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# EXTENDING HYPERSURFACES AND MEROMORPHIC FUNCTIONS*

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**Abstract:** The aim of this paper is to investigate the extension of hypersurfaces in the case where $\Omega$ is a spread domain over a locally convex space having the Levi property. From the obtained result the authors show that every meromorphic function from a spread domain $\Omega$ over a locally convex space having the Levi property with values in a sequentially complete locally convex space can be extended meromorphically to its envelope of holomorphy.

## Introduction

The problem of extension of hypersurfaces from a spread domain $\Omega$ over Stein manifolds to its envelope of holomorphy $^\wedge\Omega$ has been investigated by Dloussky [1]. For solving the problem in finite dimension, essentially, Dloussky has considered the extension of hypersurfaces from a Hartogs domain $H_2(r)$ in $C^2$ to its envelope of holomorphy $\Delta^2$, where $\Delta^2$ is the unit polydisc in $C^2$. In infinite dimension this problem is not considered until now. Hence, the aim of this paper is to examine this problem for spread domains over locally convex spaces having the Levi property. After that, based on the obtained result we consider the extension of meromorphic functions from spread domains over locally convex space having the Levi property with values in sequentially complete locally convex spaces to its envelope of holomorphy. In the case where meromorphic functions obtain scalar values the result was proved by Harita [2]. However, the method of Harita does not use for meromorphic functions with values in locally convex spaces.

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## 1 – Extending hypersurfaces

First we give the following:

**1.1 Definition.** Let $E$ be a locally convex space. $E$ is called to have the Levi property (shortly an $L$-space) if every pseudoconvex spread domain over $E$ is the domain of existence of holomorphic function.

Here a spread domain $\Omega$ over a locally convex space $E$ is pseudoconvex if the function $-\log d(z,z')$ is plurisubharmonic on $\Omega \times (E - \{0\})$ [5].

**1.2 Some examples.**

**a**) Every Lindelof locally convex space having a Schauder decomposition is an $L$-space [6].

**b**) $l^\infty(A)$ with $A$ an uncountable set is not an $L$-space [3].

The first result of this paper is the following.

**1.3 Theorem.** *Let $H$ be a hypersurface in a spread domain $(\Omega, \Phi)$ over an $L$-space $E$. Then there exists a hypersurface $\hat{}H$ in $\hat{}\Omega$ such that $\hat{}(\Omega \backslash H) \cong \hat{}\Omega \backslash \hat{}H$ where by $\hat{}\Omega$ we denote the envelope of holomorphy of $\Omega$.*

Now as in [1] we give the following.

**1.4 Definition.** Let $(\Omega, \Phi)$ be a spread domain over a locally convex space $E$ and $H$ a hypersurface in $\Omega$. We say that $(\Omega, H)$ is maximal if for every spread domain $\Omega'$ over $E$ such that $\Omega$ is open in $\Omega'$ and $\Omega' \backslash \Omega \subseteq H'$ where $H'$ is a hypersurface in $\Omega'$ we have $\Omega = \Omega'$ provided $H' \cap \Omega = H$.

For proving the Theorem 1.3 we need the following.

**1.5 Proposition.** *Let $(\Omega, \Phi)$ be a spread domain over an $L$-space $E$ and $H$ a hypersurface in $\Omega$ such that $(\Omega, H)$ is maximal and $\Omega \backslash H$ is domain of holomorphy. Then $\Omega$ is a domain of holomorphy.*

The following lemma is used for the proof of Proposition 1.5.

**1.6 Lemma** [1]. *Let $\pi: X \to Y$ be a local homeomorphism between two connected topological spaces $X$ and $Y$ and $H \subset Y$ a closed subset of $Y$ having an empty interior which does not disconnect locally $Y$. If $\pi$ has a section $\sigma$ on $Y \backslash H$ then $\pi$ is injective.*

**Proof of Proposition 1.5:** For each $t \in [0,1]$ put

$$M_t = \Big\{(z_1,z_2) \in C^2\colon\ |z_1| \le 1,\ z_2 \in [0,t]\Big\} \cup \Big\{(z_1,z_2) \in C^2\colon\ |z_1| = 1,\ z_2 \in [t,1]\Big\}\,.$$

By [2] it suffices to show that every holomorphic map $\varphi$ from a neighbourhood $U$ of $M_0$ to $\Omega$ such that $\Phi.\varphi\colon\ U \to \Phi\,\varphi(U)$ is a homeomorphism and $\Phi\,\varphi(U)$ is contained in a subspace $B$ of $E$ of dimension 2, can be extended holomorphically to a neighbourhood of $M_1$. Consider $\varphi^{-1}(H) \subset U$. From a result of Dloussky [1] we can find an analytic set $\hat{}H$ in $\hat{}U$ such that $\hat{}(U\backslash\varphi^{-1}(H)) \cong \hat{}U\backslash\hat{}H$. Since $\varphi(U\backslash\varphi^{-1}(H)) \subset \Omega\backslash H$ and, by the hypothesis, $\varphi$ can be extended to a holomorphic map $\psi$ on $\hat{}U\backslash\hat{}H$ with values in $\Omega\backslash H$. We write $E = B \oplus B^{\perp}$. Replacing $U$ by a more small neighbourhood of $M_0$ we can assume that there exists a neighbourhood $V$ of 0 in $B^{\perp}$ such that $\Phi$ has a holomorphic section $\delta\colon\ \Phi\,\varphi(U) \times V \to \Omega$. Put $\hat{}\varphi = \delta.(\Phi\,\varphi \times \mathrm{id})\colon\ U \times V \to \Omega$ and assume that $Z$ is the domain of existence of $\hat{}\varphi$ over $\hat{}U \times V$. Then $\hat{}\varphi$ has a holomorphic extension $\hat{}\widetilde{\varphi}$ on $Z$ with values in $\Omega$ and $((\hat{}U\backslash\hat{}H) \cup U) \times V \subset Z$. Now we have a following commutative diagram

$$\begin{array}{ccc} ((\hat{}U\backslash\hat{}H) \cup U) \times V & \xrightarrow{\sigma} & Z \\ & \searrow & \pi\downarrow \\ & & \hat{}U \times V \end{array}$$

By Lemma 1.6 $\pi$ is injective and, hence, $Z$ is open in $\hat{}U \times V$. Now on $\hat{}U \times V \sqcup \Omega$ we can define an equivalent relation as follows. Let $x \in \hat{}U \times V$ and $\omega \in \Omega$. We write $x \sim \omega$ if $x \in Z$ and $\hat{}\widetilde{\varphi}(x) = \omega$. Put $'\Omega = \hat{}U \times V \sqcup \Omega/\!\sim$ and

$$'\Phi = \begin{cases} \Phi & \text{on } \Omega, \\ \text{the holomorphic extension of } \Phi\,\varphi \times \mathrm{id} \text{ to } \hat{}U \times V\,. \end{cases}$$

It follows that $('\Omega, '\Phi)$ is a spread domain over $E$. Now we define a map $g\colon\ \hat{}U \times V \to {'\Omega}$ by the formula

$$g = \begin{cases} \hat{}\widetilde{\varphi} & \text{on } ((\hat{}U\backslash\hat{}H) \cup U) \times V \\ \mathrm{id} & \text{on } \hat{}U \times V\,. \end{cases}$$

We need to check that the such defined map $g$ is reasonable. Indeed, if $t \in ((\hat{}U\backslash\hat{}H) \cup U) \times V$ and, hence, $g(t) = \hat{}\widetilde{\varphi}(t) \in \Omega$, and at the same time, $g(t) \equiv t$. Put $x = \hat{}\widetilde{\varphi}(t)$ and by the equivalent relation $x \equiv t$. Hence, $g(t) = t \equiv x = \hat{}\widetilde{\varphi}(t)$. The map $g$ is holomorphic on $\hat{}U \times V$ and it is a holomorphic extension of $\hat{}\widetilde{\varphi}$. Then $'H = \hat{}H \times V \sqcup H/\!\sim$ is a hypersurface in $'\Omega$ with $\Omega \cap {'H} = H$

and $'\Omega\backslash\Omega \subseteq {'H}$. By the maximality of $\Omega$ we have $\Omega = {'\Omega}$ and, hence, $Z = {'U} \times V$. Such as, $\varphi$ can be extended to a holomorphic map on a neighbourhood of $M_1$. The proposition is proved. ∎

As in [1] we give the following

**1.7 Definition.** Let $(\Omega, \Phi)$ be a spread domain over a locally convex space $E$. A boundary point of $(\Omega, \Phi)$ is a basis of a filter $r$ consisting of connected open subsets of $\Omega$ such that

**i**) $r$ has not a limit point in $\Omega$;

**ii**) $\Phi(r)$ converges to a point in $E$;

**iii**) For every connected neighbourhood $U(x)$ of $x$, $r$ consists of one and only one connected component of $\Phi^{-1}(U)$ and every element of $r$ has a such form. By $\partial\Omega$ we denote the set of boundary points of $(\Omega, \dot{\Phi})$ and put $\check{\Omega} = \Omega \cup \partial\Omega$. We define $\check{\Phi}\colon \check{\Omega} \to E$ by $\check{\Phi}(y) = \Phi(y)$ if $y \in \Omega$, $\check{\Phi}(r) = x \in E$ if $r \in \partial\Omega$ and $x$ is the limit point of $\Phi(r)$. On $\check{\Omega}$ we give a topology defined as follows. If $x_0 \in \Omega$ then $U$ is a neighbourhood of $x_0$ in $\check{\Omega}$ if and only if $U \cap \Omega$ is a neighbourhood of $x_0$ in $\Omega$, and if $x_0 = r_0 \in \partial\Omega$ then the sets of the form $U(r_0) = V \cup \{r \in \partial\Omega\colon \exists\, V' \ni r,\, V' \subset U\}$, where $V \ni r_0$ are a basis of neighbourhoods of $r_0$. We remark that $\check{\Phi}$ is continuous on $\check{\Omega}$ equipped with this topology. At the same time, if $\lambda\colon (\Omega_1, \Phi_1) \to (\Omega_2, \Phi_2)$ is a morphism between spread domains over $E$, then it can be extended to a continuous map $\check{\lambda}\colon \check{\Omega}_1 \to \check{\Omega}_2$. Now we assume that $r \in \partial\Omega$. We say that $\partial\Omega$ is a hypersurface locally at $r$ if there exists a neighbourhood $U_r$ of $r$ in $\check{\Omega}$ such that $U_r$ is homeomorphic to $\check{\Phi}(U_r)$, $\check{\Phi}(U_r)$ is open in $E$ and $\check{\Phi}(\partial\Omega \cap U_r)$ is a hypersurface of $\check{\Phi}(U_r)$.

**1.8 Proposition.** *Let $(\Omega, \Phi)$ be a spread domain over an $L$-space $E$ and $H$ a hypersurface of $\Omega$ which is singular for a function $f \in \mathcal{O}(\Omega\backslash H)$. Then there exists a hypersurface ${}^\wedge H$ of ${}^\wedge\Omega$ such that $H = \lambda^{-1}({}^\wedge H)$ and ${}^\wedge(\Omega\backslash H) \cong {}^\wedge\Omega\backslash{}^\wedge H$, where $\lambda$ denotes the canonical map from $\Omega$ to ${}^\wedge\Omega$.*

**Proof:** Since $E$ is an $L$-space we have a following commutative diagram

$$\begin{array}{ccc} \Omega\backslash H & \xrightarrow{\lambda_H} & {}^\wedge(\Omega\backslash H) \\ & \searrow_{\lambda} & \downarrow \tilde{\lambda} \\ & & {}^\wedge\Omega \end{array}$$

Construct $('\Omega, '\lambda)$, $'\Omega = {}^\wedge(\Omega\backslash H) \cup Z$ and $Z$ denotes the set of boundary points of $\hat{}(\Omega\backslash H)$ where $Z$ is hypersurface locally at its every point. Put $'\lambda = \check{\check{\lambda}}|_{'\Omega}$, $'H = Z\cap{}'\Omega$. As in [1] $'H$ is a hypersurface of $'\Omega$ such that $'H\cap\Omega = H$ and $('\Omega, {}'H)$ is maximal. By Proposition 1.5 it follows that $'\Omega$ is a domain of holomorphy. Hence $'\Omega \cong \hat{}\Omega$. Since $'\Omega\backslash{}'H \cong \hat{}(\Omega\backslash H)$ we obtain $\hat{}(\Omega\backslash H) \cong \hat{}\Omega\backslash\hat{}H$, where $\hat{}H = {}'\lambda('H)$. The proposition is proved. ∎

**Proof of Theorem 1.3:** Now based on Proposition 1.8 and ideas of Dloussly [1] we prove Theorem 1.3. Let $'H$ denote the set of points $h \in H$ such that for every holomorphic function $f \in \mathcal{O}(\Omega\backslash H)$ there exists an open neighbourhood $V_f$ of $h$ to which $f$ can be holomorphically extended. Then $H\backslash{}'H$ is a hypersurface of $\Omega$ and singular for a holomorphic function. Indeed, by the hypothesis $\hat{}(\Omega\backslash(H\backslash{}'H))$ is the domain of existence of some holomorphic function $f$. Let $x \in H\backslash{}'H$ be an arbitrary point. Assume that there exists a neighbourhood $U_x$ of $x$ such that $f \in \mathcal{O}(U_x)$. Hence $U_x \subset \hat{}(\Omega\backslash(H\backslash{}'H))$. By the definition of $'H$ it follows that if $g \in \mathcal{O}(\Omega\backslash H)$ then $g \in \mathcal{O}(\Omega\backslash(H\backslash{}'H))$ and, hence, $g$ can be extended holomorphically to $\hat{}(\Omega\backslash(H\backslash{}'H))$. Therefore $g$ is holomorphic on $U_x$. This is impossible because for each neighbourhood $U_x$, $x \in (H\backslash{}'H)$, always there exists $g \in \mathcal{O}(\Omega\backslash H)$ such that $g$ can not be extended holomorphically to $U_x$. By Proposition 1.8 there exists a hypersurface $\hat{}H$ of $\hat{}\Omega$ such that $\hat{}(\Omega\backslash(H\backslash{}'H)) \cong \hat{}\Omega\backslash\hat{}H$. By the definition of $'H$ it implies that $\hat{}(\Omega\backslash(H\backslash{}'H)) \cong \hat{}(\Omega\backslash H)$. The Theorem 1.3 is completely proved. ∎

## 2 – The extension of meromorphic functions

In this section we give an application of Theorem 1.3. We have the following.

**2.1 Definition.** Let $E$ and $F$ be two locally convex spaces and $\Omega \subset E$ an open set. A holomorphic function $f$ defined on a dense open subset $\Omega_0$ of $\Omega$ with values in $F$ is said to be meromorphic if for every $x \in \Omega$ there exists a neighbourhood $U$ of $x$ and two holomorphic function $h\colon U \to F$, $\sigma\colon U \to C$, $\sigma \neq 0$ such that

$$f|_{U\cap\Omega_0} = \frac{h}{\sigma}|_{U\cap\Omega_0}\ .$$

Put $P(f) = \{x \in \Omega\colon f \text{ is not holomorphic at } x\}$. Now we prove the following.

**2.2 Theorem.** *Let $(\Omega, \Phi)$ be a spread domain over an $L$-space $E$. Then every meromorphic function $f : \Omega \to F$, where $F$ is a sequentially complete locally convex space, can be extended meromorphically to $\hat{}\Omega$, $\hat{}\Omega$ denotes the envelope of holomorphy of $\Omega$.*

**Proof:** By the Theorem 1.3 there exists a hypersurface $\hat{}H$ in $\hat{}\Omega$ such that $\hat{}(\Omega\backslash P(f)) \cong \hat{}\Omega\backslash\hat{}H$. Let $\widetilde{f}$ be a holomorphic extension of $f|_{\Omega\backslash P(f)}$ to $\hat{}\Omega\backslash\hat{}H$. We show that $\widetilde{f}$ is extended meromorphically to $\hat{}\Omega$.

Let $H'$ be the set of the points in $\hat{}\Omega$ where $\widetilde{f}$ is not meromorphic. For $z_0$ in $\hat{}H$ we can identify locally $\hat{}\Omega$ with a product $V \times \Delta$ ($V$ connected), $z_0$ with $(a, 0)$ and $\hat{}H$ with the zero set of some Weierstrass polynomial $P(z, \lambda)$ with roots in $\frac{1}{2}\Delta$. If there is a non empty open set $\omega$ in $V$ such that $\widetilde{f}$ is meromorphic on $\omega \times \Delta$, after shrinking $\omega$ and applying the Weierstrass division theorem we can suppose that there is an integer $N$ such that $P^N \widetilde{f}$ is holomorphic on $\omega \times \Delta$.

Then on the open set $\frac{1}{2} < |\lambda| < 1$, $P^N \widetilde{f}$ has a Laurent's expansion $\sum_{n=-\infty}^{+\infty} a_n(z)\, \lambda^n$, where $a_n(z)$ is holomorphic on $V$. But the $a_n(z)$, for $n < 0$, vanish on $\omega$ and, by the identity theorem, $a_n(z) = 0$ on $V$. Hence this Laurent's expansion is holomorphic all over $V \times \Delta$, coincides with $P^N \widetilde{f}$ on a non empty open set, so, by the analytic continuation, the coincidence is true everywhere and $P^N \widetilde{f}$ is holomorphic on $V \times \Delta$. Hence, $\widetilde{f}$ is meromorphic on $V \times \Delta$ and $z_0 \notin H'$. Since $H' \cap (\hat{}\Omega\backslash\hat{}H) = \emptyset$ and, hence, $'H \subset \hat{}H$ and from the above argument one can deduce that, at any point in $\hat{}\Omega$, the germ of $H'$ is the union of some irreducible components of the germ of $\hat{}H$ and hence is a hypersurface. Now, $\hat{}\Omega$ being pseudoconvex, $\hat{}\Omega\backslash H'$ is also pseudoconvex and by the hypothesis $E$ is an $L$-space, $\hat{}\Omega\backslash H'$ is the domain of existence, furthermore, by the hypothesis, $\hat{}\Omega\backslash H'$ contains $\Omega$ which implies $\hat{}\Omega\backslash H' \supset \hat{}\Omega$ and $H' = \emptyset$.

The Theorem 2.2 is proved. ∎

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