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AN INVERSE PROBLEM FOR A GENERAL DOUBLY--CONNECTED BOUNDED DOMAIN WITH IMPEDANCE BOUNDARY CONDITIONS

E.M.E. Zayed

Abstract: The spectral function $\theta(t) = \sum_{\nu=1}^{\infty} \exp(-t\,\lambda_\nu)$, where $\{\lambda_\nu\}_{\nu=1}^{\infty}$ are the eigenvalues of the negative Laplacian $-\Delta = -\sum_{i=1}^{2} (\frac{\partial}{\partial x^i})^2$ in the (x^1, x^2)-plane, is studied for a general doubly-connected bounded domain Ω in $\mathbb{R}^2$ together with its smooth inner boundary $\partial\Omega_1$ and its smooth outer boundary $\partial\Omega_2$, where piecewise smooth impedance boundary conditions on the two parts Γ_1, Γ_2 of $\partial\Omega_1$ and on the two parts Γ_3, Γ_4 of $\partial\Omega_2$ are considered, such that $\partial\Omega_1 = \Gamma_1 \cup \Gamma_2$ and $\partial\Omega_2 = \Gamma_3 \cup \Gamma_4$.

1 – Introduction

The underlying inverse problem is to determine some geometric quantities associated with a bounded domain, from a complete knowledge of the eigenvalues $\{\lambda_\nu\}_{\nu=1}^{\infty}$ for the negative Laplacian $-\Delta = -\sum_{i=1}^{2} (\frac{\partial}{\partial x^i})^2$ in the (x^1, x^2)-plane.

Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected bounded domain with a smooth boundary $\partial\Omega$. Consider the impedance problem

$$-\Delta u = \lambda\, u \quad \text{in} \;\; \Omega \,, \tag{1.1}$$

$$\left(\frac{\partial}{\partial n} + \gamma\right) u = 0 \quad \text{on} \;\; \partial\Omega \,, \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial\Omega$ and γ is a positive constant, with $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Denote its eigenvalues, counted according to multiplicity, by

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_\nu \le \ldots \;\to\; \infty \quad \text{as} \;\; \nu \to \infty \,. \tag{1.3}$$

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The problem of determining some geometric quantities associated with the bounded domain Ω has been discussed recently by Sleeman and Zayed [5], using the asymptotic expansion of the spectral function

$$\theta(t) = \sum_{\nu=1}^{\infty} \exp(-t\,\lambda_\nu) \qquad \text{as} \quad t \to 0^+ \ . \tag{1.4}$$

Problem (1.1)–(1.2) has been investigated by many authors (see for example the articles [1–4, 6, 7]) in the following special cases:

Case 1. $\gamma = 0$ (The Neumann problem)

$$\theta(t) = \frac{|\Omega|}{4\pi\,t} + \frac{|\partial\Omega|}{8(\pi\,t)^{1/2}} + a_0 + \frac{7}{256}\left(\frac{t}{\pi}\right)^{1/2} \int_{\partial\Omega} K^2(\sigma)\,d\sigma + O(t) \quad \text{as } t \to 0^+ \ . \tag{1.5}$$

Case 2. $\gamma \to \infty$ (The Dirichlet problem)

$$\theta(t) = \frac{|\Omega|}{4\pi\,t} - \frac{|\partial\Omega|}{8(\pi\,t)^{1/2}} + a_0 + \frac{1}{256}\left(\frac{t}{\pi}\right)^{1/2} \int_{\partial\Omega} K^2(\sigma)\,d\sigma + O(t) \quad \text{as } t \to 0^+ \ . \tag{1.6}$$

In these formulae, $|\Omega|$ is the area of Ω, $|\partial\Omega|$ is the total length of its boundary $\partial\Omega$, σ is the arc length of the counter clockwise oriented boundary $\partial\Omega$ and $K(\sigma)$ is the curvature of $\partial\Omega$. The constant term a_0 has geometric significance, e.g., if Ω is smooth and convex, then $a_0 = \frac{1}{6}$ and if Ω is permitted to have a finite number "H" of smooth convex holes, then $a_0 = (1-H)/6$.

Case 3. (The mixed problem)

If L_1 is the length of a part Γ_1 of the boundary $\partial\Omega$ with the Neumann boundary condition, and if L_2 is the length of the remaining part $\Gamma_2 = \partial\Omega \backslash \Gamma_1$ of $\partial\Omega$ with the Dirichlet boundary condition, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, then with reference to [1, 8, 9] we get

$$\begin{aligned} \theta(t) = \frac{|\Omega|}{4\pi\,t} + \frac{L_1 - L_2}{8(\pi\,t)^{1/2}} + a_0 + \frac{1}{256}\left(\frac{t}{\pi}\right)^{1/2} \Big\{ 7\int_{\Gamma_1} K^2(\sigma)\,d\sigma \\ + \int_{\Gamma_2} K^2(\sigma)\,d\sigma \Big\} + O(t) \qquad \text{as} \quad t \to 0^+ \ . \end{aligned} \tag{1.7}$$

Zayed [8] has recently discussed the equation (1.1) together with the piecewise smooth impedance boundary conditions:

$$\left(\frac{\partial}{\partial n_1} + \gamma_1\right) u = 0 \ \text{ on } \Gamma_1 \,, \qquad \left(\frac{\partial}{\partial n_2} + \gamma_2\right) u = 0 \ \text{ on } \Gamma_2 \ , \tag{1.8}$$

where $\frac{\partial}{\partial n_1}$ and $\frac{\partial}{\partial n_2}$ denote differentiations along the inward-pointing normals to Γ_1 and Γ_2 respectively, in which Γ_1 is a part of $\partial\Omega$ and $\Gamma_2 = \partial\Omega\backslash\Gamma_1$ is the remaining part of $\partial\Omega$, while the impedances γ_1 and γ_2 are positive constants. The author calculates only the first three terms of the asymptotics of the heat kernel of this problem, and shows that how the lengths of Γ_1 and Γ_2 and the impedances γ_1, γ_2 enter into the asymptotic expansions of $\theta(t)$ for small positive t.

Now, let Ω be a general doubly-connected domain in $\mathbb{R}^2$ consisting of a simply connected bounded inner domain Ω_1 with a smooth boundary $\partial\Omega_1$ and a simply connected bounded outer domain $\Omega_2 \supset \overline{\Omega}_1$ with a smooth boundary $\partial\Omega_2$ where $\overline{\Omega}_1 = \Omega_1 \cup \partial\Omega_1$. Suppose that the eigenvalues (1.3) are given for the eigenvalue equation

$$-\Delta u = \lambda\, u \quad \text{in } \Omega\,, \tag{1.9}$$

together with the impedance boundary conditions

$$\left(\frac{\partial}{\partial n_1} + \gamma_1\right)u = 0 \ \text{ on } \partial\Omega_1\,, \qquad \left(\frac{\partial}{\partial n_2} + \gamma_2\right)u = 0 \ \text{ on } \partial\Omega_2\,, \tag{1.10}$$

where γ_1 and γ_2 are positive constants.

Zayed [10] has recently discussed the problem (1.9), (1.10) and has determined only the first three terms of the asymptotic expansions of the spectral function $\theta(t)$ for small positive t. The author has determined some geometric quantities associated with the problem (1.9)–(1.10) by using (1.4).

The object of this paper is to discuss a more general inverse problem consisting of the eigenvalue equation (1.9) together with the piecewise smooth impedance boundary conditions:

$$\left(\frac{\partial}{\partial n_i} + \gamma_i\right)u = 0 \quad \text{on } \Gamma_i \ \ (i = 1,2,3,4)\,, \tag{1.11}$$

where Γ_1 is a part of the inner boundary $\partial\Omega_1$ of Ω and $\Gamma_2 = \partial\Omega_1\backslash\Gamma_1$ is the remaining part of $\partial\Omega_1$ such that $\partial\Omega_1 = \Gamma_1 \cup \Gamma_2$, while Γ_3 is a part of the outer boundary $\partial\Omega_2$ of Ω and $\Gamma_4 = \partial\Omega_2\backslash\Gamma_3$ is the remaining part of $\partial\Omega_2$ such that $\partial\Omega_2 = \Gamma_3 \cup \Gamma_4$, and the impedances γ_i $(i = 1,2,3,4)$ are positive constants.

The basic problem is to determine some geometric quantities associated with the general doubly connected domain Ω from the complete knowledge of the eigenvalues $\{\lambda_\nu\}$ for the impedance problem (1.9), (1.11) using the asymptotic expansions of the spectral function $\theta(t)$ for small positive t.

Note that our main problem (1.9)–(1.11) can be considered as a generalization of those obtained by Zayed [8, 9, 10].

2 – Statement of results

Suppose that the inner boundary $\partial\Omega_1$ of Ω is given locally by the equations $x^i = y^i(\sigma_1)$ $(i = 1, 2)$, in which σ_1 is the arc length of the counterclock-wise oriented inner boundary $\partial\Omega_1$ and $y^i(\sigma_1) \in C^\infty(\partial\Omega_1)$. Suppose also that the outer boundary $\partial\Omega_2$ of Ω is given locally by the equations $x^i = y^i(\sigma_2)$ $(i = 1, 2)$, in which σ_2 is the arc length of the counterclock-wise oriented outer boundary $\partial\Omega_2$ and $y^i(\sigma_2) \in C^\infty(\partial\Omega_2)$.

Let $k_1(\sigma_1)$ and $k_2(\sigma_2)$ be the curvatures of $\partial\Omega_1$ and $\partial\Omega_2$ respectively. Let L_1 and L_2 be the lengths of the parts Γ_1 and Γ_2 of $\partial\Omega_1$ respectively and let L_3 and L_4 be the lengths of the parts Γ_3 and Γ_4 of $\partial\Omega_2$ respectively. Then, the results of our main problem (1.9), (1.11) can be summarized in the following cases:

Case 1. $(0 < \gamma_1 \ll 1,\ \gamma_2 \gg 1,\ 0 < \gamma_3 \ll 1,\ \gamma_4 \gg 1)$

$$\begin{aligned}
\theta(t) = {} & \frac{|\Omega|}{4\pi\, t} + \frac{1}{8(\pi\, t)^{1/2}} \Big\{ \Big[L_1 - \Big(L_2 + \gamma_2^{-1} \int_{\Gamma_2} k_1(\sigma_1)\, d\sigma_1 \Big) \Big] \\
& + \Big[L_3 - \Big(L_4 + \gamma_4^{-1} \int_{\Gamma_4} k_2(\sigma_2)\, d\sigma_2 \Big) \Big] \Big\} + \frac{1}{2\pi} \left(\gamma_1 L_1 - \gamma_3 L_3 \right) \\
& + \frac{1}{256} \left(\frac{t}{\pi} \right)^{1/2} \Big\{ 7 \int_{\Gamma_1} \Big[k_1^2(\sigma_1) - \frac{64}{7} \Big(\frac{\pi\, \gamma_1}{L_1} - \gamma_1^2 \Big) \Big] d\sigma_1 \\
& + \int_{\Gamma_2} \Big[k_1^2(\sigma_1) - \Big(\frac{2\pi}{L_2} \Big)^3 \gamma_2^{-1} \Big] d\sigma_1 + 7 \int_{\Gamma_3} \Big[k_2^2(\sigma_2) - \frac{64}{7} \Big(\frac{\pi\, \gamma_3}{L_3} - \gamma_3^2 \Big) \Big] d\sigma_2 \\
& + \int_{\Gamma_4} \Big[k_2^2(\sigma_2) - \Big(\frac{2\pi}{L_4} \Big)^3 \gamma_4^{-1} \Big] d\sigma_2 \Big\} + O(t) \quad \text{as} \ \ t \to 0^+ \,.
\end{aligned} \tag{2.1}$$

Case 2. $(0 < \gamma_1 \ll 1,\ \gamma_2 \gg 1,\ \gamma_3 \gg 1,\ 0 < \gamma_4 \ll 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.1) with the interchanges $\gamma_3 \leftrightarrow \gamma_4$, $\Gamma_3 \leftrightarrow \Gamma_4$ and $L_3 \leftrightarrow L_4$.

Case 3. $(\gamma_1, \gamma_2 \gg 1,\ 0 < \gamma_3, \gamma_4 \ll 1)$

$$\begin{aligned}
\theta(t) = {} & \frac{|\Omega|}{4\pi\, t} + \frac{1}{8(\pi\, t)^{1/2}} \Big\{ \sum_{i=3}^{4} L_i - \sum_{i=1}^{2} \Big[L_i + \gamma_i^{-1} \int_{\Gamma_i} k_1(\sigma_1)\, d\sigma_1 \Big] \Big\} \\
& - \frac{1}{2\pi} \sum_{i=3}^{4} \gamma_i L_i + \frac{1}{256} \left(\frac{t}{\pi} \right)^{1/2} \Big\{ \sum_{i=1}^{2} \int_{\Gamma_i} \Big[k_1^2(\sigma_1) - \Big(\frac{2\pi}{L_i} \Big)^3 \gamma_i^{-1} \Big] d\sigma_1 \\
& + 7 \sum_{i=3}^{4} \int_{\Gamma_i} \Big[k_2^2(\sigma_2) - \frac{64}{7} \Big(\frac{\pi\, \gamma_i}{L_i} - \gamma_i^2 \Big) \Big] d\sigma_2 \Big\} + O(t) \quad \text{as} \ \ t \to 0^+ \,.
\end{aligned} \tag{2.2}$$

Case 4. $(0 < \gamma_1,\, \gamma_2 \ll 1,\, \gamma_3, \gamma_4 \gg 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.2) with the interchanges $\gamma_1 \leftrightarrow \gamma_3$, $\gamma_2 \leftrightarrow \gamma_4$, $L_1 \leftrightarrow L_3$, $L_2 \leftrightarrow L_4$, $\Gamma_1 \leftrightarrow \Gamma_3$, $\Gamma_2 \leftrightarrow \Gamma_4$ and $k_1(\sigma_1) \leftrightarrow k_2(\sigma_2)$.

Case 5. $(\gamma_1 \gg 1,\, 0 < \gamma_2 \ll 1,\, \gamma_3 \gg 1,\, 0 < \gamma_4 \ll 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.1) with the interchanges $\gamma_1 \leftrightarrow \gamma_2$, $\gamma_3 \leftrightarrow \gamma_4$, $L_1 \leftrightarrow L_2$, $L_3 \leftrightarrow L_4$, $\Gamma_1 \leftrightarrow \Gamma_2$ and $\Gamma_3 \leftrightarrow \Gamma_4$.

Case 6. $(\gamma_1 \gg 1,\, 0 < \gamma_2 \ll 1,\, 0 < \gamma_3 \ll 1,\, \gamma_4 \gg 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.1) with the interchanges $\gamma_1 \leftrightarrow \gamma_2$, $L_1 \leftrightarrow L_2$ and $\Gamma_1 \leftrightarrow \Gamma_2$.

Case 7. $(0 < \gamma_1 \ll 1,\, \gamma_2 \gg 1,\, \gamma_3, \gamma_4 \gg 1)$

$$
\begin{aligned}
\theta(t) = {} & \frac{|\Omega|}{4\pi t} + \frac{1}{8(\pi t)^{1/2}}\Big\{\Big[L_1 - \Big(L_2 + \gamma_2^{-1}\int_{\Gamma_2} k_1(\sigma_1)\, d\sigma_1\Big)\Big] \\
& - \sum_{i=3}^{4}\Big(L_i + \gamma_i^{-1}\int_{\Gamma_i} k_2(\sigma_2)\, d\sigma_2\Big)\Big\} - \frac{\gamma_1 L_1}{2\pi} \\
& + \frac{1}{256}\Big(\frac{t}{\pi}\Big)^{1/2}\Big\{7\int_{\Gamma_1}\Big[k_1^2(\sigma_1) - \frac{64}{7}\Big(\frac{\pi\gamma_1}{L_1} - \gamma_1^2\Big)\Big] d\sigma_1 \\
& + \int_{\Gamma_2}\Big[k_1^2(\sigma_1) - \Big(\frac{2\pi}{L_2}\Big)^3\gamma_2^{-1}\Big] d\sigma_1 + \sum_{i=3}^{4}\int_{\Gamma_i}\Big[k_2^2(\sigma_2) - \Big(\frac{2\pi}{L_i}\Big)^3\gamma_i^{-1}\Big] d\sigma_2\Big\} \\
& + O(t) \quad \text{as} \quad t \to 0^+ \,.
\end{aligned}
\tag{2.3}
$$

Case 8. $(\gamma_1 \gg 1,\, 0 < \gamma_2 \ll 1,\, \gamma_3, \gamma_4 \gg 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.3) with the interchanges $\gamma_1 \leftrightarrow \gamma_2$, $L_1 \leftrightarrow L_2$ and $\Gamma_1 \leftrightarrow \Gamma_2$.

Case 9. $(\gamma_1, \gamma_2 \gg 1,\, 0 < \gamma_3 \ll 1,\, \gamma_4 \gg 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.3) with the interchanges $\gamma_1 \leftrightarrow \gamma_3$, $\gamma_2 \leftrightarrow \gamma_4$, $L_1 \leftrightarrow L_3$, $L_2 \leftrightarrow L_4$, $\Gamma_1 \leftrightarrow \Gamma_3$, $\Gamma_2 \leftrightarrow \Gamma_4$ and $k_1(\sigma_1) \leftrightarrow k_2(\sigma_2)$.

Case 10. $(\gamma_1, \gamma_2 \gg 1,\, \gamma_3 \gg 1,\, 0 < \gamma_4 \ll 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.3) with the interchanges $\gamma_1 \leftrightarrow \gamma_4$, $\gamma_2 \leftrightarrow \gamma_3$, $L_1 \leftrightarrow L_4$, $L_2 \leftrightarrow L_3$, $\Gamma_1 \leftrightarrow \Gamma_4$, $\Gamma_2 \leftrightarrow \Gamma_3$ and $k_1(\sigma_1) \leftrightarrow k_2(\sigma_2)$.

Case 11. $(\gamma_1 \gg 1,\ 0 < \gamma_2 \ll 1,\ 0 < \gamma_3, \gamma_4 \ll 1)$

$$\begin{aligned}
\theta(t) = {} & \frac{|\Omega|}{4\pi t} + \frac{1}{8(\pi t)^{1/2}}\Big\{\Big[L_2 - \Big(L_1 + \gamma_1^{-1}\int_{\Gamma_1} k_1(\sigma_1)\,d\sigma_1\Big)\Big] + \sum_{i=3}^{4} L_i\Big\} \\
& + \frac{1}{2\pi}\Big(\gamma_2 L_2 - \sum_{i=3}^{4}\gamma_i L_i\Big) \\
& + \frac{1}{256}\Big(\frac{t}{\pi}\Big)^{1/2}\Big\{\int_{\Gamma_1}\Big[K_1^2(\sigma_1) - \Big(\frac{2\pi}{L_1}\Big)^3\gamma_1^{-1}\Big]d\sigma_1 \\
& + 7\int_{\Gamma_2}\Big[k_1^2(\sigma_1) - \frac{64}{7}\Big(\frac{\pi\,\gamma_2}{L_2} - \gamma_2^2\Big)\Big]d\sigma_1 \\
& + 7\sum_{i=3}^{4}\int_{\Gamma_i}\Big[k_2^2(\sigma_2) - \frac{64}{7}\Big(\frac{\pi\,\gamma_i}{L_i} - \gamma_i^2\Big)\Big]d\sigma_2\Big\} + O(t) \quad \text{as } t \to 0^+ .
\end{aligned} \tag{2.4}$$

Case 12. $(0 < \gamma_1 \ll 1,\ \gamma_2 \gg 1,\ 0 < \gamma_3, \gamma_4 \ll 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.4) with the interchanges $\gamma_1 \leftrightarrow \gamma_2$, $L_1 \leftrightarrow L_2$ and $\Gamma_1 \leftrightarrow \Gamma_2$.

Case 13. $(0 < \gamma_1, \gamma_2 \ll 1,\ \gamma_3 \gg 1,\ 0 < \gamma_4 \ll 1)$

$$\begin{aligned}
\theta(t) = {} & \frac{|\Omega|}{4\pi t} + \frac{1}{8(\pi t)^{1/2}}\Big\{\sum_{i=1}^{2} L_i + \Big[L_4 - \Big(L_3 + \gamma_3^{-1}\int_{\Gamma_3} k_2(\sigma_2)\,d\sigma_2\Big)\Big]\Big\} \\
& + \frac{1}{2\pi}\Big(\sum_{i=1}^{2}\gamma_i L_i - \gamma_4 L_4\Big) + \frac{1}{256}\Big(\frac{t}{\pi}\Big)^{1/2}\Big\{7\sum_{i=1}^{2}\int_{\Gamma_i}\Big[k_1^2(\sigma_1) \\
& - \frac{64}{7}\Big(\frac{\pi\,\gamma_i}{L_i} - \gamma_i^2\Big)\Big]d\sigma_1 + \int_{\Gamma_3}\Big[k_2^2(\sigma_2) - \Big(\frac{2\pi}{L_3}\Big)^3\gamma_3^{-1}\Big]d\sigma_2 \\
& + 7\int_{\Gamma_4}\Big[k_2^2(\sigma_2) - \frac{64}{7}\Big(\frac{\pi\,\gamma_4}{L_4} - \gamma_4^2\Big)\Big]d\sigma_2\Big\} + O(t) \quad \text{as } t \to 0^+ .
\end{aligned} \tag{2.5}$$

Case 14. $(0 < \gamma_1, \gamma_2 \ll 1,\ 0 < \gamma_3 \ll 1,\ \gamma_4 \gg 1)$

In this case, the asymptotic expansion of $\theta(t)$ has the same form (2.5) with the interchanges $\gamma_3 \leftrightarrow \gamma_4$, $L_3 \leftrightarrow L_4$ and $\Gamma_3 \leftrightarrow \Gamma_4$.

Case 15. $(0 < \gamma_i \ll 1,\, i = 1,2,3,4)$

$$\begin{aligned}\theta(t) &= \frac{|\Omega|}{4\pi\, t} + \frac{\sum_{i=1}^{4} L_i}{8(\pi\, t)^{1/2}} + \frac{1}{2\pi}\Big(\sum_{i=1}^{2}\gamma_i L_i - \sum_{i=3}^{4}\gamma_i L_i\Big)\\ &+ \frac{7}{256}\left(\frac{t}{\pi}\right)^{1/2}\Big\{\sum_{i=1}^{2}\int_{\Gamma_i}\Big[k_1^2(\sigma_1) - \frac{64}{7}\Big(\frac{\pi\,\gamma_i}{L_i} - \gamma_i^2\Big)\Big]\,d\sigma_1\\ &+ \sum_{i=3}^{4}\int_{\Gamma_i}\Big[k_2^2(\sigma_2) - \frac{64}{7}\Big(\frac{\pi\,\gamma_i}{L_i} - \gamma_i^2\Big)\Big]\,d\sigma_2\Big\} + O(t) \quad \text{as } t \to 0^+ .\end{aligned} \tag{2.6}$$

Case 16. $(\gamma_i \gg 1,\, i = 1,2,3,4)$

$$\begin{aligned}\theta(t) &= \frac{|\Omega|}{4\pi\, t} - \frac{1}{8(\pi\, t)^{1/2}}\Big\{\sum_{i=1}^{2}\Big(L_i + \gamma_i^{-1}\int_{\Gamma_i} k_1(\sigma_1)\,d\sigma_1\Big)\\ &+ \sum_{i=3}^{4}\Big(L_i + \gamma_i^{-1}\int_{\Gamma_i} k_2(\sigma_2)\,d\sigma_2\Big)\Big\}\\ &+ \frac{1}{256}\left(\frac{t}{\pi}\right)^{1/2}\Big\{\sum_{i=1}^{2}\int_{\Gamma_i}\Big[k_1^2(\sigma_1) - \Big(\frac{2\pi}{L_i}\Big)^3\gamma_i^{-1}\Big]\,d\sigma_1\\ &+ \sum_{i=3}^{4}\int_{\Gamma_i}\Big[k_2^2(\sigma_2) - \Big(\frac{2\pi}{L_i}\Big)^3\gamma_i^{-1}\Big]\,d\sigma_2\Big\} + O(t) \quad \text{as } t \to 0^+ .\end{aligned} \tag{2.7}$$

With reference to the formulas (1.5)–(1.7) and to the articles [8], [10] [11] the asymptotic expansions (2.1)–(2.7) may be interpreted as:

- **i)** Ω is a general doubly connected bounded domain in $\mathbb{R}^2$ and we have the piecewise smooth impedance boundary conditions (1.11) with small/large impedances γ_i $(i = 1,2,3,4)$ as indicated in the specifications of the sixteen respective cases, where we notice that γ_i small approaches Neumann boundary conditions, while γ_i large approaches Dirichlet boundary conditions.
- **ii)** For the first four terms, Ω is a general doubly connected bounded domain in $\mathbb{R}^2$ of area $|\Omega|$.

In (2.1), it has $H = 1 - \frac{3}{\pi}(\gamma_1 L_1 - \gamma_3 L_3)$ holes, the part Γ_1 of $\partial\Omega_1$ is of length L_1 and of curvature $[k_1^2(\sigma_1) - \frac{64}{7}(\frac{\pi\,\gamma_1}{L_1} - \gamma_1^2)]^{1/2}$ together with the Neumann boundary condition, while the remaining part $\Gamma_2 = \partial\Omega_1 \backslash \Gamma_1$ of $\partial\Omega_1$ is of length $(L_2+\gamma_2^{-1}\int_{\Gamma_2} k_1(\sigma_1)\,d\sigma_1)$ and of curvature $[k_1^2(\sigma_1)-(\frac{2\pi}{L_2})^3\,\gamma_2^{-1}]^{1/2}$ together with the Dirichlet boundary condition. Similarly, the part Γ_3 of $\partial\Omega_2$ is of length L_3 and of curvature $[k_2^2(\sigma_2)-\frac{64}{7}(\frac{\pi\,\gamma_3}{L_3}-\gamma_3^2)]^{1/2}$ together with the Neumann boundary condition, while the remaining part $\Gamma_4=\partial\Omega_2\backslash\Gamma_3$ of $\partial\Omega_2$ is of length $(L_4+\gamma_4^{-1}\int_{\Gamma_4} k_2(\sigma_2)\,d\sigma_2)$ and of curvature $[k_2^2(\sigma_2)-(\frac{2\pi}{L_4})^3\,\gamma_4^{-1}]^{1/2}$ together with the Dirichlet boundary condition, provided H is an integer.

In (2.2), it has $H=1+\frac{3}{\pi}\sum_{i=3}^{4}\gamma_i L_i$ holes, the part Γ_1 of $\partial\Omega_1$ is of length $(L_1+\gamma_1^{-1}\int_{\Gamma_1} k_1(\sigma_1)\,d\sigma_1)$ and of curvature $[k_1^2(\sigma_1)-(\frac{2\pi}{L_1})^3\,\gamma_1^{-1}]^{1/2}$, while the remaining part $\Gamma_2=\partial\Omega_1\backslash\Gamma_1$ of $\partial\Omega_1$ is of length $(L_2+\gamma_2^{-1}\int_{\Gamma_2} k_1(\sigma_1)\,d\sigma_1)$ and of curvature $[k_1^2(\sigma_1)-(\frac{2\pi}{L_2})^3\,\gamma_2^{-1}]^{1/2}$ together with the Dirichlet boundary conditions on Γ_1 and Γ_2. Similarly, the part Γ_3 of $\partial\Omega_2$ is of length L_3 and of curvature $[k_2^2(\sigma_2)-\frac{64}{7}(\frac{\pi\,\gamma_3}{L_3}-\gamma_3^2)]^{1/2}$ while the remaining part $\Gamma_4=\partial\Omega_2\backslash\Gamma_3$ of $\partial\Omega_2$ is of length L_4 and curvature $[k_2^2(\sigma_2)-\frac{64}{7}(\frac{\pi\,\gamma_4}{L_4}-\gamma_4^2)]^{1/2}$ together with the Neumann boundary conditions on Γ_3 and Γ_4, provided H is an integer.

In (2.3), it has $H=1+\frac{3}{\pi}\gamma_1 L_1$ holes, the part Γ_1 of $\partial\Omega_1$ is of length L_1 and of curvature $[k_1^2(\sigma_1)-\frac{64}{7}(\frac{\pi\,\gamma_1}{L_1}-\gamma_1^2)]^{1/2}$ together with the Neumann boundary condition, while the remaining part $\Gamma_2=\partial\Omega_1\backslash\Gamma_1$ of $\partial\Omega_1$ is of length $(L_2+\gamma_2^{-1}\int_{\Gamma_2} k_1(\sigma_1)\,d\sigma_1)$ and of curvature $[k_1^2(\sigma_1)-(\frac{2\pi}{L_2})^3\,\gamma_2^{-1}]^{1/2}$ together with the Dirichlet boundary condition. Similarly, the parts Γ_3 and Γ_4 of $\partial\Omega_2$ are respectively of lengths $(L_3+\gamma_3^{-1}\int_{\Gamma_3} k_2(\sigma_2)\,d\sigma_2)$, $(L_4+\gamma_4^{-1}\int_{\Gamma_4} k_2(\sigma_2)\,d\sigma_2)$ and of curvatures $[k_2^2(\sigma_2)-(\frac{2\pi}{L_3})^3\,\gamma_3^{-1}]^{1/2}$, $[k_2^2(\sigma_2)-(\frac{2\pi}{L_4})^3\,\gamma_4^{-1}]^{1/2}$ together with the Dirichlet boundary conditions on Γ_3 and Γ_4, provided H is an integer.

In (2.4), it has $H=1-\frac{3}{\pi}(\gamma_2 L_2-\sum_{i=3}^{4}\gamma_i L_i)$ holes, the part Γ_2 of $\partial\Omega_1$ is of length L_2 and of curvature $[k_1^2(\sigma_1)-\frac{64}{7}(\frac{\pi\,\gamma_2}{L_2}-\gamma_2^2)]^{1/2}$ together with the Neumann boundary condition, while the remaining part $\Gamma_1=\partial\Omega_1\backslash\Gamma_2$ of $\partial\Omega_1$ is of length $(L_1+\gamma_1^{-1}\int_{\Gamma_1} k_1(\sigma_1)\,d\sigma_1)$ and of curvatures $[k_1^2(\sigma_1)-(\frac{2\pi}{L_1})^3\,\gamma_1^{-1}]^{1/2}$ together with the Dirichlet boundary condition. Similarly, the parts Γ_3 and Γ_4 of $\partial\Omega_2$ are respectively of lengths L_3, L_4 and curvatures $[k_2^2(\sigma_2)-\frac{64}{7}(\frac{\pi\,\gamma_3}{L_3}-\gamma_3^2)]^{1/2}$, $[k_2^2(\sigma_2)-\frac{64}{7}(\frac{\pi\,\gamma_4}{L_4}-\gamma_4^2)]^{1/2}$ together with the Neumann boundary conditions on Γ_3 and Γ_4, provided H is an integer.

In (2.5), it has $H=1-\frac{3}{\pi}(\sum_{i=1}^{2}\gamma_i L_i-\gamma_4 L_4)$ holes, the parts Γ_1 and Γ_2 of $\partial\Omega_1$ are of lengths L_1, L_2 and of curvatures $[k_1^2(\sigma_1)-\frac{64}{7}(\frac{\pi\,\gamma_1}{L_1}-\gamma_1^2)]^{1/2}$, $[k_1^2(\sigma_1)-\frac{64}{7}(\frac{\pi\,\gamma_2}{L_2}-\gamma_2^2)]^{1/2}$ together with the Neumann boundary conditions on Γ_1 and Γ_2. Similarly, the part Γ_4 of $\partial\Omega_2$ is of length L_4 and curvature $[k_2^2(\sigma_2)-\frac{64}{7}(\frac{\pi\,\gamma_4}{L_4}-\gamma_4^2)]^{1/2}$ together with the Neumann boundary condition, while the remaining part $\Gamma_3=\partial\Omega_2\backslash\Gamma_4$ of $\partial\Omega_2$ is of length $(L_3+\gamma_3^{-1}\int_{\gamma_3} k_2(\sigma_2)\,d\sigma_2)$ and of curvature $[k_2^2(\sigma_2)-(\frac{2\pi}{L_3})^3\,\gamma_3^{-1}]^{1/2}$ together with the Dirichlet boundary condition, provided H is an integer.

In (2.6), it has $H = 1 - \frac{3}{\pi}(\sum_{i=1}^{2} \gamma_i L_i - \sum_{i=3}^{4} \gamma_i L_i)$ holes, the parts Γ_1 and Γ_2 of $\partial\Omega_1$ are respectively of lengths L_1, L_2 and of curvatures $[k_1^2(\sigma_1) - \frac{64}{7}(\frac{\pi\,\gamma_1}{L_1} - \gamma_1^2)]^{1/2}$, $[k_1^2(\sigma_1) - \frac{64}{7}(\frac{\pi\,\gamma_2}{L_2} - \gamma_2^2)]^{1/2}$ together with the Neumann boundary conditions on Γ_1 and Γ_2. Similarly, the parts Γ_3 and Γ_4 of $\partial\Omega_2$ are respectively of lengths L_3, L_4 and of curvatures $[k_2^2(\sigma_2) - \frac{64}{7}(\frac{\pi\,\gamma_3}{L_3} - \gamma_3^2)]^{1/2}$, $[k_2^2(\sigma_2) - \frac{64}{7}(\frac{\pi\,\gamma_4}{L_4} - \gamma_4^2)]^{1/2}$ together with the Neumann boundary conditions on Γ_3 and Γ_4, provided H is an integer.

In (2.7), it has only one hole (i.e., $H = 1$), the parts Γ_1 and Γ_2 of $\partial\Omega_1$ are respectively of lengths $(L_1 + \gamma_1^{-1} \int_{\Gamma_1} k_1(\sigma_1)\, d\sigma_1)$, $(L_2 + \gamma_2^{-1} \int_{\Gamma_2} k_1(\sigma_1)\, d\sigma_1)$ and of curvatures $[k_1^2(\sigma_1) - (\frac{2\pi}{L_1})^3\, \gamma_1^{-1}]^{1/2}$, $[k_1^2(\sigma_1) - (\frac{2\pi}{L_2})^3\, \gamma_2^{-1}]^{1/2}$, together with the Dirichlet boundary conditions on Γ_1 and Γ_2. Similarly, the parts Γ_3 and Γ_4 of $\partial\Omega_2$ are respectively of lengths $(L_3 + \gamma_3^{-1} \int_{\Gamma_3} k_2(\sigma_2)\, d\sigma_2)$, $(L_4 + \gamma_4^{-1} \int_{\Gamma_4} k_2(\sigma_2)\, d\sigma_2)$ and of curvatures $[k_2^2(\sigma_2) - (\frac{2\pi}{L_3})^3\, \gamma_3^{-1}]^{1/2}$, $[k_2^2(\sigma_2) - (\frac{2\pi}{L_4})^3\, \gamma_4^{-1}]^{1/2}$ together with the Dirichlet boundary conditions on Γ_3 and Γ_4.

3 – Formulation of the mathematical problem

With reference to [2], [5], [7] one can show that the spectral function $\theta(t)$ associated with the problem (1.9), (1.11) is given by the formula

$$\theta(t) = \iint_\Omega G(\mathbf{x}, \mathbf{x}; t)\, d\mathbf{x} \ , \tag{3.1}$$

where $G(\mathbf{x}_1, \mathbf{x}_2; t)$ is the Green's function for the heat equation

$$\Delta u = \frac{\partial u}{\partial t} \ , \tag{3.2}$$

subject to the piecewise smooth impedance boundary conditions

$$\left(\frac{\partial}{\partial n_i} + \gamma_i\right) G(\mathbf{x}_1, \mathbf{x}_2; t) = 0 \quad \text{ for } \ x_1 \in \Gamma_i \ \ (i = 1, 2, 3, 4) \ , \tag{3.3}$$

and the initial condition

$$\lim_{t \to 0^+} G(\mathbf{x}_1, \mathbf{x}_2; t) = \delta(\mathbf{x}_1 - \mathbf{x}_2) \ , \tag{3.4}$$

where $\delta(\mathbf{x}_1 - \mathbf{x}_2)$ is the Dirac delta function located at the source point $\mathbf{x}_2$.

Let us write

$$G(\mathbf{x}_1, \mathbf{x}_2; t) = G_0(\mathbf{x}_1, \mathbf{x}_2; t) + \chi(\mathbf{x}_1, \mathbf{x}_2; t) \ , \tag{3.5}$$

where

$$G_0(\mathbf{x}_1,\mathbf{x}_2;t) = (4\pi\, t)^{-1} \exp\left\{\frac{|\mathbf{x}_1-\mathbf{x}_2|^2}{4t}\right\}\ , \tag{3.6}$$

is the "fundamental solution" of the heat equation (3.2), while $\chi(\mathbf{x}_1,\mathbf{x}_2;t)$ is the "regular solution" chosen in such a way that $G(\mathbf{x}_1,\mathbf{x}_2;t)$ satisfies the piecewise smooth impedance boundary conditions (3.3).

On setting $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$ we find that

$$\theta(t) = \frac{|\Omega|}{4\pi\, t} + R(t)\ , \tag{3.7}$$

where

$$R(t) = \iint_\Omega \chi(\mathbf{x};\mathbf{x};t)\, d\mathbf{x}\ , \tag{3.8}$$

The problem now is to determine the asymptotic expansion of $R(t)$ as $t \to 0^+$. In what follows, we shall use Laplace transforms with respect to t, and use s^2 as the Laplace transform parameter; thus we define

$$\overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2) = \int_0^{+\infty} e^{-s^2t} G(\mathbf{x}_1,\mathbf{x}_2;t)\, dt\ . \tag{3.9}$$

An application of the Laplace transform to the heat equation (3.2) shows that $\overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ satisfies the membrane equation

$$(\Delta - s^2)\, \overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2) = -\delta(\mathbf{x}_1-\mathbf{x}_2) \quad \text{in } \ \Omega\ , \tag{3.10}$$

together with the piecewise smooth impedance boundary conditions

$$\left(\frac{\partial}{\partial n_i} + \gamma_i\right) \overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2) = 0, \qquad \text{for } \ \mathbf{x}_1 \in \Gamma_i \ \ (i = 1,2,3,4)\ . \tag{3.11}$$

The asymptotic expansion of $R(t)$ as $t \to 0^+$ may then be deduced directly from the asymptotic expansion of $\overline{R}(s^2)$ as $s \to \infty$, where

$$\overline{R}(s^2) = \iint_\Omega \overline{\chi}(\mathbf{x},\mathbf{x};s^2)\, d\mathbf{x}\ . \tag{3.12}$$

4 – Construction of the Green's function

It is well known (see [4, 5, 7]) that the membrane equation (3.10) has the fundamental solution

$$\overline{G}_0(\mathbf{x}_1,\mathbf{x}_2;s^2) = \frac{1}{2\pi}\,K_0(sr_{\mathbf{x}_1\mathbf{x}_2})\ , \tag{4.1}$$

where $r_{\mathbf{x}_1\mathbf{x}_2} = |\mathbf{x}_1 - \mathbf{x}_2|$ is the distance between the points $\mathbf{x}_1 = (x_1^1, x_1^2)$, $\mathbf{x}_2 = (x_2^1, x_2^2)$ of the region Ω and K_0 is the modified Bessel function of the second kind and of zero order. The existence of this solution enables us to construct integral equations for $\overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ satisfying the piecewise smooth impedance boundary conditions (3.11) for small/large impedances γ_i $(i = 1,2,3,4)$ as indicated in the specifications of the sixteen respective cases. Therefore, Green's theorem gives:

Case 1. $(0 < \gamma_1 \ll 1,\ \gamma_2 \gg 1,\ 0 < \gamma_3 \ll 1,\ \gamma_4 \gg 1)$

In this case, we have the integral equation

$$\begin{aligned}
\overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2) &= \frac{1}{2\pi}\,K_0(sr_{\mathbf{x}_1\mathbf{x}_2}) + \\
&+ \frac{1}{\pi}\int_{\Gamma_1} \overline{G}(\mathbf{x}_1,\mathbf{y};s^2)\Big\{\frac{\partial}{\partial n_{1\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{x}_2}) + \gamma_1 K_0(sr_{\mathbf{y}\mathbf{x}_2})\Big\}\,d\mathbf{y} \\
&- \frac{1}{\pi}\int_{\Gamma_2} \frac{\partial}{\partial n_{2\mathbf{y}}}\overline{G}(\mathbf{x}_1,\mathbf{y};s^2)\Big\{K_0(s\,r_{\mathbf{y}\mathbf{x}_2}) + \gamma_2^{-1}\frac{\partial}{\partial n_{2\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{x}_2})\Big\}\,d\mathbf{y} \\
&- \frac{1}{\pi}\int_{\Gamma_3} \overline{G}(\mathbf{x}_1,\mathbf{y};s^2)\Big\{\frac{\partial}{\partial n_{3\mathbf{y}}}K_0(s\,r_{\mathbf{y}\mathbf{x}_2}) + \gamma_3 K_0(sr_{\mathbf{y}\mathbf{x}_2})\Big\}\,d\mathbf{y} \\
&+ \frac{1}{\pi}\int_{\Gamma_4} \frac{\partial}{\partial n_{4\mathbf{y}}}\overline{G}(\mathbf{x}_1,\mathbf{y};s^2)\Big\{K_0(sr_{\mathbf{y}\mathbf{x}_2}) + \gamma_4^{-1}\frac{\partial}{\partial n_{4\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{x}_2})\Big\}\,d\mathbf{y}\ .
\end{aligned} \tag{4.2}$$

Similarly, the integral equations of $\overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ for the other fifteen cases can be found easily.

On applying the iteration methods (see [8], [9]) to the integral equation (4.2), we obtain the Green's function $\overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ which has a regular part in the following form:

$$\begin{aligned}
\overline{\chi}(\mathbf{x}_1,\mathbf{x}_2;s^2) &= \frac{1}{2\pi^2}\int_{\Gamma_1} K_0(sr_{\mathbf{x}_1\mathbf{y}})\Big\{\frac{\partial}{\partial n_{1\mathbf{y}}}K_0(s\,r_{\mathbf{y}\mathbf{x}_2}) + \gamma_1 K_0(sr_{\mathbf{y}\mathbf{x}_2})\Big\}\,d\mathbf{y} - \\
&- \frac{1}{2\pi^2}\int_{\Gamma_2} \frac{\partial}{\partial n_{2\mathbf{y}}}K_0(sr_{\mathbf{x}_1\mathbf{y}})\Big\{K_0(s\,r_{\mathbf{y}\mathbf{x}_2}) + \gamma_2^{-1}\frac{\partial}{\partial n_{2\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{x}_2})\Big\}\,d\mathbf{y} -
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&- \frac{1}{2\pi^2} \int_{\Gamma_3} K_0(s\, r_{\mathbf{x}_1\mathbf{y}}) \Big\{ \frac{\partial}{\partial n_{3\mathbf{y}}} K_0(sr_{\mathbf{y}\mathbf{x}_2}) + \gamma_3 K_0(s\, r_{\mathbf{y}\mathbf{x}_2}) \Big\}\, d\mathbf{y} \\
&+ \frac{1}{2\pi^2} \int_{\Gamma_4} \frac{\partial}{\partial n_{4\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}}) \Big\{ K_0(s\, r_{\mathbf{y}\mathbf{x}_2}) + \gamma_4^{-1} \frac{\partial}{\partial n_{4\mathbf{y}}} K_0(sr_{\mathbf{y}\mathbf{x}_2}) \Big\}\, d\mathbf{y} \\
&+ \frac{1}{2\pi^2} \int_{\Gamma_1}\int_{\Gamma_1} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, M_{\gamma_1}(\mathbf{y},\mathbf{y}') \Big\{ \frac{\partial}{\partial n_{1\mathbf{y}'}} K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) + \gamma_1 K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}\, d\mathbf{y}' \\
&+ \frac{1}{2\pi_i^2} \int_{\Gamma_2}\int_{\Gamma_2} \frac{\partial}{\partial n_{2\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, M_{\gamma_2^{-1}}(\mathbf{y},\mathbf{y}') \Big\{ K_0(sr_{\mathbf{y}'\mathbf{x}_2}) + \gamma_2^{-1} \frac{\partial}{\partial n_{2\mathbf{y}'}} K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}\, d\mathbf{y}' \\
&+ \frac{1}{2\pi^2} \int_{\Gamma_3}\int_{\Gamma_3} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L_{\gamma_3}(\mathbf{y},\mathbf{y}') \Big\{ \frac{\partial}{\partial n_{3\mathbf{y}'}} K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) + \gamma_3 K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}\, d\mathbf{y}' \\
&+ \frac{1}{2\pi^2} \int_{\Gamma_4}\int_{\Gamma_4} \frac{\partial}{\partial n_{4\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L_{\gamma_4^{-1}}(\mathbf{y},\mathbf{y}') \Big\{ K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) \\
&\qquad + \gamma_4^{-1} \frac{\partial}{\partial n_{4\mathbf{y}'}} K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}\, d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \int_{\Gamma_1} \Big\{ \int_{\Gamma_2} \frac{\partial}{\partial n_{2\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, M^*_{\gamma_2^{-1}}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \Big\} \Big\{ \frac{\partial}{\partial n_{1\mathbf{y}'}} K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) \\
&\qquad + \gamma_1 K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \int_{\Gamma_2} \Big\{ \int_{\Gamma_1} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, M^*_{\gamma_1}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \Big\} \Big\{ K_0(sr_{\mathbf{y}'\mathbf{x}_2}) + \gamma_2^{-1} \frac{\partial}{\partial n_{2\mathbf{y}'}} K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \int_{\Gamma_1} \Big\{ \int_{\Gamma_3} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L^*_{\gamma_3}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \Big\} \Big\{ \frac{\partial}{\partial n_{1\mathbf{y}'}} K_0(sr_{\mathbf{y}'\mathbf{x}_2}) + \gamma_1 K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \int_{\Gamma_3} \Big\{ \int_{\Gamma_1} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L_{\gamma_1}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \Big\} \Big\{ \frac{\partial}{\partial n_{3\mathbf{y}'}} K_0(sr_{\mathbf{y}'\mathbf{x}_2}) + \gamma_3 K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}' \\
&+ \frac{1}{2\pi^2} \Big\{ \int_{\Gamma_4} \frac{\partial}{\partial n_{4\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, M^*_{\gamma_4^{-1}}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \Big\} \Big\{ \frac{\partial}{\partial n_{1\mathbf{y}'}} K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) + \gamma_1 K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}' \\
&+ \frac{1}{2\pi^2} \int_{\Gamma_4} \Big\{ \int_{\Gamma_1} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L^*_{\gamma_1}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \Big\} \Big\{ K_0(sr_{\mathbf{y}'\mathbf{x}_2}) + \gamma_4^{-1} \frac{\partial}{\partial n_{4\mathbf{y}'}} K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}' \\
&+ \frac{1}{2\pi^2} \int_{\Gamma_2} \Big\{ \int_{\Gamma_3} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, M^*_{\gamma_3}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \Big\} \Big\{ K_0(sr_{\mathbf{y}'\mathbf{x}_2}) + \gamma_2^{-1} \frac{\partial}{\partial n_{2\mathbf{y}'}} K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}' \\
&+ \frac{1}{2\pi^2} \int_{\Gamma_3} \Big\{ \int_{\Gamma_2} \frac{\partial}{\partial n_{2\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L^*_{\gamma_2^{-1}}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \Big\} \Big\{ \frac{\partial}{\partial n_{3\mathbf{y}'}} K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) \\
&\qquad + \gamma_3 K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \Big\}\, d\mathbf{y}'
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2\pi^2} \int_{\Gamma_2} \left\{ \int_{\Gamma_4} \frac{\partial}{\partial n_{4\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, M_{\gamma_4^{-1}}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \right\} \left\{ K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \right. \\
&\qquad \left. + \gamma_2^{-1} \frac{\partial}{\partial n_{2\mathbf{y}'}} K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) \right\} d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \int_{\Gamma_4} \left\{ \int_{\Gamma_2} \frac{\partial}{\partial n_{2\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L_{\gamma_2^{-1}}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \right\} \left\{ K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \right. \\
&\qquad \left. + \gamma_4^{-1} \frac{\partial}{\partial n_{4\mathbf{y}'}} K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) \right\} d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \int_{\Gamma_3} \left\{ \int_{\Gamma_4} \frac{\partial}{\partial n_{4\mathbf{y}}} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L^*_{\gamma_4^{-1}}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \right\} \left\{ \frac{\partial}{\partial n_{3\mathbf{y}'}} K_0(s\, r_{\mathbf{y}'\mathbf{x}_2}) \right. \\
&\qquad \left. + \gamma_3 K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \right\} d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \int_{\Gamma_4} \left\{ \int_{\Gamma_3} K_0(sr_{\mathbf{x}_1\mathbf{y}})\, L^+_{\gamma_3}(\mathbf{y},\mathbf{y}')\, d\mathbf{y} \right\} \left\{ K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \right. \\
&\qquad \left. + \gamma_4^{-1} \frac{\partial}{\partial n_{4\mathbf{y}'}} K_0(sr_{\mathbf{y}'\mathbf{x}_2}) \right\} d\mathbf{y}' \ ,
\end{aligned}
$$

where

(4.4) $$M_{\gamma_1}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} K^{(\nu)}_{\gamma_1}(\mathbf{y}',\mathbf{y}) \ ,$$

(4.5) $$K^{(0)}_{\gamma_1}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{ \frac{\partial}{\partial n_{1\mathbf{y}}} K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_1 K_0(sr_{\mathbf{y}\mathbf{y}'}) \right\} ,$$

(4.6) $$M_{\gamma_2^{-1}}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} (-1)^{\nu} K^{(\nu)}_{\gamma_2^{-1}}(\mathbf{y}',\mathbf{y}) \ ,$$

(4.7) $$K^{(0)}_{\gamma_2^{-1}}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{ \frac{\partial}{\partial n_{2\mathbf{y}'}} K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_2^{-1} \frac{\partial^2}{\partial n_{2\mathbf{y}}\, \partial n_{2\mathbf{y}'}} K_0(sr_{\mathbf{y}\mathbf{y}'}) \right\} ,$$

(4.8) $$L_{\gamma_3}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} (-1)^{\nu} K^{(\nu)}_{\gamma_3}(\mathbf{y}',\mathbf{y}) \ ,$$

where $K^{(0)}_{\gamma_3}(\mathbf{y}',\mathbf{y})$ has the same form (4.5) with the interchanges $\gamma_1 \leftrightarrow \gamma_3$ and $n_1 \leftrightarrow n_3$,

(4.9) $$L^{-1}_{\gamma_4}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} K^{(\nu)}_{\gamma_4^{-1}}(\mathbf{y}',\mathbf{y}) \ ,$$

where $K^{(0)}_{\gamma_4^{-1}}(\mathbf{y}',\mathbf{y})$ has the same form (4.7) with the interchanges $\gamma_2 \leftrightarrow \gamma_4$ and

$n_2 \leftrightarrow n_4$,

$$M^*_{\gamma_2^{-1}}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} {}^*K^{(\nu)}_{\gamma_2^{-1}}(\mathbf{y}',\mathbf{y}) \ , \tag{4.10}$$

$$^*K^{(0)}_{\gamma_2^{-1}}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_2^{-1}\frac{\partial}{\partial n_{2\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{y}'})\right\} \ , \tag{4.11}$$

$$M^*_{\gamma_1}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}\, {}^*K^{(\nu)}_{\gamma_1}(\mathbf{y}',\mathbf{y}) \ , \tag{4.12}$$

$$^*K^{(0)}_{\gamma_1}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{\frac{\partial^2}{\partial n_{1\mathbf{y}}\,\partial n_{2\mathbf{y}'}}K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_1\frac{\partial}{\partial n_{1\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{y}'})\right\} \ , \tag{4.13}$$

$$L^*_{\gamma_3}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} {}^*K^{(\nu)}_{\gamma_3}(\mathbf{y}',\mathbf{y}) \ , \tag{4.14}$$

$$^*K^{(0)}_{\gamma_3}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{\frac{\partial^2}{\partial n_{3\mathbf{y}}\,\partial n_{1\mathbf{y}'}}K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_3\frac{\partial}{\partial n_{3\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{y}'})\right\} \ , \tag{4.15}$$

$$L_{\gamma_1}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}\, K^{(\nu)}_{\gamma_1}(\mathbf{y}',\mathbf{y}) \ , \tag{4.16}$$

where $K^{(0)}_{\gamma_1}(\mathbf{y}',\mathbf{y})$ has the same form (4.5),

$$M^*_{\gamma_4^{-1}}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} {}^*K^{(\nu)}_{\gamma_4^{-1}}(\mathbf{y}',\mathbf{y}) \ , \tag{4.17}$$

where ${}^*K^{(0)}_{\gamma_4^{-1}}(\mathbf{y}',\mathbf{y})$ has the same form (4.11) with the interchanges $\gamma_2 \leftrightarrow \gamma_4$ and $n_2 \leftrightarrow n_4$,

$$L^*_{\gamma_1}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} {}^+K^{(\nu)}_{\gamma_1}(\mathbf{y}',\mathbf{y}) \ , \tag{4.18}$$

$$^+K^{(0)}_{\gamma_1}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{\frac{\partial^2}{\partial n_{1\mathbf{y}}\,\partial n_{4\mathbf{y}'}}K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_1\frac{\partial}{\partial n_{1\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{y}'})\right\} \ , \tag{4.19}$$

$$M^*_{\gamma_3}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}\, {}^{**}K^{(\nu)}_{\gamma_3}(\mathbf{y}',\mathbf{y}) \ , \tag{4.20}$$

where ${}^{**}K^{(0)}_{\gamma_3}(\mathbf{y}',\mathbf{y})$ has the same form (4.15) with the interchanges $n_1 \leftrightarrow n_2$,

$$L^*_{\gamma_2^{-1}}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}\, {}^*K^{(\nu)}_{\gamma_2^{-1}}(\mathbf{y}',\mathbf{y}) \ , \tag{4.21}$$

where ${}^{*}K^{(0)}_{\gamma_2^{-1}}(\mathbf{y}',\mathbf{y})$ has the same form (4.11)

$$M_{\gamma_4^{-1}}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}\, K^{(\nu)}_{\gamma_4^{-1}}(\mathbf{y}',\mathbf{y}) \,, \tag{4.22}$$

where $K^{(0)}_{\gamma_4^{-1}}(\mathbf{y}',\mathbf{y})$ has the same form (4.7) with the interchanges $\gamma_2 \leftrightarrow \gamma_4$, $n_2 \leftrightarrow n_4$,

$$L_{\gamma_2^{-1}}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} K^{(\nu)}_{\gamma_2^{-1}}(\mathbf{y}',\mathbf{y}) \,, \tag{4.23}$$

where $K^{(0)}_{\gamma_2^{-1}}(\mathbf{y}',\mathbf{y})$ has the same form (4.7),

$$L^{*}_{\gamma_4^{-1}}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}\, {}^{*}K^{(\nu)}_{\gamma_4^{-1}}(\mathbf{y}',\mathbf{y}) \,, \tag{4.24}$$

where ${}^{*}K^{(0)}_{\gamma_4^{-1}}(\mathbf{y}',\mathbf{y})$ has the same form (4.11) with the interchanges $\gamma_2 \leftrightarrow \gamma_4$, $n_2 \leftrightarrow n_4$,

$$L^{+}_{\gamma_3}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty} {}^{+}K^{(\nu)}_{\gamma_3}(\mathbf{y}',\mathbf{y}) \,, \tag{4.25}$$

where ${}^{+}K^{(0)}_{\gamma_3}(\mathbf{y}',\mathbf{y})$ has the same form (4.19) with the interchanges $\gamma_1 \leftrightarrow \gamma_3$, $n_1 \leftrightarrow n_3$.

In these formulae, we note that $K^{(\nu)}_{\gamma_i}(\mathbf{y}',\mathbf{y})$ being the iterates of the kernels $K^{(0)}_{\gamma_i}(\mathbf{y}',\mathbf{y})$ $(i=1,2,3,4)$ respectively.

Similarly, we can find $\overline{\chi}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ for the other fifteen cases.

On the basis of (4.3), the function $\overline{\chi}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ will be estimated for $s\to\infty$ together with small/large impedances γ_i $(i=1,2,3,4)$. The case when $\mathbf{x}_1$ and $\mathbf{x}_2$ lie in the neighbourhood of the parts Γ_1, Γ_2 of the inner boundary $\partial\Omega_1$ of Ω or in the neighbourhood of the parts Γ_3, Γ_4 of the outer boundary $\partial\Omega_2$ of Ω is particularly interesting. In what follows, we shall use coordinates similar to those obtained in Pleijel [4], Sleeman and Zayed [5] and Zayed [8, 9, 10] to examine this case.

5 – Coordinates in the neighbourhood of the boundary

Let $h_i > 0$ $(i=1,2,3,4)$ be sufficiently small. Let n_i $(i=1,2,3,4)$ be the minimum distances from a point $\mathbf{x}=(x^1,x^2)$ of the region Ω to the parts Γ_i

$(i=1,2,3,4)$ respectively. Let $\mathbf{n}_i(\sigma_1)$ $(i=1,2)$ denote the inward drawn unit normals to the parts Γ_i $(i=1,2)$ of the inner boundary $\partial\Omega_1$ of Ω respectively, while $\mathbf{n}_i(\sigma_2)$ $(i=3,4)$ denote the inward drawn unit normals to the parts Γ_i $(i=3,4)$ of the outer boundary $\partial\Omega_2$ of Ω respectively. Then, we note that the coordinates in the neighbourhood of the parts Γ_1, Γ_2 of $\partial\Omega_1$ and its diagrams (see [9]) are in the same form as in Section 5.2 of Zayed [9] with the interchanges $n_1 \leftrightarrow n_i$, $h_1 \leftrightarrow h_i$, $I_1 \leftrightarrow I_i$, $\mathcal{D}(I_1) \leftrightarrow \mathcal{D}(I_i)$ and $\delta_1 \leftrightarrow \delta_i$ $(i=1,2)$. Thus, we have the same formulae (5.2.1)–(5.2.5) of Section 5.2 in [9] with the interchanges $n_1 \leftrightarrow n_i$, $\mathbf{n}_1(\sigma_1) \leftrightarrow \mathbf{n}_i(\sigma_1)$, $\mathbf{t}_1(\sigma_1) \leftrightarrow \mathbf{t}_i(\sigma_1)$ $(i=1,2)$.

Similarly, the coordinates in the neighbourhood of the parts Γ_3, Γ_4 of $\partial\Omega_2$ and its diagrams (see [9]) are in the same form as in Section 5.1 of Zayed [9] with the interchanges $n_2 \leftrightarrow n_i$, $h_2 \leftrightarrow h_i$, $I_2 \leftrightarrow I_i$, $\mathcal{D}(I_2) \leftrightarrow \mathcal{D}(I_i)$ and $\delta_2 \leftrightarrow \delta_i$ $(i=3,4)$. Thus, we have the same formulae (5.1.1)–(5.1.5) of Section 5.1 in [9] with the interchanges $n_2 \leftrightarrow n_i$, $\mathbf{n}_2(\sigma_2) \leftrightarrow \mathbf{n}_i(\sigma_2)$, $\mathbf{t}_2(\sigma_2) \leftrightarrow \mathbf{t}_i(\sigma_2)$ $(i=3,4)$.

6 – Some local expansions

It now follows that the local expansions of the functions

$$K_0(sr_{\mathbf{xy}})\,, \quad \frac{\partial}{\partial n_{i\mathbf{y}}} K_0(sr_{\mathbf{xy}}) \quad (i=1,2,3,4)\,, \tag{6.1}$$

when the distance between $\mathbf{x}$ and $\mathbf{y}$ is small, are very similar to those obtained in Sections 4 and 5 of [5] (see, also Section 6 in [9]). Consequently, for small/large impedances γ_i $(i=1,2,3,4)$ the local behaviour of the kernels

$$K^{(0)}_{\gamma_i}(\mathbf{y}',\mathbf{y}),\quad {}^{*}K^{(0)}_{\gamma_i}(\mathbf{y}',\mathbf{y}),\quad {}^{+}K^{(0)}_{\gamma_i}(\mathbf{y}',\mathbf{y}) \quad (i=1,3)\,, \qquad {}^{**}K^{(0)}_{\gamma_3}(\mathbf{y}',\mathbf{y})\,, \tag{6.2}$$

and

$$K^{(0)}_{\gamma_i^{-1}}(\mathbf{y}',\mathbf{y}),\quad {}^{*}K^{(0)}_{\gamma_i^{-1}}(\mathbf{y}',\mathbf{y}) \quad (i=2,4)\,, \tag{6.3}$$

when the distance between $\mathbf{y}$ and $\mathbf{y}'$ is small, follows directly from the knowledge of the local expansions of the functions (6.1).

Definition 1. Let $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ be points in the upper half-plane $\xi^2>0$ of the (ξ^1,ξ^2)-plane, then we define

$$\rho_{12} = \sqrt{(\xi_1^1-\xi_2^1)^2 + (\xi_1^2+\xi_2^2)^2}\ .$$

An $e^{\lambda}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s)$-function is defined for points $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ belong to sufficiently small domains $\mathcal{D}(I_i)$ $(i = 1, 2, 3, 4)$ except when $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 \in I_i$ $(i = 1, 2, 3, 4)$, where λ is called the degree of this function. For every positive integer Λ, it has the local expansion (see [4], [5], [8], [9]):

$$e^{\lambda}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s) = \sum^{*} f(\xi_1^1)\,(\xi_1^2)^{p_1}\,(\xi_2^2)^{p_2}\left(\frac{\partial}{\partial \xi_1^1}\right)^{\ell_1}\left(\frac{\partial}{\partial \xi_1^2}\right)^{\ell_2} K_0(s\,\rho_{12}) + R^{\Lambda}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s)\ , \tag{6.4}$$

where $\sum^{*}$ denotes a sum of a finite number of terms in which $f(\xi_1^1)$ is an infinitely differentiable function. In this expansion p_1, p_2, ℓ_1, ℓ_2 are integers where $p_1 \geq 0$, $p_2 \geq 0$, $\ell_1 \geq 0$, $\lambda = \min(p_1 + p_2 - q)$, $q = \ell_1 + \ell_2$ and the minimum is taken over all terms which occur in the summation $\sum^{*}$. The remainder $R^{\Lambda}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s)$ has continuous derivatives of all order $d \leq \Lambda$ satisfying

$$D^d R^{\Lambda}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s) = O(s^{-\Lambda}\, e^{-As\rho_{12}}) \quad \text{as} \quad s \to \infty\ , \tag{6.5}$$

where A is a positive constant.

Thus, using methods similar to those obtained in Sections 6–10 of [5], we can show that the functions (6.1) are e^{λ}-functions with degree $\lambda = 0, -1$ respectively. Consequently, for small impedances γ_i $(i = 1, 3)$ the functions (6.2) are e^{λ}-functions with degrees $\lambda = 0, -1, -1, -1$, respectively while, for large impedances γ_i $(i = 2, 4)$ the functions (6.3) are e^{λ}-functions with degrees $\lambda = 0, 1$ respectively (see also [8]).

Definition 2. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are points in large domains $\Omega + \Gamma_i$ $(i = 1, 2, 3, 4)$, then we define

$$r_{12} = \min_{\mathbf{y}}(r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \quad \text{if } \ \mathbf{y} \in \Gamma_1\ ,$$

$$R_{12} = \min_{\mathbf{y}}(r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \quad \text{if } \ \mathbf{y} \in \Gamma_2\ ,$$

$$r_{12}^{*} = \min_{\mathbf{y}}(r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \quad \text{if } \ \mathbf{y} \in \Gamma_3\ ,$$

and

$$R_{12}^{*} = \min_{\mathbf{y}}(r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \quad \text{if } \ \mathbf{y} \in \Gamma_4\ .$$

An $E^{\lambda}(\mathbf{x}_1, \mathbf{x}_2; s)$-function is defined and infinitely differentiable with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$ when these points belong to large domains $\Omega + \Gamma_i$ $(i = 1, 2, 3, 4)$ except when $\mathbf{x}_1 = \mathbf{x}_2 \in \Gamma_i$ $(i = 1, 2, 3, 4)$. Thus, the E^{λ}-function has a similar local expansion of the e^λ-function (see [4], [5], [8]). With the help of Sections 8 and 9 in [15], it is easily seen that the formula (4.3) is an $E^0(\mathbf{x}_1,\mathbf{x}_2;s)$-function and consequently we get

$$\begin{aligned}\overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2) &= O\Big\{\Big[1+|\log sr_{12}|\Big]\,e^{-A_1 s r_{12}}\Big\}\\ &+O\Big\{\Big[1+|\log sR_{12}|\Big]\,e^{-A_2 s R_{12}}\Big\}\\ &+O\Big\{\Big[1+|\log sr^*_{12}|\Big]\,e^{-A_3 s r^*_{12}}\Big\}\\ &+O\Big\{\Big[1+|\log sR^*_{12}|\Big]\,e^{-A_4 s R^*_{12}}\Big\}\ ,\end{aligned}\tag{6.6}$$

which is valid for $s\to\infty$ and for small/large impedances γ_i $(i=1,2,3,4)$ as indicated in the specification of case 1, where A_i $(i=1,2,3,4)$ are positive constants. Formula (6.6) shows that $\overline{G}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ is exponentially small for $s\to\infty$. Similar statements are true in the other fifteen cases.

With reference to Section 10 in [5], if the e^λ-expansions of the functions (6.1)–(6.3) are introduced into (4.3) and if we use formulae similar to (6.4), (6.9) of Section 6 in [5], we obtain the following local behaviour of $\overline{\chi}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ when r_{12}, R_{12}, r^*_{12} and R^*_{12} are small, which is valid for $s\to\infty$ and for small γ_1, γ_3 and large γ_2, γ_4:

$$\overline{\chi}(\mathbf{x}_1,\mathbf{x}_2;s^2)=\sum_{i=1}^{4}\overline{\chi}_i(\mathbf{x}_1,\mathbf{x}_2;s^2)\ ,\tag{6.7}$$

where

a) if $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_1)$, then

$$\overline{\chi}_1(\mathbf{x}_1,\mathbf{x}_2;s^2)=-\frac{1}{2\pi}\left\{1-\gamma_1\left(\frac{\partial}{\partial\xi_1^2}\right)^{-1}\right\}K_0(s\,\rho_{12})+O\Big\{s^{-1}\,e^{-A_1 s\rho_{12}}\Big\}\ ;\tag{6.8}$$

b) if $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_2)$, then

$$\overline{\chi}_2(\mathbf{x}_1,\mathbf{x}_2;s^2)=\frac{1}{2\pi}\left\{1-\gamma_2^{-1}\left(\frac{\partial}{\partial\xi_1^2}\right)\right\}K_0(s\,\rho_{12})+O\Big\{s^{-1}\,e^{-A_2 s\rho_{12}}\Big\}\ ;\tag{6.9}$$

c) if $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_3)$, then

$$\overline{\chi}_3(\mathbf{x}_1,\mathbf{x}_2;s^2)=\frac{1}{2\pi}\left\{1-\gamma_3\left(\frac{\partial}{\partial\xi_1^2}\right)^{-1}\right\}K_0(s\,\rho_{12})+O\Big\{s^{-1}\,e^{-A_3 s\rho_{12}}\Big\}\ ;\tag{6.10}$$

d) if $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_4)$, then

$$\overline{\chi}_4(\mathbf{x}_1,\mathbf{x}_2;s^2)=-\frac{1}{2\pi}\left\{1-\gamma_4^{-1}\left(\frac{\partial}{\partial\xi_1^2}\right)\right\}K_0(s\,\rho_{12})+O\Big\{s^{-1}\,e^{-A_4 s\rho_{12}}\Big\}\ .\tag{6.11}$$

When $r_{12} \geq \delta_1 > 0$, $R_{12} \geq \delta_2 > 0$, $r^*_{12} \geq \delta_3 > 0$ and $R^*_{12} \geq \delta_4 > 0$, the function $\overline{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ is of order $O(e^{-Bs})$ as $s \to \infty$, where B is a positive constant. Thus, since

$$\lim_{r_{12}\to 0} \frac{r_{12}}{\rho_{12}} = \lim_{R_{12}\to 0} \frac{R_{12}}{\rho_{12}} = \lim_{r^*_{12}\to 0} \frac{r^*_{12}}{\rho_{12}} = \lim_{R^*_{12}\to 0} \frac{R^*_{12}}{\rho_{12}} = 1 \ ,$$

(see [8], [9]), then we have the asymptotic formulae (6.9)–(6.11) with ρ_{12} in the small domains $\mathcal{D}(I_i)$ $(i = 1,2,3,4)$ being replaced by r_{12}, R_{12}, r^*_{12} and R^*_{12} in the large domains $\Omega + \Gamma_i$ $(i = 1,2,3,4)$ respectively.

Similar formulae for the other fifteen cases can be found.

7 – Construction of our results

Since for $\xi^2 \geq h_i$ $(i=1,2,3,4)$ the functions $\overline{\chi}_i(\mathbf{x},\mathbf{x}; s^2)$ are of order $O(e^{-2sA_i h_i})$ $(i = 1,2,3,4)$, the integral over the region Ω of the function $\overline{\chi}(\mathbf{x},\mathbf{x}; s^2)$ can be approximated in the following way (see (3.12)):

$$\begin{aligned}\overline{R}(s^2) = &\sum_{i=3}^{4} \int_{\xi^3=0}^{h_i} \int_{\xi^1=0}^{L_i} \overline{\chi}_i(\mathbf{x},\mathbf{x}; s^2)\, \{1 - k_2(\xi^1)\,\xi^2\}\, d\xi^1\, d\xi^2 \\ &- \sum_{i=1}^{2} \int_{\xi^3=0}^{h_i} \int_{\xi^1=0}^{L_i} \overline{\chi}_i(\mathbf{x},\mathbf{x}; s^2)\, \{1 + k_1(\xi^1)\,\xi^2\}\, d\xi^1\, d\xi^2 \\ &+ \sum_{i=1}^{4} O\{e^{-2sA_i h_i}\} \quad \text{as} \quad s \to \infty \ .\end{aligned} \tag{7.1}$$

If the e^λ-expansions of $\overline{\chi}_i(\mathbf{x},\mathbf{x}; s^2)$ $(i = 1,2,3,4)$ are introduced into (7.1), one obtains an asymptotic series of the form

$$\overline{R}(s^2) = \sum_{n=1}^{p} a_n\, s^{-n} + O(s^{-p-1}) \quad \text{as} \quad s \to \infty \ , \tag{7.2}$$

where the coefficients a_n, for all sixteen cases, are calculated from the e^λ-expansions with the help of formula (10.3) of Section 10 in [9] (see also [4], [5]).

On inverting Laplace transforms and using (3.7), we arrive at our results (2.1)–(2.7).

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E.M.E. Zayed,
Mathematics Department, Faculty of Science,
Zagazig University, Zagazig – EGYPT