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A NOTE ON OCTAHEDRAL SPHERICAL FOLDINGS

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Abstract: We show that the set of octahedral folding classes is a differentiable manifold of dimension six.

1 – Introduction

An *isometric folding* of S^2 is a map $f\colon\ S^2 \to S^2$ which sends piecewise geodesic segments on S^2 to piecewise geodesic segments on S^2 of the same length.

The points $x \in S^2$ where f fails to be differentiable are the *singularities* of f. We shall denote by $\sum f$ the set of singularities of f.

It is known [2] that for each $x \in \sum f$ the singularities of f near x form the image of an even number of geodesic rays emanating from x making alternate angles $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_n$, where

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = \pi \ .$$

This condition on the angles will be called the *angle folding relation*. Thus the set of singularities of an isometric folding of S^2 can be regarded as a graph on M satisfying the angle-folding relation.

2 – Octahedral foldings

A non-trivial isometric folding f of S^2 is an *octahedral spherical folding* or simply an *octahedral folding* if the underlying graph of its singularity set is an

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octahedral graph, that is, $\sum f$ partitions S^2 into 8 triangles. One of the simplest octahedral foldings is given by the map $\mathcal{S}\colon\ S^2 \to S^2$ with $\mathcal{S}(x,y,z) = (|x|,|y|,|z|)$ referred to as the *standard folding*.

Let $\mathcal{O}$ denote the set of all octahedral foldings and $\mathcal{G}$ be the quotient space $\mathcal{O}/\sim$ obtained from $\mathcal{O}$ by introducing the equivalence relation $f \sim g$ iff there exist an isometry $\Theta\colon\ S^2 \to S^2$ such that $\Theta(\sum f) = \sum g$.

The equivalence relation $\sim$ may be extended in a natural way to an equivalence relation on the set $\mathcal{T}$ of tilings of S^2 whose underlying graph is an octahedral graph obeying the angle folding relation.

Proposition 2.1. *The map $\Psi\colon\ \mathcal{G} \to \mathcal{T}/\sim$ given by $\Psi([f]) = [T]$, where T is the spherical tiling whose underlying graph is $\sum f$, is bijective.*

Proof: We first show that Ψ is surjective.

Let $[T] \in \mathcal{T}/\sim$. Denote by Δ_i, $i = 1,2,...,8$, e_j, $j = 1,2,...,12$ and v_k, $k = 1,2,...,6$, the faces, edges and vertices of T respectively and label them as indicated in fig. 1.

For each $j = 1,2,...,12$, let ϱ_j be the reflection in the great circle containing e_j. Consider the map $f\colon\ S^2 \to S^2$ given by

$$f|_{\Delta_1} = id|_{\Delta_1} \quad (id \text{ denotes the identity map on } S^2)\ ,$$

$$f|_{\Delta_2} = \varrho_1|_{\Delta_2}, \quad f|_{\Delta_3} = (\varrho_1 \circ \varrho_2)|_{\Delta_3}, \quad 4|_{\Delta_4} = \varrho_4|_{\Delta_4}, \quad f|_{\Delta_5} = \varrho_5|_{\Delta_5},$$

$$f|_{\Delta_6} = (\varrho_5 \circ \varrho_9)|_6, \quad f|_{\Delta_7} = (\varrho_5 \circ \varrho_9 \circ \varrho_{10})|_{\Delta_7}, \quad f|_{\Delta_8} = (\varrho_5 \circ \varrho_{12})|_{\Delta_8}\ .$$

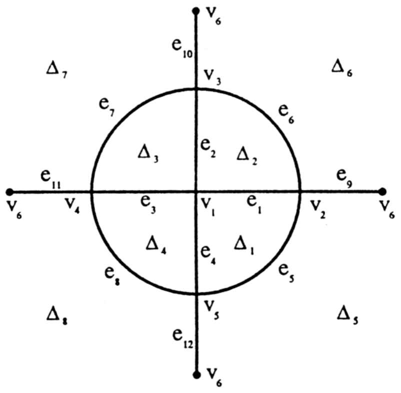

Fig. 1

Since, every vertex of T satisfies the angle folding relation and every $\varrho_j(e_j) = e_j$, for $j = 1, 2, ..., 12$ we conclude that f is a well defined map. Moreover $f|_{\Delta_j}$ is an isometry for each $j = 1, 2, ..., 12$, and so f is an isometric folding. Clearly $\Psi([f]) = [T]$.

Assume now that $\Psi([f]) = \Psi([g]) = [T]$. It is straightforward to show the existence of a spherical isometry Θ such that $\sum f = \Theta(\sum g)$. ∎

Consider the map $\Gamma\colon \mathcal{T}/\sim \to \mathbb{R}^{24}$ given by $\Gamma([T]) = (\varphi_1, \varphi_2, ..., \varphi_{24})$, $\varphi_1 \leq \varphi_2 \leq ... \leq \varphi_{24}$, where φ_i, $i = 1, 2, ..., 24$, are the 24 angles of the eight triangular faces of T and topologies $\mathcal{T}/\sigma$ with the topology induced by Γ. Use now the map Ψ defined in Proposition 2.1 to topologies $\mathcal{G}$.

Let $[T] \in \mathcal{T}/\sigma$. The tiling T has 8 triangular faces T_j, $j = 1, 2, ..., 8$, corresponding to 24 angles. Since the angle folding relation has to be fulfilled only 12 angles need to be specified in principle. However we shall show that there is a particular set of 6 angles whose knowledge is enough to determine the other six.

Proposition 2.2. *If the vertices $v_1, v_2, ..., v_5$ of the spherical pattern T indicated bellow obey the angle folding relation then $\theta + \theta^* = \pi$.*

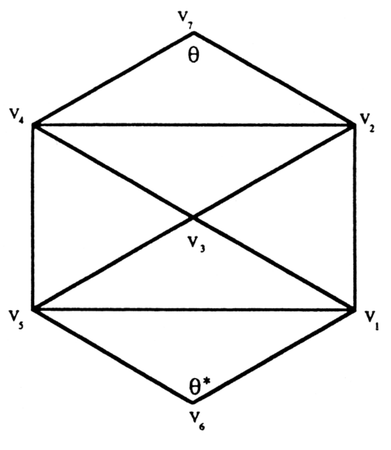

Fig. 2

Proof: Let $v_1, v_2, ..., v_7$ be the vertices of such a spherical tiling T. It is then possible to label T as indicated in figure 3, where φ_i, $\overline{\varphi}_i = \pi - \varphi_i$, $i = 1, ..., 6$, θ and θ^* stand for the angles and $a, b, ..., h$ for the edges.

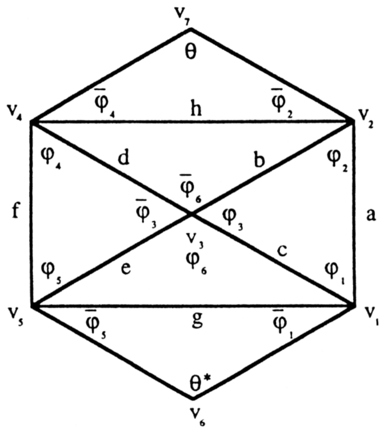

Fig. 3

Using spherical trigonometry, we may conclude, on the one hand,

$$\cos h = \frac{\cos\theta + \cos\varphi_2\cos\varphi_4}{\sin\varphi_2\sin\varphi_4}\ ,$$

and on the other hand

$$\begin{aligned}\cos h &= \cos b\cos d + \sin b\sin d\cos\overline{\varphi}_6\\ &= \left(\frac{\cos\varphi_1 + \cos\varphi_2\cos\varphi_3}{\sin\varphi_2\sin\varphi_3}\right)\left(\frac{\cos\varphi_5 - \cos\varphi_3\cos\varphi_4}{\sin\varphi_3\sin\varphi_4}\right) - \sin b\sin d\cos\varphi_6\end{aligned}$$

and so

$$\begin{aligned}\cos\theta = {} & \frac{(\cos\varphi_1 + \cos\varphi_2\cos\varphi_3)\,(\cos\varphi_5 - \cos\varphi_3\cos\varphi_4)}{\sin^2\varphi_3}\\ & - \sin b\sin d\sin\varphi_2\sin\varphi_4\cos\varphi_6 - \cos\varphi_2\cos\varphi_4\ .\end{aligned} \tag{1}$$

Also

$$\cos g = \frac{\cos\theta^* + \cos\varphi_1\cos\varphi_5}{\sin\varphi_1\sin\varphi_5}$$

and

$$\begin{aligned}\cos g &= \cos c\cos e + \sin c\sin e\cos\varphi_6\\ &= \frac{(\cos\varphi_2 + \cos\varphi_1\cos\varphi_3)}{\sin\varphi_1\sin\varphi_3}\,\frac{(\cos\varphi_4 - \cos\varphi_3\cos\varphi_5)}{\sin\varphi_3\sin\varphi_5} + \sin c\sin e\cos\varphi_6\end{aligned}$$

hence

$$\cos\theta^* = \frac{(\cos\varphi_2 + \cos\varphi_1\cos\varphi_3)\,(\cos\varphi_4 - \cos\varphi_3\cos\varphi_5)}{\sin^2\varphi_3} + \sin c\sin e\sin\varphi_1\sin\varphi_5\cos\varphi_6 - \cos\varphi_1\cos\varphi_5\ . \tag{2}$$

taking into account that

$$\frac{\sin b}{\sin c} = \frac{\sin\varphi_1}{\sin\varphi_2} \quad \text{and} \quad \frac{\sin d}{\sin e} = \frac{\sin\varphi_5}{\sin\varphi_4}$$

we can rewrite (1) as follows

$$\cos\theta = \frac{(\cos\varphi_1 + \cos\varphi_2\cos\varphi_3)\,(\cos\varphi_5 - \cos\varphi_3\cos\varphi_4)}{\sin^2\varphi_3} - \sin c\sin e\sin\varphi_1\sin\varphi_5\cos\varphi_6 - \cos\varphi_2\cos\varphi_4\ . \tag{3}$$

From (2) and (3) we have $\cos\theta^* = -\cos\theta$, that is, $\theta^* = \pi - \theta$. ∎

Corollary 2.1. *If five of the six vertices of an octahedral tiling T obey the angle folding relation then $T \in \mathcal{T}$.*

Observe that in Proposition 2.2 the spherical tiling does not need to be an octahedral tiling.

Proposition 2.3. *The space $\mathcal{T}/\sim$ is a differentiable manifold of dimension 6.*

Proof: Given the angles φ_j, $j = 1, 2, ..., 6$, we may construct an octahedral tiling T obeying the angle folding relation as follows. Denote by $v_1, v_2, ..., v_6$ the vertices of T. Up to an isometry, we may assume that $v_3 = (0, 0, 1)$, v_1 is in the great circle determined by the plane $y = 0$ and v_2 is in the hemisphere corresponding to $y > 0$.

The vertices v_1 and v_2 are uniquely determined if we require that, with v_3, they are the vertices of a spherical triangle T_1 with angles $\varphi_1 = \angle(\widehat{v_1v_2}, \widehat{v_1v_3})$, $\varphi_2 = \angle(\widehat{v_1v_2}, \widehat{v_2v_3})$ and $\varphi_3 = \angle(\widehat{v_1v_3}, \widehat{v_2v_3})$, where $\angle$ means angle and $\widehat{vw}$ means the geodesic segment joining v to w.

The angles φ_j, $j = 4, 5, 6$, determine, in a unique way, points v_4 and v_5 on S^2 such that

- **i)** v_2, v_3, v_4 are the vertices of a triangle T_2 adjacent to T_1 at $\widehat{v_2v_3}$, with $\overline{v}_6 = \pi - \varphi_j = \angle(\widehat{v_2v_3}, \widehat{v_3v_4})$;

ii) v_3, v_4, v_5 are the vertices of a triangle T_3 adjacent to T_2 at $\widehat{v_3v_4}$, with angles $\overline{\varphi}_3 = \pi - \varphi_3 = \angle(v_3v_4, \widehat{v_3v_5})$, $\varphi_4 = \angle(\widehat{v_3v_4}, \widehat{v_3v_5})$ and $\varphi_5 = \angle(\widehat{v_3v_4}, \widehat{v_3v_5})$.

Let a, b, c be the edges of T_1 opposite the angles φ_3, φ_1 and φ_2 respectively and d, e, f be the edges of T_3 opposite the angles φ_5, φ_4 and $\overline{\varphi}_3$ respectively. Then the edges of T_2 are b, d and h, where h is opposite $\overline{\varphi}_6$, and the edges of T_4, the triangle adjacent to T_1 and T_3, are e, c and g where g is the edge opposite to φ_6.

In this construction the vertex v_3 obeys automatically the angle folding relation. Since the vertices v_2 and v_4, also obey that relation we are led to the construction of a unique triangle T_5 adjacent to T_2 at $\widehat{v_2v_4}$, with angles $\overline{\varphi}_2$, $\overline{\varphi}_4$ and θ, where $\theta = \arccos(\cos h \sin\varphi_2 \sin\varphi_4 - \cos\varphi_2 \cos\varphi_4)$, and vertices v_2, v_4 and v_6. Similarly, working with v_1 and v_5 we obtain a unique triangle T_6 adjacent to T_4 at $\widehat{v_1v_5}$. The angles of T_6 are $\overline{\varphi}_4$, $\overline{\varphi}_5$ and θ^* where $\theta^* = \arccos(\cos g \sin\varphi_1 \sin\varphi_5 - \cos\varphi_1 \cos\varphi_5)$ and its vertices are v_1, v_3 and v_7. By Proposition 2.2 one has $\theta^* = \pi - \theta$.

We shall have an octahedral spherical f-tiling if (and only if) $v_6 = v_7$. We show this next.

Observe that T_5 has h, i and j as edges where i and j are the edges opposite to $\overline{\varphi}_4$ and $\overline{\varphi}_2$ respectively and T_6 has g, l and m as edges where l and m are respectively the edges opposite to $\overline{\varphi}_1$ and $\overline{\varphi}_5$.

Let ψ_1 be the angle $\angle(a,i)$ and ψ_2 be the angle $\angle(a,m)$. up to an isometry there exists a unique spherical triangle A such that a and i are edges of A and $\psi_1 = \angle(a,i)$ is one of its angles. Let m^* be the edge of A opposite to ψ_1 and δ be the angle of A opposed to i. Using spherical trigonometry we have

$$\begin{aligned}\cos m^* &= \cos a \cos i + \sin a \sin i \cos\psi_1 \\ &= \left(\frac{\cos\varphi_3 + \cos\varphi_1\cos\varphi_2}{\sin\varphi_1 \sin\varphi_2}\right)\left(\frac{-\cos\varphi_4 + \cos\varphi_1\cos\theta}{\sin\varphi_2 \sin\theta}\right) + \sin a \sin i \cos\psi_1 \ .\end{aligned}$$

Since

$$\cos\overline{\psi}_1 = \frac{\cos d - \cos b \cos h}{\sin b \sin h}$$

and

$$\frac{\sin a \sin i}{\sin b \sin h} = \frac{\sin\varphi_3 \sin\varphi_4}{\sin\varphi_1 \sin\theta} \ ,$$

we have

$$\begin{aligned}\cos m^* = &\left(\frac{\cos\varphi_3 + \cos\varphi_1\cos\varphi_2}{\sin\varphi_1 \sin\varphi_2}\right)\left(\frac{-\cos\varphi_4 + \cos\varphi_1\cos\theta}{\sin\varphi_2 \sin\theta}\right) \\ &+ \frac{\sin\varphi_3 \sin\varphi_4}{\sin\varphi_1 \sin\theta}\left(-\cos d + \cos b \cos h\right) \ .\end{aligned}$$

Using the fact that

$$\cos d = \frac{\cos\varphi_5 - \cos\varphi_3\cos\varphi_4}{\sin\varphi_3\sin\varphi_4}\,, \qquad \cos b = \frac{\cos\varphi_1 + \cos\varphi_2\cos\varphi_3}{\sin\varphi_2\sin\varphi_3}$$

and

$$\cos h = \frac{\cos\theta + \cos\varphi_2\cos\varphi_4}{\sin\varphi_2\sin\varphi_4}\,,$$

one has

$$\begin{aligned}
\cos m^* &= \frac{(\cos\varphi_3 + \cos\varphi_1\cos\varphi_2)\,(-\cos\varphi_4 - \cos\varphi_2\cos\theta)}{\sin\varphi_1\sin^2\varphi_2\sin\theta} \\
&\quad + \frac{1}{\sin\varphi_1\sin\theta}\Big(-\cos\varphi_5 + \cos\varphi_3\cos\varphi_4 \\
&\quad + \frac{(\cos\varphi_1 + \cos\varphi_2\cos\varphi_3)\,(\cos\theta + \cos\varphi_2\cos\varphi_4)}{\sin^2\varphi_2}\Big) \\
&= \frac{1}{\sin\varphi_1\sin\theta}\Big(\frac{-\cos\varphi_3\cos\varphi_4 - \cos\varphi_1\cos^2\varphi_2\cos\theta}{\sin^2\varphi_2} \\
&\quad - \cos\varphi_5 + \cos\varphi_3\cos\varphi_4 + \frac{\cos\varphi_1\cos\theta}{\sin^2\varphi_2} \\
&\quad + \cos\varphi_2\cos\varphi_3\cos\theta + \cos^2\varphi_2\cos\varphi_3\cos\varphi_4\Big) \\
&= \frac{1}{\sin\varphi_1\sin\theta}\Big(\frac{(\cos^2\varphi_2 - 1)\cos\varphi_3\cos\varphi_4}{\sin^2\varphi_2} + \cos\varphi_3\cos\varphi_4 \\
&\quad + \frac{(1-\cos^2\varphi_2)\cos\varphi_1\cos\theta}{\sin^2\varphi_2} - \cos\varphi_5\Big) \\
&= \frac{-\cos\varphi_5 + \cos\varphi_1\cos\theta}{\sin^2\varphi_2} = \cos m\,.
\end{aligned} \tag{1}$$

On the other hand

$$\begin{aligned}
\cos\delta &= \frac{\cos i - \cos a\cos m}{\sin a\sin m} \\
&= \frac{1}{\sin a\sin m}\Big(\frac{-\cos\varphi_4 - \cos\varphi_2\cos\theta}{\sin\varphi_2\sin\theta} \\
&\quad - \Big(\frac{\cos\varphi_3 + \cos\varphi_1\cos\varphi_2}{\sin\varphi_1\sin\varphi_2}\Big)\Big(\frac{-\cos\varphi_5 + \cos\varphi_1\cos\theta}{\sin\varphi_1\sin\theta}\Big)\Big) \\
&= \frac{1}{\sin^2\varphi_1\sin a\sin m}\Big(-\cos\varphi_4\sin^2\varphi_1 - \cos\varphi_2\cos\theta + \cos\varphi_3\cos_5 \\
&\quad - \cos\varphi_1\cos\varphi_3\cos\theta + \cos\varphi_1\cos\varphi_2\cos\varphi_5\Big)
\end{aligned} \tag{2}$$

and

$$\cos\psi_2 = -\left(\frac{\cos e - \cos g \cos c}{\sin g \sin c}\right)$$
$$= -\frac{1}{\sin g \sin c}\left(\frac{\cos\varphi_4 - \cos\varphi_3\cos\varphi_5}{\sin\varphi_3\sin\varphi_5}\right.$$
$$\left.\frac{-\cos\theta + \cos\varphi_1\cos\varphi_5}{\sin\varphi_1\sin\varphi_5}\,\frac{\cos\varphi_2 + \cos\varphi_1\cos\varphi_3}{\sin\varphi_1\sin\varphi_3}\right)$$

Since,

$$\frac{\sin g}{\sin m} = \frac{\sin\theta}{\sin\varphi_5} \quad \text{and} \quad \frac{\sin c}{\sin a} = \frac{\sin\varphi_2}{\sin\varphi_3}$$

one have

$$\begin{aligned} \cos\psi_2 = &-\frac{1}{\sin a \sin m \sin^2\varphi_1}\Big(\cos\varphi_4\sin^2\varphi_1 + \cos\varphi_2\cos\theta - \cos\varphi_3\varphi_5 \\ &+ \cos\varphi_1\cos\varphi_3\cos\theta - \cos\varphi_1\cos\varphi_2\cos\varphi_5\Big)\ . \end{aligned} \tag{3}$$

Taking into account (1), (2) and (3) one has $m = m^*$, $\delta = \psi_2$ and so $v_6 = v_7$.

We may now conclude that the map $\Phi : \mathcal{T}/\sim \to]0,\pi[\times\ldots\times]0,\pi[$, defined by $\Phi([T]) = (\varphi_1, \varphi_2, \ldots, \varphi_6)$, is a homeomorphism and consequently $\mathcal{T}/\sim$ is a differentiable manifold of dimension 6. ∎

S. Robertson [2] has conjectured that the set of all spherical foldings is connected. We do not have an answer to this question yet, but Proposition 2.3 enable us to prove

Corollary 2.2. *The set $\mathcal{G}$ of all octahedral foldings (up to an isometry) is connected (in fact, path-connected).*

Proof: Using Propositions 2.3 and 2.1, we can give to $\mathcal{G}$ a structure of a differentiable manifold with dimension 6. In fact, with notations as above, the map $\Upsilon = \Phi\circ\Psi\colon \mathcal{G} \to]0,\pi[\times\ldots\times]0,\pi[$ given by $\Upsilon([f]) = (\varphi_1, \varphi_2, \ldots, \varphi_6)$ gives rise to a differentiable structure for $\mathcal{G}$.

Let $[f] \in \mathcal{G}$, $\Upsilon([f]) = (\varphi_1, \varphi_2, \ldots, \varphi_6)$ and $\gamma : [0,1] \to]0,\pi[\times\ldots\times]0,\pi[$ the map given by $\gamma(t) = ((1-t)\varphi_1 + t\frac{\pi}{2},\ (1-t)\varphi_2 + t\frac{\pi}{2},\ \ldots,\ (1-t)\varphi_6 + t\frac{\pi}{2})$. Then $\overline{\gamma} = \Upsilon^{-1}\circ\gamma\colon [0,1] \to \mathcal{G}$ is a path joining $[f]$ to the standard folding S. Therefore $\mathcal{G}$ is path-connected. ∎

REFERENCES

[**1**] d'Azevedo Breda, A.M. – *Isometric foldings*, Ph.D. Thesis, Southampton University, U.K., 1989.

[**2**] Robertson, S.A. – *Isometric foldings of Riemannian manifolds*, Proc. Royal Soc. Edinburgh (1977), 275–284.

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