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INTEGRAL REPRESENTATIONS OF GRAPHS *

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Abstract: Following the definition of graph representation modulo an integer given by Erdös and Evans in [1], we call degree of a representation to the number of prime factors in the prime factorization of its modulo. Here we study the smallest possible degree for a representation of a graph.

The starting point for this research is the concept of representation introduced in [1], and the proposed study of relations between properties of graphs and properties of their representations.

Let $G=(V,E)$ be a graph with n vertices $v_1,...,v_n$. The graph G is said to be *representable* modulo a positive integer b if there exist distinct integers $a_1,...,a_n$ such that $0\le a_i<b$, and g.c.d.$\{a_i-a_j,b\}=1$ if and only if v_i and v_j are adjacent. We say that $\{a_1,...,a_n\}$ is a *representation* of G modulo b. We call *degree* of the representation to the number of prime factors, counting multiplicities, in the prime factorization of b. The concept of degree was not mentioned in [1] explicitly. However we can see in the proof of the theorem of [1] that there always exists a representation of degree equal to the number of edges of the complement of a graph that results from G by adjoining an isolated vertex. We shall see that there exist representations of smaller degree. We call *representation degree* of G, $d_r(G)$, to the smallest possible degree for a representation of G.

We say that a function $\phi\colon E\to X$ is *transitive* if, for every $(v_i,v_j),(v_j,v_k)\in E$ such that $\phi(v_i,v_j)=\phi(v_j,v_k)=x$, we have $(v_i,v_k)\in E$ and $\phi(v_i,v_k)=x$. For example, if $\phi\colon E\to X$ is one-to-one, then ϕ is transitive. Given a set Y, $\#Y$

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denotes its cardinal number. We call *degree* of a transitive function $\phi\colon E \to X$ to $\#\phi(E)$ and we call *transitive degree* of G, $d_t(G)$, to the smallest $\#\phi(E)$, when ϕ runs over the transitive functions defined in E. It is not difficult to prove some properties of $d_t(G)$. For example:

Proposition 1. *$d_t(G) \le \#E$.*

Proposition 2. *$d_t(G) = \max_H d_t(H)$, where H runs over the maximal connected subgraphs of G.*

Proposition 3. *Suppose that G is connected. Then*

a) *$d_t(G) = 0$ if and only if $\#V = 1$.*

b) *$d_t(G) = 1$ if and only if $\#V \ge 2$ and G is complete.*

c) *$d_t(G) = \#E$ if and only if there exists a vertex incident with all the edges of G.*

Let $G' = (V', E')$ be the complement of G. The following theorems are our main results. We shall prove them later.

Theorem 4. *Let ϕ be a transitive function defined in E' of degree $d \ge 2$. Then there exists a representation of G of degree d.*

Corollary 5. *If $d_t(G') \ge 2$, then $d_r(G) \le d_t(G') \le \#E'$.*

Corollary 5 is not always true when $d_t(G') \le 1$. The following proposition shows this and is easy to prove.

Proposition 6.

a) *$d_r(G) = 0$ if and only if $\#V = 1$.*

b) *$d_r(G) = 1$ if and only if $\#V \ge 2$ and G is complete.*

c) *$d_r(G) \le 1$ if and only if $d_t(G') = 0$.*

d) *If $d_t(G') = 1$, then $d_r(G) = 2$.*

Theorem 7. *Suppose that G' does not have any subgraph isomorphic to K_3. If G has a representation of degree d, then there exists a transitive function defined in E' of degree $\le d$.*

Counter-example. If G' has subgraphs isomorphic to K_3, then Theorem 7 is not always true, as the following example shows. Suppose that G is a graph with 5 vertices and only one edge. Then $R = \{0, 3, 5, 15, 30\}$ is a representation of G modulo $b = 3 \times 5 \times 7 = 105$. It is not difficult to see that any transitive function defined in E' has degree greater than 3.

Corollary 8. *If $d_t(G') \geq 2$ and G' does not have any subgraph isomorphic to K_3, then $d_r(G) = d_t(G')$.*

Let $M(G')$ be the maximum number of edges incident with one vertex in G'.

Theorem 9. *Suppose that G' has no cycles. Then*

a) *$d_t(G') = M(G')$.*

b) *If at least one of the maximal connected subgraphs of G' has at least 3 vertices, then*

$$d_r(G) = d_t(G') = M(G') \ . \tag{1}$$

Corollary 10. *If G' is a tree and $n \neq 2$, then (1) holds.*

Now we are going to prove the theorems above. We split the proof of Theorem 4 into several lemmas.

Lemma 11. *Suppose that $\phi\colon E' \to X$ is a transitive function with $\#\phi(E') = 1$. Let δ be a positive integer. Then there exists a positive prime $p > \delta$ and there exist distinct nonnegative integers $a_1, ..., a_n$ such that $(v_i, v_j) \in E'$ if and only if p divides $a_i - a_j$, $i, j \in \{1, ..., n\}$, $i \neq j$.*

Proof: Let $H_1, ..., H_t$ be the maximal connected subgraphs of G'. Without loss of generality, suppose that $H_s = \{v_{k_1+\cdots+k_{s-1}+1}, ..., v_{k_1+\cdots+k_s}\}$, $k_s = \#H_s$, $s \in \{1, ..., t\}$. Let p be a prime $> \max\{t, \delta\}$. If $i = k_1 + \cdots + k_{s-1} + j$, $1 \leq j \leq k_s$, let $a_i = s + j\,p$. Since $\#\phi(E') = 1$, the graphs H_i are complete. It is easy to conclude that the lemma is satisfied. ∎

Lemma 12. *Let α and β be integers with* g.c.d.$\{\alpha, \beta\} = 1$. *Let p be a prime. Then there exists at most one $\epsilon \in \{0, ..., p-1\}$ such that $\epsilon\beta + \alpha \in (p)$, where (p) denotes the principal ideal, of the ring of the integers, generated by p.*

Proof: Firstly, suppose that p divides β. Then p does not divide α and, therefore, $\epsilon\beta + \alpha \notin (p)$, for every integer ϵ. Now suppose that p does not divide β

and that there exist $\epsilon_1, \epsilon_2 \in \{1,...,p-1\}$ such that $\epsilon_1 \neq \epsilon_2$ and $\epsilon_1\beta+\alpha$, $\epsilon_2\beta+\alpha \in (p)$. Then $(\epsilon_1-\epsilon_2)\beta \in (p)$. As p is prime, p divides $\epsilon_1-\epsilon_2$ or p divides β, what is impossible. ∎

Lemma 13. *Let $\alpha_1,...,\alpha_s,\beta_1,...,\beta_s$ be integers such that* g.c.d.$\{\alpha_j,\beta_j\}=1$, $j \in \{1,...,s\}$. *Let $b=p_1\cdots p_r$, where $p_1,...,p_r$ are positive primes. If* $\min\{p_i : 1 \le i \le r\} > s\,r$, *then there exists an integer γ such that*

$$\text{g.c.d.}\{\gamma\beta_j+\alpha_j,\, b\} = 1\,, \qquad j \in \{1,...,s\}\ . \tag{2}$$

Proof: Let $m=\min\{p_i\}$. From the previous lemma, it can easily be deduced that there exists $\gamma \in \{0,...,m-1\}$ such that $\gamma\beta_j+\alpha_j \notin (p_i)$, $j \in \{1,...,s\}$, $i \in \{1,...,r\}$. That is, γ satisfies (2). ∎

Lemma 14. *Let $\phi : E' \to X$ be a transitive function. Suppose that $d=\#\phi(E') \ge 2$ and $\phi(E')=\{x_1,...,x_d\}$. Let δ be a positive integer. Then there exist distinct positive primes $p_1,...,p_d$ and there exist distinct integers $a_1,...,a_n$ such that:*

- **i**) $0 \le a_i < p_1\cdots p_d$, $i \in \{1,...,n\}$.
- **ii**) g.c.d.$\{a_i-a_j,\, p_1\cdots p_d\}=1$ *if and only if* $(v_i,v_j) \notin E'$, $i,j \in \{1,...,n\}$, $i \neq j$.
- **iii**) g.c.d.$\{a_i-a_j,\, p_1\cdots p_d\}=p_u$ *if and only if* $(v_i,v_j) \in E'$, *and* $\phi(v_i,v_j)=x_u$, $i,j \in \{1,...,n\}$, $i \neq j$, $u \in \{1,...,d\}$.
- **iv**) $\min\{p_1,...,p_d\} > \delta$.

Proof: By induction on n. As $d \ge 2$, we have $n \ge 3$. Let $G_0=(V_0,E_0)$ be the subgraph that we obtain from G' deleting v_n and all the edges incident with v_n. Without loss of generality, we assume that $E_0 \neq E'$ and $\phi(E_0)=\{x_1,...,x_e\}$. We choose $p_1,...,p_e$ and $a_1,...,a_{n-1}$ as follows. Note that $e \le 1$ when $n=3$.

If $e \ge 2$, then, by the induction assumption, there exist distinct primes $p_1,...,p_e$ and there exist distinct integers $a_1,...,a_{n-1}$ such that:

- **i**$_0$) $0 \le a_i < p_1\cdots p_e$, $i \in \{1,...,n-1\}$.
- **ii**$_0$) g.c.d.$\{a_i-a_j,\, p_1\cdots p_e\}=1$ if and only if $(v_i,v_j) \notin E_0$, $i,j \in \{1,...,n-1\}$, $i \neq j$.
- **iii**$_0$) g.c.d.$\{a_i-a_j,\, p_1\cdots p_e\}=p_u$ if and only if $(v_i,v_j) \in E_0$, and $\phi(v_i,v_j)=x_u$, $i,j \in \{1,...,n-1\}$, $i \neq j$, $u \in \{1,...,e\}$.
- **iv**$_0$) $\min\{p_1,...,p_e\} > \max\{\delta,(n-1)d\}$.

If $e = 1$, then, according to Lemma 11, there exists a prime p_1 and there exist distinct nonnegative integers $a_1, ..., a_{n-1}$ satisfying ii$_0$), iii$_0$) and iv$_0$).

If $e = 0$, take $a_i = i - 1$, $i \in \{1, ..., n-1\}$.

In any case $e \geq 0$, we choose primes $p_{e+1}, ..., p_d$ such that:

I) $p_1, ..., p_d$ are distinct.

II) None of the primes $p_{e+1}, ..., p_d$ divide $a_i - a_j$, $i, j \in \{1, ..., n-1\}$, $i \neq j$.

III) $\min\{p_1, ..., p_d\} > \max\{\delta, (n-1)d\}$.

Without loss of generality, suppose that $v_1, ..., v_t$ are the vertices of G' incident with v_n. Let $x_{k_i} = \phi(v_i, v_n)$, $i \in \{1, ..., t\}$. Without loss of generality, suppose that $k_1, ..., k_r$ are pairwise distinct and $k_i \in \{k_1, ..., k_r\}$ whenever $i \in \{r+1, ..., t\}$.

According to the Chinese Remainder Theorem, there exists an integer z such that

$$z - a_j \in (p_{k_j}), \quad j \in \{1, ..., r\} . \tag{3}$$

Let $i \in \{r+1, ..., t\}$ and suppose that $k_i = k_j$, wherej $j \in \{1, ..., r\}$. As ϕ is transitive, $(v_i, v_j) \in E'$ and $\phi(v_i, v_j) = x_{k_i}$. Therefore $k_i \in \{1, ..., e\}$. From iii$_0$), it follows that $a_i - a_j \in (p_{k_i})$. Thus $z - a_i = (z - a_j) + (a_j - a_i) \in (p_{k_i})$.

Now suppose that $z - a_i \in (p_{k_j})$, with $i \in \{1, ..., n-1\}$, $j \in \{1, ..., r\}$. From (3), $a_i - a_j \in (p_{k_j})$. Bearing in mind II), ii$_0$) and iii$_0$), we conclude that $k_j \in \{1, ..., e\}$, $(v_i, v_j) \in E_0$ and $\phi(v_i, v_j) = x_{k_j}$. From the transitivity of ϕ, $(v_i, v_n) \in E'$ and $\phi(v_i, v_n) = x_{k_j}$. Therefore, $i \in \{1, ..., t\}$ and $k_i = k_j$.

It is not difficult to prove that

$$\text{g.c.d.}\{p_{k_1} \cdots p_{k_r},\, z - a_i\} = p_{k_i}, \quad i \in \{1, .., t\} , \tag{4}$$

$$\text{g.c.d.}\{p_{k_1} \cdots p_{k_r},\, z - a_i\} = 1, \quad i \in \{t+1, ..., n-1\} . \tag{5}$$

Using Lemma 13, it follows from (4), (5) and III) that there exists an integer γ such that

$$\text{g.c.d.}\left\{\gamma\,\frac{p_{k_1} \cdots p_{k_r}}{p_{k_i}} + \frac{z - a_i}{p_{k_i}},\, b\right\} = 1, \quad i \in \{1, ..., t\} , \tag{6}$$

$$\text{g.c.d.}\left\{\gamma\, p_{k_1} \cdots p_{k_r} + z - a_i,\, b\right\} = 1, \quad i \in \{t+1, .., n-1\} , \tag{7}$$

where $b = p_1 \cdots p_d$. Let $a_n = \gamma\, p_{k_1} \cdots p_{k_r} + z + wb$, where w is an integer chosen so that $0 \leq a_n < b$. Then (6) and (7) take the forms

$$\text{g.c.d.}\{a_n - a_i,\, b\} = p_{k_i}, \quad i \in \{1, ..., t\} ,$$

$$\text{g.c.d.}\{a_n - a_i, b\} = 1, \quad i \in \{t+1, ..., n-1\} .$$

Clearly a_n is different from a_i, $i \in \{1, ..., n-1\}$, and conditions i)–iv) are satisfied. ∎

Now Theorem 4 follows immediately from Lemma 14.

Proof of Theorem 7: Let $R = \{a_1, ..., a_n\}$ be a representation of G modulo $b = p_1 \cdots p_d$, where $p_1, ..., p_d$ are primes. Suppose that $a_1, ..., a_n$ are ordered so that g.c.d.$\{a_i - a_j, b\} = 1$ if and only if $(v_i, v_j) \in E$. For each $(v_i, v_j) \in E'$, let $\phi(v_i, v_j)$ be an element of $\{p_1, ..., p_d\}$ such that $\phi(v_i, v_j)$ divides $a_i - a_j$. It is easy to see that $\phi\colon E' \to \{p_1, ..., p_d\}$ is transitive. ∎

Proof of Theorem 9: a) For each $i \in \{1, ..., n\}$, we denote by $E_i(G')$ the set of all the edges incident with v_i in G'. Given a transitive function $\phi\colon E' \to X$, the restriction of ϕ to $E_i(G')$ is one-to-one. Therefore, $\#\phi(E') \geq \#E_i(G')$. Consequently, $d_t(G') \geq \max\{\#E_i(G')\}$.

Now we prove that $d_t(G') \leq \max\{\#E_i(G')\}$ by induction on $\#E'$. If E' is empty, this is trivial. Suppose that $\#E' \geq 1$. Then there exists $i \in \{1, ..., n\}$ such that $\#E_i(G') = 1$. Without loss of generality, assume that $E_n(G') = \{(v_{n-1}, v_n)\}$. Let $\widetilde{G} = (V, \widetilde{E})$, where $\widetilde{E} = E' \backslash \{(v_{n-1}, v_n)\}$. By the induction assumption, $d_t(\widetilde{G}) \leq \max\{\#E_i(\widetilde{G})\}$. Let $\psi\colon \widetilde{E} \to X$ be a transitive function of degree $d_t(\widetilde{G})$. If there exists $i \in \{1, ..., n-2\}$ such that $\#E_{n-1}(\widetilde{G}) < \#E_i(\widetilde{G})$ $(= \#E_i(G'))$, let x be an element of $\psi(E_i(\widetilde{G})) \backslash \psi(E_{n-1}(\widetilde{G}))$. If $\#E_i(\widetilde{G}) \leq \#E_{n-1}(\widetilde{G})$, $i \in \{1, ..., n-2\}$, let x be an element that does not belong to X. Let $\phi\colon E' \to X \cup \{x\}$ be the extension function of ψ satisfying $\phi(v_{n-1}, v_n) = x$. It is easy to see that ϕ is transitive and

$$d_t(G') \leq \#\phi(E') \leq \max\{\#E_i(G')\} \ .$$

b) Since G' is acyclic, the hypothesis of b) is equivalent to $d_t(G') \geq 2$. Thus b) follows from a) and Corollary 8. ∎

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