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ON A RESULT OF WILLIAMSON

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Abstract: In this paper we generalize a result of Williamson on the structure of the kernel of a symmetrizer and also obtain some other related results.

1 – Introduction

Let S_m be the full symmetric group of degree m and $c(\sigma)$ an arbitrary nonzero function from S_m into the complex field $\mathbb{C}$. Given an $m\times m$ matrix $X=[x_{ij}]$ we define its generalized matrix function $d_c(X)$ by

$$d_c(X)=\sum_{\sigma\in S_m} c(\sigma)\prod_{i=1}^m x_{i\sigma(i)}\ .$$

When $c=\lambda$ is a character of a subgroup G of S_m, we will write d_c as d^G_λ. We denote by $\Gamma_{m,n}$ the set of maps from $\{1,...,m\}$ into $\{1,...,n\}$. If $\alpha\in\Gamma_{m,n}$, we identify it with the m-tuple $(\alpha(1),...,\alpha(m))$. For an $n\times n$ matrix $A=[a_{ij}]$ and $\alpha,\beta\in\Gamma_{m,n}$, $A[\alpha|\beta]$ will denote the $m\times m$ matrix whose (i,j) element is $a_{\alpha(i),\beta(j)}$. For $\alpha\in\Gamma_{m,n}$, $\sigma\in S_m$, we write $\alpha\,\sigma=(\alpha(\sigma(1)),...,\alpha(\sigma(m)))$. We also write $e=(1,...,m)$.

Let V be an n-dimensional unitary vector space over $\mathbb{C}$, and $\bigotimes^m V$ be its m-th tensor power. If $\sigma\in S_m$, there exists a unique linear operator $P(\sigma)$ on $\bigotimes^m V$ such that

$$P(\sigma)\,x_1\otimes...\otimes x_m=x_{\sigma^{-1}(1)}\otimes...\otimes x_{\sigma^{-1}(m)}\ .$$

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The linear mapping

$$T_c = \sum_{\sigma \in S_m} c(\sigma)\, P(\sigma)$$

will be called a symmetrizer. The star product $x_1 * ... * x_m$ is, by definition, $T_c(x_1 \otimes ... \otimes x_m)$.

As we know, the characterization of the kernel of the symmetrizer T_c is equivalent to that of the following set (see [2] and [4])

$$\mathcal{N}(c) = \left\{ A \in M_m(\mathbb{C}) \mid d_c(AX) = 0,\ \forall\, X \right\} . \tag{1.1}$$

In [7], Williamson proves a fundamental combinatorial property of cyclic permutations of finite sequences of integers and considers an application of this result to the characterization of the kernel of T_c when c is a homomorphism from G into $\mathbb{C}$.

If Δ is an orbit of the subgroup G, let G^Δ be the subgroup of G restricted to Δ. Following Williamson, we denote by $\mathcal{G}$ the class of all subgroups G of S_m such that if Δ is any orbit of G the G^Δ is cyclic. For $\alpha \in \Gamma_{m,n}$ we shall denote by G_α that subgroup of G defined by

$$G_\alpha = \left\{ \sigma \in G \mid \alpha(\sigma(i)) = \alpha(i),\ i = 1, ..., m \right\} .$$

For any homogeneous tensor $w = y_1 \otimes ... \otimes y_m$, α is called an indicator of w if $\alpha(i) = \alpha(j)$ if and only if y_i and y_j are linearly dependent. Now we are able to state the following two results of Williamson [7]:

Theorem 1.1. *Let $G \in \mathcal{G}$. For any $\gamma \in \Gamma_{m,n}$ such that γ has at least two elements, there exists $\omega \in \Gamma_{m,n}$ such that:*

i) range $\omega \subseteq$ range γ;

ii) $\gamma(i) \neq \omega(i)$, $i = 1, ..., m$;

iii) *For each $\sigma \notin G_\gamma$ there is an integer j, $1 \le j \le m$, such that $\gamma(\sigma(j)) = \omega(j)$.*

Theorem 1.2. *Let $G \in \mathcal{G}$ and $w = y_1 \otimes ... \otimes y_m$. If λ is any homomorphism of G into $\mathbb{C}$ and γ an indicator of w, then $y_1 \otimes ... \otimes y_m$ is in the kernel of T_λ iff $\sum_{\sigma \in G_\gamma} \lambda(\sigma) = 0$.*

In this paper, we generalize Theorem 1.2 to arbitrary functions c. We also obtain some other related results.

2 – Results

Let $E = \{e_1, ..., e_n\}$ be an orthonormal basis of V. For $\alpha \in \Gamma_{m,n}$, let $x^{\otimes}_{\alpha} = x_{\alpha(1)} \otimes ... \otimes x_{\alpha(m)}$ and $T_c(x^{\otimes}_{\alpha}) = x^*_{\alpha}$. Define as in [3].

$$b(\pi) = \sum_{\sigma \in S_m} c(\sigma\, \pi)\, \overline{c(\sigma)}\,, \qquad c \in S_m \;.$$

Particularly, when $c = \lambda$ is a character of the subgroup G, we have

$$b(\pi) = \sum_{\sigma \in G} \lambda(\sigma\, \pi)\, \overline{\lambda(\sigma)} = \frac{|G|}{\lambda(id)}\, \lambda(\pi) \;. \tag{2.1}$$

If $v^* = v_1 * ... * v_m$ with $v_i = \sum_{l=1}^{n} a_{il}\, e_l \in V$, then

$$v^* = \sum_{\alpha \in \Gamma_{m,n}} a_\alpha\, T_c\Big(e_{\alpha(1)} \otimes ... \otimes e_{\alpha(m)}\Big) = \sum_{\alpha \in \Gamma_{m,n}} a_\alpha\, e^*_\alpha$$

with $a_\alpha = a_{1\alpha(1)} \cdots a_{m\alpha(m)}$, and $\|e^*_\alpha\|^2 = \sum_{\sigma \in G_\alpha} b(\sigma)$.

We have already known that (see [1], [6]):

$$e^*_\alpha = 0 \quad \text{iff} \quad \sum_{\pi \in G_\alpha} c(\sigma\, \pi) = 0, \quad \forall\, \sigma \in S_m \;. \tag{2.2}$$

When $c = \lambda$ is a character of the subgroup $G \subseteq S_m$, a stronger result can be obtained. In fact we have

Proposition 2.1. *Let λ be a character of the subgroup G. Then $e^*_\alpha = 0$ iff $\sum_{\sigma \in G_\alpha} \lambda(\pi\, \sigma\, \tau) = 0$ for $\forall\, \pi, \tau \in G$.*

Proof: At first, we have

$$(e^*_{\alpha\tau}, e^*_{\beta\pi}) = \frac{\lambda(id)}{|G|} \sum_{\sigma \in G} \lambda(\sigma)\, \delta_{\alpha, \beta\pi\sigma\tau^{-1}}$$

and

$$\|e^*_\alpha\|^2 = \frac{\lambda(id)}{|G|} \sum_{\sigma \in G_\alpha} \lambda(\sigma) \;.$$

The "if" part is easy.

The "only if" part: When $e^*_\alpha = 0$, then for arbitrary $\sigma \in G$, $e^*_{\alpha\sigma} = 0$ and for $\forall\, \pi, \tau \in G$,

$$\begin{aligned}(e^*_{\alpha\tau}, e^*_{\alpha\pi}) &= \frac{\lambda(id)}{|G|} \sum_{\sigma \in G} \lambda(\sigma)\, \delta_{\alpha, \alpha\pi\sigma\tau^{-1}} \\ &= \frac{\lambda(id)}{|G|} \sum_{\sigma \in G_\alpha} \lambda(\pi^{-1}\, \sigma\, \tau) = 0 \;. \blacksquare\end{aligned}$$

With this result, we can prove

Proposition 2.2. *Let λ be a character of the subgroup G. Then $\sum_{\sigma\in G_\alpha}\lambda(\sigma)=0$ iff $\sum_{\sigma\in G}\lambda(\sigma)\,\phi(\alpha\sigma)=0$ for arbitrary $\phi\colon \Gamma_{m,n}\to C$.*

Proof: The "if" part: Since $\sum_{\sigma\in G}\lambda(\sigma)\,\phi(\alpha\sigma)=0$ for arbitrary $\phi\colon \Gamma_{m,n}\to C$, let

$$\phi(\beta)=\begin{cases}1, & \text{if } \beta=\alpha,\\ 0, & \text{otherwise}\ .\end{cases}$$

Then we have

$$\sum_{\sigma\in G}\lambda(\sigma)\,\phi(\alpha\,\sigma)=\sum_{\sigma\in G_\alpha}\lambda(\sigma)=0\ .$$

The "only if" part: Let $\tau_1,\dots,\tau_r$ be a system of right coset representatives of G_α in G. Using Proposition 2.1,

$$\begin{aligned}\sum_{\sigma\in G}\lambda(\sigma)\,\phi(\alpha\,\sigma)&=\sum_{j=1}^{r}\sum_{\sigma\in G_\alpha}\lambda(\sigma\,\tau_j)\,\phi(\alpha\,\sigma\,\tau_j)\\ &=\sum_{j=1}^{r}\Big(\sum_{\sigma\in G_\alpha}\lambda(\sigma\,\tau_j)\Big)\,\phi(\alpha\,\tau_j)=0\ .\ \blacksquare\end{aligned}$$

The next result can be similarly proved:

Corollary 2.3. *Let λ be a character of the subgroup G. Then $\sum_{\sigma\in G_\alpha}\lambda(\sigma)=0$ iff $\sum_{\sigma\in G}\lambda(\sigma^{-1})\,\phi(\alpha\sigma)=0$ for arbitrary $\phi\colon \Gamma_{m,n}\to C$.*

Now we come to discuss the case when c is an arbitrary function. With (2.2), we can prove

Proposition 2.4. *If $e^*_\alpha=0$, then $A[\alpha|e]\in\mathcal{N}(c)$.*

Proof: Let $\tau_1,\dots,\tau_r$ be a system of left coset representatives of G_α in S_m, using (2.2), for arbitrary $\beta \in \Gamma_{m,n}$, we have

$$\begin{aligned} d_c A[\alpha|\beta] &= \sum_{\sigma \in S_m} c(\sigma) \prod_{i=1}^{m} a_{\alpha(i),\beta(\sigma(i))} \\ &= \sum_{\sigma \in S_m} c(\sigma) \prod_{i=1}^{m} a_{\alpha(\sigma^{-1}(i)),\beta(i)} \\ &= \sum_{j=1}^{r} \sum_{\sigma \in G_\alpha} c(\tau_j\, \sigma) \prod_{i=1}^{m} a_{\alpha(\sigma^{-1}\tau_j^{-1}(i)),\beta(i)} \\ &= \sum_{j=1}^{r} \Big(\sum_{\sigma \in G_\alpha} c(\tau_j\, \sigma) \Big) \prod_{i=1}^{m} a_{\alpha(\tau_j^{-1}(i)),\beta(i)} = 0 \ . \end{aligned}$$

Noting that (see [2] or [5])

$$d_c(A[\alpha|e]\, X) = \sum_{\beta \in \Gamma_{m,n}} d_c A[\alpha|\beta] \prod_{i=1}^{m} x_{\beta(i)i} \ ,$$

we arrive at $d_c(A[\alpha|e]\, X) = 0$ for arbitrary X. ∎

Bearing in mind the definition of the indicator of a homogeneous tensor, recalling the remark preceding (1.1) and using Proposition 2.4, we can easily prove the following

Corollary 2.5. *Let γ be an indicator of $w = y_1 \otimes ... \otimes y_m$. If $e^*_\gamma = 0$, then $y_1 \otimes ... \otimes y_m$ is in the kernel of T_c.*

Now we are in a position to prove the main result of this paper.

Proposition 2.6. *Let G be in $\mathcal{G}$, c an arbitrary function from G into $\mathbb{C}$ and $b(\pi) = \sum_{\sigma \in G} c(\sigma\, \pi)\, \overline{c(\sigma)}$. Let γ be an indicator of $v^{\otimes} = v_1 \otimes ... \otimes v_m$. Then $v^* = 0$ if and only if $\sum_{\sigma \in G_\gamma} b(\sigma) = \nu(\gamma) = 0$.*

Proof: The "if" part follows immediately from Corollary 2.5. For the "only if" part, our proof parallels that of [7], with some slight modifications. As in [7], we assume that $\nu(\gamma) \neq 0$. If $\gamma(1) = ... = \gamma(m)$, then $G_\gamma = G$ and

$$\begin{aligned} \nu(\gamma) = \sum_{\sigma \in G_\gamma} b(\sigma) = \sum_{\sigma \in G} b(\sigma) &= \sum_{\sigma \in G} \sum_{\pi \in G} c(\pi\, \sigma)\, \overline{c(\pi)} \\ &= \sum_{\pi \in G} \Big(\sum_{\sigma \in G} c(\pi\, \sigma) \Big) \overline{c(\pi)} = \Big\| \sum_{\pi \in G} c(\pi) \Big\|^2 \neq 0 \ . \end{aligned}$$

Hence

$$v^* = T_c(v_1 \otimes \dots \otimes v_m) = K v_\gamma^\otimes \sum_{\sigma \in G} c(\sigma) \neq 0 \ ,$$

where K is a nonzero constant.

Assume γ has at least two elements. Let $\tau_1 = id, \dots, \tau_r$ be a system of left coset representatives of G_γ in G. Then

$$\begin{aligned} \sum_{\sigma \in G} b(\sigma)\, P(\sigma)\, v^\otimes &= \sum_{\sigma \in G} b(\sigma)\, P(\sigma)\, (v_1 \otimes \dots \otimes v_m) \\ &= K \sum_{\sigma \in G} b(\sigma)\, P(\sigma)\, v_\gamma^\otimes \\ &= K \sum_{i=1}^{r} \sum_{\sigma \in G_\gamma} b(\tau_i\, \sigma)\, P(\tau_i\, \sigma)\, v_\gamma^\otimes \\ &= K \sum_{i=1}^{r} \sum_{\sigma \in G_\gamma} b(\tau_i\, \sigma)\, P(\tau_i)\, v_\gamma^\otimes \ . \end{aligned} \tag{2.3}$$

Let $z^\otimes = z_1 \otimes \dots \otimes z_m$. Then from (2.3), we have

$$\begin{aligned} (v^*, z^*) &= \Big(\sum_{\sigma \in G} b(\sigma)\, P(\sigma)\, v^\otimes,\ z^\otimes \Big) \\ &= K \sum_{i=1}^{r} \sum_{\sigma \in G_\gamma} b(\tau_i\, \sigma)\, (P(\tau_i)\, v_\gamma^\otimes,\ z^\otimes) \ . \end{aligned} \tag{2.4}$$

By Theorem 1.1, there exists an ω such that

i) $\operatorname{range} \omega \subseteq \operatorname{range} \gamma$;

ii) $\gamma(i) \neq \omega(i)$, $i = 1, \dots, m$;

iii) For each σ not in G_γ there is an i such that $\gamma(\sigma(i)) = \omega(i)$.

Now we may choose $z^\otimes = z_1 \otimes \dots \otimes z_m$ such that:

$$(v_{\gamma(i)}, z_i) = 1 \quad \text{and} \quad (v_{\omega(i)}, z_i) = 0 \qquad \text{for} \quad i = 1, \dots, m \ . \tag{2.5}$$

This is possible since γ is an indicator for $v^\otimes$, and also because $\operatorname{range} \omega \subseteq \operatorname{range} \gamma$, $\gamma(i) \neq \omega(i)$ implies $v_{\gamma(i)}$ and $v_{\omega(i)}$ are linearly independent.

For any i we have

$$(P(\tau_i) v_\gamma^\otimes,\ z^\otimes) = \prod_{t=1}^{m} (v_{\gamma(\tau_i^{-1}(t))}, z_t) \ .$$

When $i \geq 2$, using iii) of Theorem 1.1, there exists j such that $\gamma(\tau_i^{-1}(j)) = \omega(j)$, and the term

$$(v_{\gamma(\tau_i^{-1}(j))}, z_j) = (v_{\omega(j)}, z_j) = 0 \ .$$

So for any $i \geq 2$, $(P(\tau_i) v_\gamma^\otimes, z^\otimes) = 0$.

For $i = 1$, according to (2.5), $(v_\gamma^\otimes, z^\otimes) = 1$ and from (2.4),

$$(v^*, z^*) = K \sum_{\sigma \in G_\gamma} b(\sigma) = K\,\nu(\gamma) \neq 0 \ .$$

Therefore, $T_c(v^\otimes) = v^* \neq 0$. This ends the proof. ∎

REFERENCES

[1] Gong, M.-P. – *Notes on semigroups defined by* $d_c(AX) = 0$, to appear.

[2] Gong, M.-P. – *Semigroups defined by generalized Schur functions*, to appear.

[3] Merris, R. and Dias da Silva, J.A. – Generalized Schur functions, *J. Algebra,* 35 (1975), 442–448.

[4] Oliveira, G.N. and Dias da Silva, J.A. – Conditions for equality of decomposable symmetric tensors, II, *Linear Alg. Appl.,* 28 (1979), 161–176.

[5] Oliveira, G.N. and Dias da Silva, J.A. – Equality of decomposable symmetrized tensors and $*$-matrix groups, *Linear Alg. Appl.,* 49 (1983), 191–219.

[6] Dias da Silva, J.A. – On the Schur inequality, *Linear and Multilin. Algebra,* 7 (1979), 343–357.

[7] Williamson, S.G. – A combinatorial property of finite sequences with an application to tensor algebra, *J. Comb. Theory,* 1 (1966), 401–410.

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