{22} + \varepsilon\, C_{32} + (C_{32} + \varepsilon\, C_{33})\, E_2 \quad \text{and}$$
$$\zeta_\varepsilon = C_{21} + \varepsilon\, C_{31} + (C_{23} + \varepsilon\, C_{33})\, E_1\,, \quad \varepsilon = \pm 1\ .$$

In [6], a computational technique for the Gaussian curvatures K and $\widehat{K}$ of the K-manifold M^k_3 and the immersion $\widehat{M}^k_2$ is given. Thus, we have

$$\begin{aligned} &K = \mathrm{Det}(b^2_{\alpha\beta})/\Delta^-\ , \\ &\widehat{K} = \Big(\mathrm{Det}(F^1_{\mu\nu}) + \mathrm{Det}(F^2_{\mu\nu})\Big)\ /\ \mathrm{Det}(\widehat{a}_{\mu\nu})\,, \quad \text{respectively}\ . \end{aligned} \tag{21}$$

5 – CK-curves

On the K-manifold M^k_3, if there exist two independent relations between the forms θ^α as the following

$$\theta^\mu = \phi^\mu\,\theta^3\,, \quad \omega^3_2 = \phi^3\,\theta^3\,, \quad \phi^3 = B_\mu\,\phi^\mu + B_3\ . \tag{22}$$

The system (22) represent a K-curve (ruled surface in $\overline{P}_3$) immersed in the K-manifold M^k_3 or for brevity, a CK-curve.

Exterior differentiation of (22) and using Cartan's lemma, we get

$$d\phi^\alpha = \Sigma\,\widetilde{M}_{\alpha\beta}\,\theta_\beta + F^\alpha\,\theta^3\ , \tag{23}$$

where $\widetilde{M}_{\alpha\beta}$ are the elements of an attitude matrix attached to the CK-curve (22) and are given by

$$3\widetilde{M}_{11} = -3\widetilde{M}_{32} = -\widetilde{M}_{33} = 3\,\phi^3\,, \quad \widetilde{M}_{12} = \widetilde{M}_{13} = -2\,\phi^1\ ,$$
$$\widetilde{M}_{21} = 1\,, \quad \widetilde{M}_{22} = -\widetilde{M}_{23} = -\phi^2\,, \quad \widetilde{M}_{31} = 0\ .$$

The functions ϕ^α, F^α are invariants in the 1st, 2nd order contact elements $\widetilde{U}_1$, $\widetilde{U}_2$ ($\widetilde{U}_2 \subset \widetilde{U}_1$) of the K-point A_{12} on the CK-curve (22) respectively.

The δ-variations of the invariants ϕ^α ($F^\alpha(\delta) = 0$) are given by

$$\delta\,\phi^\alpha = \Sigma\,\widetilde{M}_{\alpha\beta}\,\pi_\beta\ . \tag{24}$$

For a general CK-curve, the matrix ($\widetilde{M}_{\alpha\beta}$) has rank three. The curves (22) are singular of rank one and two if the following conditions

$$(I_1) \qquad \phi^3 = 0,\ \ \phi^1 \neq 0\,, \qquad B_\mu\,\phi^\mu = -B_3\ ,$$

$$(I_2) \qquad \phi^3 = 0\ \ \phi^1 = 0\,, \qquad \phi^2 = -(B_3/B_2),\ \ B_2 \neq 0\ ,$$

are satisfied respectively.

We denote the classes of CK-curves according to the conditions I_μ by C_μ respectively.

In the following, we consider differential projective invariants of all orders on the C_μ curves.

For this purpose, we derive the projective Frenet–Serret formulae and the differential equations of the classes C_μ of CK-curves.

The class C_1 is characterized by the differential equations

$$\begin{aligned} &\theta^1 = 0\,, \qquad \omega_2^3 = 0\,, \qquad \theta^2 = \phi^2\,\theta^3\ ,\\ &D\theta^3 \equiv 0\ (\operatorname{mod}\theta^3)\,, \qquad B_2\,\phi^2 + \phi^3 = 0\ . \end{aligned} \tag{25}$$

The motion along a CK-curve of the class C_1 is given by

$$\begin{aligned} dA_{12} &\equiv -\theta^3(\phi^2\,A_{24} + A_{14}) \quad (\operatorname{mod} A_{12})\ ,\\ d^2A_{12} &\equiv F^2(\theta^3)^2\,A_{24} \quad (\operatorname{mod} A_{12}, dA_{12})\ ,\\ d^3A_{12} &\equiv 0 \quad (\operatorname{mod} A_{12}, dA_{12}, d^2A_{12})\ . \end{aligned}$$

Thus, the CK-curves of the class C_1 are plane curves in the K-plane (A_{12}, A_{14}, A_{24}) with projective curvature equal to the invariant F^2. Thus, we have proved the following

Theorem 2. *The CK-curves of the class C_1 are plane CK-curves (developable ruled surfaces of a line complex in $\overline{P}_3$) with curvature equal to the invariant F^2.*

The CK-curves of the class C_2 are characterized by the system of differential equations

$$\theta^\mu = \phi^\mu\,\theta^3,\ \ \omega_2^3 = 0,\ \ B_\mu\,\phi^\mu = -B_3,\ \ D\theta^3 \equiv 0 \quad (\operatorname{mod}\theta^3)\ . \tag{26}$$

From (26) and (12) we get

$$\theta_\alpha = (C_{\alpha\mu}\,\phi^\mu + C_{\alpha 3})\,\theta^3 \ . \tag{27}$$

Putting $\widehat{F}^\mu = F^\mu/(2\phi^1)$, $d/d\theta^3 \equiv D$, $h^\varepsilon = (C_{3\mu} + \varepsilon C_{2\mu})\,\phi^\mu + C_{33} + \varepsilon\, C_{23}$, $\varepsilon = \pm 1$.

From (3), (6), (7), (26) and (27) we get

$$D^r A_{12} \equiv \Omega^r\, Q_r \pmod{A_{12}, Q_1, Q_2, ..., Q_{r-1}} \qquad (r = 1, 2, ..., 5)\ , \tag{28}$$

where $Q_1 = \phi^\mu\, N_\mu - A_{14}$, $Q_2 = A_{34} - \widehat{F}^\mu\, N_\mu$,

$$Q_3 = A_{24} + f\, A_{23}\,, \qquad Q_4 = N_1\ ,$$

$$Q_5 = Q_2 + \widehat{F}^\mu\, N_\mu \quad \text{and}$$

$$f = \Big(F^1 (D \log|\widehat{F}^1| + 2\,h^1 - \widehat{F}^1)\Big)/\Omega^3\ .$$

The invariants Ω^r are given by the relations

$$\begin{aligned}
&\Omega^1 = -1\,, \qquad \Omega^2 = -2\,\phi^1\ ,\\
&\Omega^3 = F^2\Big(D \log|\widehat{F}^2| - \widehat{F}^1 + h^{-1}\Big)\ ,\\
&\Omega^4 = \Omega^3\Big(Df + f\Big((C_{31} + 3C_{21})\,\phi^1 + (C_{32} + 3C_{22})\,\phi^2\\
&\qquad + C_{33} + 3C_{23}\Big) + (f\widehat{F}^2 - \widehat{F}^1)\Big)\ ,\\
&\Omega^5 = -\Omega^4\ .
\end{aligned} \tag{29}$$

The invariants Ω^i $(i \neq 1)$ are called the projective curvatures of CK-curves (non developable ruled surfaces in $\overline{P}_3$) of the class C_2. Thus, we have proved the following

Theorem 3. *The infinitesimal displacements of the Frenet–Serret frame $\{A_{12}, Q_r\}$ are given by (28) and the projective curvatures are given by (29).*

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