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A NOTE ON THE SEQUENCE $(W_n)_{n\geq 0}$ OF A.F. HORADAM

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1. In 1965, A.F. Horadam [2] considered the sequence $(W_n)_{n\geq 0}$, defined by the following conditions:

$$\begin{aligned} W_n &= W_n(a,b;p,q) \\ &= p\cdot W_{n-1} - q\cdot W_{n-2}\,, \qquad W_0 = a, \quad W_1 = b\,, \end{aligned} \tag{1.1}$$

where $n\in\mathbb{N}$, $n\geq 2$, and $a,b,p,q\in\mathbf{Z}$.

For this sequence, one has (if $p^2-4q\neq 0$)

$$W_n = \frac{A\cdot\alpha^n - B\cdot\beta^n}{\alpha-\beta}\,, \tag{1.2}$$

where $A = b-a\cdot\beta$, $B = b-a\cdot\alpha$, α and β being the roots of the associated characteristic equation of the sequence $(W_n)_{n\geq 0}$:

$$\lambda^2 - p\cdot\lambda + q = 0 \tag{1.3}$$

(see [5], p. 138).

The sequence $(W_n)_{n\geq 0}$ generalizes the Fibonacci sequence $(F_n)_{n\geq 0}$, since

$$\begin{aligned} F_n = W_n(0,1;1,-1) &= \frac{\alpha^n-\beta^n}{\alpha-\beta} \\ &= \frac{1}{\sqrt{5}}\cdot\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \qquad n\in\mathbb{N}\,. \end{aligned} \tag{1.4}$$

The sequence $(W_n)_{n\geq 0}$ generalizes, also, other important sequences, for instance:

a) the fundamental Lucas sequence $(H_n)_{n\geq 0}$, where

$$\begin{aligned} H_n = W_n(0,1;p,q) &= \frac{\alpha^n-\beta^n}{\alpha-\beta} = \\ &= \frac{1}{\sqrt{p^2-4q}}\cdot\left[\left(\frac{p+\sqrt{p^2-4q}}{2}\right)^n - \left(\frac{p-\sqrt{p^2-4q}}{2}\right)^n\right]\,; \end{aligned} \tag{1.5}$$

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b) the Lucas sequence $(L_n)_{n\geq 0}$, where

$$(1.6)\qquad \begin{aligned} L_n = W_n(2,1;1,-1) &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\ &+ \frac{2}{\sqrt{5}}\cdot\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]; \end{aligned}$$

c) the Pell sequence $(P_n)_{n\geq 0}$, where

$$(1.7)\qquad P_n = W_n(0,1;2,-1) = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\sqrt{2}}{4}\cdot\left[\left(\frac{1+\sqrt{2}}{2}\right)^n - \left(\frac{1-\sqrt{2}}{2}\right)^n\right];$$

d) the Tagiuri sequence $(\mathcal{T}_n)_{n\geq 0}$, where

$$(1.8)\qquad \begin{aligned} \mathcal{T}_n = W_n(a,b;1,-1) &= \frac{1}{\sqrt{5}}\cdot\left\{b\cdot\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]\right. \\ &\left. - a\cdot\left[\frac{1-\sqrt{5}}{2}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1+\sqrt{5}}{2}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n\right]\right\}, \qquad n \in \mathbb{N}. \end{aligned}$$

Also, we have

$$(1.9)\qquad E_W \overset{\text{def.}}{=} -A\cdot B = p\cdot a\cdot b - q\cdot a^2 - b^2 .$$

2. If we put $p = 2x$, $q = 1$, $a = 1$, $b = x$ or $b = 2x$, $x \in \mathbb{C}$, in (1.1), we obtain the sequences of Chebyshev polynomials of the first and second kind, respectively:

$$(2.1)\qquad T_n(x) = W_n(1,x;2x,1), \qquad n \geq 0, \quad x \in \mathbb{C},$$

$$(2.2)\qquad U_n(x) = W_n(1,2x;2x,1), \qquad n \geq 0, \quad x \in \mathbb{C}.$$

Also, we obtain

$$(2.1')\qquad E_T = x^2 - 1, \qquad x \in \mathbb{C},$$

respectively

$$(2.2')\qquad E_U = -1 .$$

Some important properties of the polynomials (2.1) and (2.2) are given by the formulas

$$(2.3)\qquad T_n(\cos\varphi) = \cos n\varphi, \qquad n \in \mathbb{N}, \quad \varphi \in \mathbb{C},$$

and

$$U_{n-1}(\cos\varphi) = \frac{\sin n\varphi}{\sin\varphi}, \quad n \in \mathbb{N}^*, \quad \varphi \in \mathbb{C}, \quad \sin\varphi \neq 0 \ . \tag{2.4}$$

From (2.1) and (2.2) it follows that the sequence of Chebyshev polynomials of the first kind, $(T_n)_{n\geq 0}$, and the sequence of Chebyshev polynomials of the second kind, $(U_n)_{n\geq 0}$, have the same properties as the sequence $(W_n)_{n\geq 0}$ (see [3], [4], [5], [8], [10]).

3. Conversely, we have:

A) $$\alpha,\beta = \frac{p\pm\sqrt{p^2-4q}}{2} = \sqrt{q}\cdot\left(\frac{p}{2\cdot\sqrt{q}}\pm\sqrt{\left(\frac{p}{2\cdot\sqrt{q}}\right)^2-1}\right) = \sqrt{q}\cdot(\cos\varphi\pm i\cdot\sin\varphi)\ ,$$

where $\cos\varphi = \frac{p}{2\cdot\sqrt{q}}$, $\varphi \in \mathbb{C}$;

$$\begin{aligned} H_n = W_n(0,1;p,q) &= \frac{\alpha^n-\beta^n}{\alpha-\beta} = (\sqrt{q})^{n-1}\cdot\frac{\sin n\varphi}{\sin\varphi} = \\ &= q^{\frac{n-1}{2}}\cdot U_{n-1}(\cos\varphi) = q^{\frac{n-1}{2}}\cdot U_{n-1}\left(\frac{p}{2\cdot\sqrt{q}}\right)\ . \end{aligned}$$

Hence

$$H_n = q^{\frac{n-1}{2}}\cdot U_{n-1}\left(\frac{p}{2\cdot\sqrt{q}}\right), \quad n \in \mathbb{N}^*\ . \tag{3.1}$$

B) $$W_n = \frac{1}{\alpha-\beta}\cdot(A\cdot\alpha^n - B\cdot\beta^n) =$$

$$\begin{aligned} &= \frac{1}{\alpha-\beta}\cdot\Big[A\cdot(\sqrt{q})^n\cdot(\cos n\varphi + i\cdot\sin n\varphi) - B\cdot(\sqrt{q})^n\cdot(\cos n\varphi - i\cdot\sin n\varphi)\Big] = \\ &= \frac{(\sqrt{q})^n}{\alpha-\beta}\cdot\Big[(A-B)\cdot\cos n\varphi + i\cdot(A+B)\cdot\sin n\varphi\Big] \\ &= q^{\frac{n}{2}}\cdot\left[\frac{A-B}{\alpha-\beta}\cdot\cos n\varphi + i\cdot\frac{A+B}{\alpha-\beta}\cdot\sin n\varphi\right] \\ &= q^{\frac{n}{2}}\cdot\left[a\cdot\cos n\varphi + \frac{2\cdot b - a\cdot p}{2\sqrt{q}\cdot\sin\varphi}\cdot\sin n\varphi\right] \\ &= q^{\frac{n}{2}}\cdot\left[a\cdot T_n(\cos\varphi) + \frac{2\cdot b - a\cdot p}{2\cdot\sqrt{q}}\cdot U_{n-1}(\cos\varphi)\right] \\ &= q^{\frac{n}{2}}\cdot\left[a\cdot T_n\left(\frac{p}{2\cdot\sqrt{q}}\right) + \frac{2\cdot b - a\cdot p}{2\cdot\sqrt{q}}\cdot U_{n-1}\left(\frac{p}{2\cdot\sqrt{q}}\right)\right]\ . \end{aligned}$$

So that, we obtain

$$(3.2)\quad W_n = q^{\frac{n}{2}} \cdot \left[a \cdot T_n\left(\frac{p}{2\cdot\sqrt{q}}\right) + \frac{2\cdot b - a\cdot p}{2\cdot\sqrt{q}} \cdot U_{n-1}\left(\frac{p}{2\cdot\sqrt{q}}\right)\right], \quad n \in \mathbb{N}^* .$$

Also, we get

$$(3.3)\qquad W_n = q^{\frac{n}{2}} \cdot \left[\frac{b}{\sqrt{q}} \cdot U_{n-1}\left(\frac{p}{2\cdot\sqrt{q}}\right) - a \cdot U_{n-2}\left(\frac{p}{2\cdot\sqrt{q}}\right)\right], \quad n \in \mathbb{N} ,$$

$n \geq 2$, since $T_m(x) = x \cdot U_{m-1}(x) - U_{m-2}(x)$, $(\forall)\, x \in \mathbb{C}$, $(\forall)\, m \in \mathbb{N}$, $m \geq 2$.
On the other hand,

$$\begin{aligned}
W_n &= \frac{(\sqrt{q})^n}{\alpha-\beta} \cdot \Big[(A-B)\cdot\cos n\varphi + i\cdot(A+B)\cdot\sin n\varphi\Big] \\
&= \frac{q^{\frac{n}{2}}}{q^{\frac{1}{2}}\cdot 2\cdot i\cdot\sin\varphi} \cdot \left[\frac{A-B}{\sqrt{(A-B)^2+(i\cdot(A+B))^2}}\cdot\cos n\varphi\right. \\
&\quad \left. + \frac{i\cdot(A+B)}{\sqrt{(A-B)^2+(i\cdot(A+B))^2}}\cdot\sin n\varphi\right] \cdot \sqrt{(A-B)^2+(i\cdot(A+B))^2} \\
&= \frac{2\cdot\sqrt{E}\cdot q^{\frac{n}{2}}}{\sqrt{p^2-4q}} \cdot \left[\frac{A-B}{2\sqrt{E}}\cdot\cos n\varphi + i\cdot\frac{A+B}{2\sqrt{E}}\cdot\sin n\varphi\right] \\
&= \frac{2\sqrt{E}\cdot q^{\frac{n}{2}}}{\sqrt{p^2-4q}} \cdot \Big[\cos\phi\cdot\cos n\varphi - \sin\phi\cdot\sin n\varphi\Big] \\
&= \frac{2\sqrt{E}\cdot q^{\frac{n}{2}}}{\sqrt{p^2-4q}} \cdot \cos(n\varphi+\phi) = \frac{2\sqrt{E}\cdot q^{\frac{n}{2}}}{\sqrt{p^2-4q}} \cdot T_n\left(\cos\left(\varphi+\frac{\phi}{n}\right)\right) ,
\end{aligned}$$

where $\cos\varphi = \frac{p}{2\sqrt{q}}$, $\varphi \in \mathbb{C}$, and $\cos\phi = \frac{A-B}{2\sqrt{E}}$, $\phi \in \mathbb{C}$ (see [6], [9]).
Consequently,

$$(3.4)\qquad W_n(a,b;p,q) = \frac{2\sqrt{E}\cdot q^{\frac{n}{2}}}{\sqrt{p^2-4q}} \cdot \cos(n\varphi+\phi)\,, \quad n \in \mathbb{N} ,$$

where $\cos\varphi = \frac{p}{2\sqrt{q}}$, $\varphi \in \mathbb{C}$, and $\cos\phi = \frac{a\cdot\sqrt{p^2-4q}}{2\sqrt{E}}$, $\phi \in \mathbb{C}$.

Remarks I. We have:

a) $F_n = W_n(0,1;1,-1) \overset{(3.3)}{=} i^n \cdot \frac{1}{i} \cdot U_{n-1}\left(\frac{1}{2i}\right) = i^{n-1} \cdot U_{n-1}\left(-\frac{i}{2}\right) =$

$$= i^{n-1}\cdot(-1)^{n-1}\cdot U_{n-1}\left(\frac{i}{2}\right) = i^{3(n-1)}\cdot U_{n-1}\left(\frac{i}{2}\right), \quad n \in \mathbb{N}^* .$$

Hence,

$$F_{n+1} = i^{3n} \cdot U_n\left(\frac{i}{2}\right), \quad n \in \mathbb{N} . \tag{3.5}$$

b) $L_n = W_n(2,1;1,-1) \overset{(3.2)}{=} 2 \cdot i^n \cdot T_n\left(-\frac{i}{2}\right) =$

$$= 2 \cdot i^n \cdot (-1)^n \cdot T_n\left(\frac{i}{2}\right) = 2 \cdot i^{3n} \cdot T_n\left(\frac{i}{2}\right), \quad n \in \mathbb{N} .$$

So that, we have

$$L_n = i^{3n} \cdot 2 \cdot T_n\left(\frac{i}{2}\right), \quad n \in \mathbb{N} . \tag{3.6}$$

c) $P_n = W_n(0,1;2,-1) = i^{n-1} \cdot U_{n-1}\left(\frac{1}{i}\right) = i^{n-1} \cdot U_{n-1}(-i) =$

$$= i^{n-1} \cdot (-1)^{n-1} \cdot U_{n-1}(i) = i^{3(n-1)} \cdot U_{n-1}(i), \quad n \in \mathbb{N}^* .$$

Hence,

$$P_{n+1} = i^{3n} \cdot U_n(i), \quad n \in \mathbb{N} . \tag{3.7}$$

d) $\mathcal{T}_n = W_n(a,b;1,-1) \overset{(3.2)}{=} i^n \cdot \left[a \cdot T_n\left(\frac{1}{2i}\right) + \frac{2b-a}{2i} \cdot U_{n-1}\left(\frac{1}{2i}\right)\right]$

$$= i^n \cdot \left[a \cdot T_n\left(-\frac{i}{2}\right) + \frac{2b-a}{2i} \cdot U_{n-1}\left(-\frac{i}{2}\right)\right]$$

$$= i^n \cdot \left[a \cdot (-1)^n \cdot T_n\left(\frac{i}{2}\right) + \frac{2b-a}{2i} \cdot (-1)^{n-1} \cdot U_{n-1}\left(\frac{i}{2}\right)\right]$$

$$= i^n \cdot \left[a \cdot (-1)^n \cdot \frac{1}{2} \cdot i^n \cdot L_n + \frac{2b-a}{2i} \cdot (-1)^{n-1} \cdot i^{n-1} \cdot F_n\right] =$$

$$= \frac{a}{2} \cdot L_n + \frac{2b-a}{2} \cdot F_n = a \cdot \frac{L_n - F_n}{2} + b \cdot F_n = a \cdot F_{n-1} + b \cdot F_n, \quad n \in \mathbb{N}^* ,$$

since $L_n - F_n = 2F_{n-1}$, $(\forall)\, n \in \mathbb{N}^*$.

Remarks II. We observe that the identity

$$W_n \cdot W_{n+r+s} - W_{n+r} \cdot W_{n+s} = E_W \cdot q^n \cdot H_r \cdot H_s \tag{$*$}$$

(see [4]), is a consequence of the formula (3.4).

Indeed, we have:

i) $$W_n \cdot W_{n+r+s} = \frac{2\sqrt{E}}{\sqrt{p^2-4q}} \cdot q^{\frac{n}{2}} \cdot \cos(n\varphi + \phi) \cdot \frac{2\sqrt{E}}{\sqrt{p^2-4q}} \cdot q^{\frac{n+r+s}{2}} \cdot \cos\Big((n+r+s)\,\varphi + \phi\Big) = \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot \cos(n\varphi+\phi) \cdot \cos\Big((n+r+s)\,\varphi+\phi\Big)\ ;$$

ii) $$W_{n+r} \cdot W_{n+s} = \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot \cos\Big((n+r)\,\varphi+\phi\Big) \cdot \cos\Big((n+s)\,\varphi+\phi\Big), \quad n, r, s \in \mathbb{N}\ .$$

From i) and ii) we obtain

$$\begin{aligned}
&W_n \cdot W_{n+r+s} - W_{n+r} \cdot W_{n+s} = \\
&= \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot \Big[\cos(n\varphi+\phi) \cdot \cos\Big((n+r+s)\,\varphi+\phi\Big) \\
&\quad - \cos\Big((n+r)\,\varphi+\phi\Big) \cdot \cos\Big((n+s)\,\varphi+\phi\Big)\Big] \\
&= \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot \left[\frac{\cos\Big((2n+r+s)\cdot\varphi+2\phi\Big) + \cos(r+s)\,\varphi}{2}\right. \\
&\quad \left. - \frac{\cos\Big((2n+r+s)\,\varphi+2\phi\Big) + \cos(r-s)\,\varphi}{2}\right] \\
&= \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot \frac{\cos(r+s)\,\varphi - \cos(r-s)\,\varphi}{2} \\
&= \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot (-1) \cdot \sin r\varphi \cdot \sin s\varphi \\
&= (-\sin^2\varphi) \cdot \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot \frac{\sin r\varphi}{\sin\varphi} \cdot \frac{\sin s\varphi}{\sin\varphi} \\
&= (\cos^2\varphi - 1) \cdot \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot U_{r-1}(\cos\varphi) \cdot U_{s-1}(\cos\varphi) \\
&= \left[\left(\frac{p}{2\sqrt{q}}\right)^2 - 1\right] \cdot \frac{4E}{p^2-4q} \cdot q^{\frac{2n+r+s}{2}} \cdot U_{r-1}\left(\frac{p}{2\sqrt{q}}\right) \cdot U_{s-1}\left(\frac{p}{2\sqrt{q}}\right) \\
&= E \cdot q^{\frac{2n+r+s-2}{2}} \cdot U_{r-1}\left(\frac{p}{2\sqrt{q}}\right) \cdot U_{s-1}\left(\frac{p}{2\sqrt{q}}\right)\ ,
\end{aligned}$$

i.e.,

$$(\alpha)\quad W_n \cdot W_{n+r+s} - W_{n+r} \cdot W_{n+s} = \\ = E \cdot q^{\frac{2n+r+s-2}{2}} \cdot U_{r-1}\left(\frac{p}{2\sqrt{q}}\right) \cdot U_{s-1}\left(\frac{p}{2\sqrt{q}}\right), \quad n, r, s \in \mathbb{N}^* .$$

On the other hand, we have

$$E \cdot q^n \cdot H_r \cdot H_s = E \cdot q^n \cdot q^{\frac{r-1}{2}} \cdot U_{r-1}\left(\frac{p}{2\sqrt{q}}\right) \cdot q^{\frac{s-1}{2}} \cdot U_{s-1}\left(\frac{p}{2\sqrt{q}}\right) \\ = E \cdot q^{\frac{2n+r+s-2}{2}} \cdot U_{r-1}\left(\frac{p}{2\sqrt{q}}\right) \cdot U_{s-1}\left(\frac{p}{2\sqrt{q}}\right) ,$$

i.e.,

$$(\beta)\quad E\cdot q^n \cdot H_r \cdot H_s = E\cdot q^{\frac{2n+r+s-2}{2}} \cdot U_{r-1}\left(\frac{p}{2\sqrt{q}}\right) \cdot U_{s-1}\left(\frac{p}{2\sqrt{q}}\right), \quad n, r, s \in \mathbb{N}^* .$$

From (α) and (β) we obtain the identity $(*)$, q.e.d..

4. It is well-known that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$, then

$$(4.1)\qquad A^n = \begin{cases} \dfrac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} - \delta_A \cdot \dfrac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} \cdot I_2, & \lambda_1 \neq \lambda_2 , \\ n \cdot \lambda^{n-1} \cdot A - (n-1) \cdot \delta_A \cdot \lambda^{n-2} \cdot I_2, & \lambda_1 = \lambda_2 = \lambda , \end{cases}$$

where $\delta_A = \det A = ad - bc$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, λ_1 and λ_2 being the roots of the associated characteristic equation of the matrix A:

$$(4.1')\qquad \lambda^2 - (a+d) \cdot \lambda + \delta_A = 0 .$$

On the other hand, we have

$$\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = W_n(0, 1; a+d, \delta_A) = H_n = \delta_A^{\frac{n-1}{2}} \cdot U_{n-1}\left(\frac{a+d}{2\sqrt{\delta_A}}\right), \quad \delta_A \neq 0 .$$

Hence,

$$(4.2)\qquad \begin{aligned} A^n &= \delta_A^{\frac{n-1}{2}} \cdot \left[U_{n-1}\left(\frac{a+d}{2\sqrt{\delta_A}}\right) \cdot A - \delta_A^{\frac{1}{2}} \cdot U_{n-2}\left(\frac{a+d}{2\sqrt{\delta_A}}\right) \cdot I_2\right] \\ &= \delta^{\frac{n-1}{2}} \cdot \left[U_{n-1}\left(\frac{a+d}{2\sqrt{\delta_A}}\right) \cdot \left(A - \frac{a+d}{2} \cdot I_2\right) + \delta_A^{\frac{1}{2}} \cdot T_n\left(\frac{a+d}{2\sqrt{\delta_A}}\right) \cdot I_2\right] , \end{aligned}$$

if $\lambda_1 \neq \lambda_2$ and $\delta_A \neq 0$.

For $\delta_A = 1$ we obtain, from (4.2),

$$A^n = U_{n-1}\left(\frac{a+d}{2}\right)\cdot \mathbf{A} - U_{n-2}\left(\frac{a+d}{2}\right)\cdot I_2$$
$$(4.3) \qquad = U_{n-1}\left(\frac{a+d}{2}\right)\cdot\left(A - \frac{a+d}{2}\cdot I_2\right) + T_n\left(\frac{a+d}{2}\right)\cdot I_2 \ .$$

Also, we have, for $\delta_A = -1$,

$$(4.4) \qquad A^n = i^{3(n-1)}\cdot\left[U_{n-1}\left(i\cdot\frac{a+d}{2}\right)\cdot\mathbf{A} + i\cdot U_{n-2}\left(\frac{a+d}{2}\cdot i\right)\cdot I_2\right] \ ,$$

where $i^2 = -1$, $n \in \mathbb{N}^*$.

From (4.4) we deduce, for $a + d = 1$,

$$(4.5) \qquad \mathbf{A}^n = F_n \cdot A + F_{n-1}\cdot I_2\,, \qquad n \in \mathbb{N}^* \ ,$$

where $(F_n)_{n\geq 0}$ is the Fibonacci sequence.

Remarks III.

a) For $A = \left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$ we have $\delta_A = -1$ and $a + d = 1$. Hence

$$A^n = F_n\cdot A + F_{n-1}\cdot I_2 = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1}\end{pmatrix} \ ,$$

i.e.,

$$(4.6) \qquad \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1}\end{pmatrix}, \qquad (\forall)\, x \in \mathbb{N}^* \ ,$$

(see [1]).

From (4.6) we deduce

$$F_{n+1}\cdot F_{n-1} - F_n^2 = \det\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = \left(\det\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\right)^n = (-1)^n \ ,$$

i.e.,

$$(4.7) \qquad F_{n+1}\cdot F_{n-1} - F_n^2 = (-1)^n\,, \qquad n \in \mathbb{N}^* \ .$$

The identity (4.7) is in fact the Catalan's Identity for $m = n$ and $r = 1$ (see (4.1)).

b) For $A = \left(\begin{smallmatrix}3 & 1\\ -1 & 0\end{smallmatrix}\right)$ we have $a + d = 3$ and $\delta_A = 1$.

Hence,

$$\begin{aligned}
A^n &= U_{n-1}\left(\frac{3}{2}\right)\cdot A - U_{n-2}\left(\frac{3}{2}\right)\cdot I_2\\
&= U_{n-1}\left(-\left(-\frac{3}{2}\right)\right)\cdot A - U_{n-2}\left(-\left(-\frac{3}{2}\right)\right)\cdot I_2\\
&= (-1)^{n-1}\cdot\left[U_{n-1}\left(-\frac{3}{2}\right)\cdot A + U_{n-2}\left(-\frac{3}{2}\right)\cdot I_2\right]\\
&= (-1)^{n-1}\cdot\left[U_{n-1}\left(T_2\left(\frac{i}{2}\right)\right)\cdot A + U_{n-2}\left(T_2\left(\frac{i}{2}\right)\right)\cdot I_2\right]\\
&= (-1)^{n-1}\cdot\left[\frac{U_{2n-1}\left(\frac{i}{2}\right)}{U_{2-1}\left(\frac{i}{2}\right)}\cdot A + \frac{U_{2(n-1)-1}\left(\frac{i}{2}\right)}{U_{2-1}\left(\frac{i}{2}\right)}\cdot I_2\right]\\
&= (-1)^{n-1}\cdot\frac{1}{i}\cdot\left[U_{2n-1}\left(\frac{i}{2}\right)\cdot A + U_{2n-3}\left(\frac{i}{2}\right)\cdot I_2\right]\\
&= i^{2n-3}\cdot\left[U_{2n-1}\left(\frac{i}{2}\right)\cdot A + U_{2n-3}\left(\frac{i}{2}\right)\cdot I_2\right]\\
&= i^{2n-3}\cdot\left[i^{2n-1}\cdot F_{2n}\cdot A + i^{2n-3}\cdot F_{2(n-1)}\cdot I_2\right]\\
&= F_{2n}\cdot A - F_{2(n-1)}\cdot I_2 = \begin{pmatrix} F_{2(n+1)} & F_{2n}\\ -F_{2n} & -F_{2(n-1)}\end{pmatrix},
\end{aligned}$$

i.e.,

$$\begin{pmatrix} 3 & 1\\ -1 & 0\end{pmatrix}^n = \begin{pmatrix} F_{2(n+1)} & F_{2n}\\ -F_{2n} & -F_{2(n-1)}\end{pmatrix}, \tag{4.8}$$

where $n \in \mathbb{N}^*$ and $(F_n)_{n\geq 0}$ is the Fibonacci sequence (see [13]).

From (4.8) we get

$$\begin{aligned}
1 = \left(\det\begin{pmatrix} 3 & 1\\ -1 & 0\end{pmatrix}\right)^n &= \det\begin{pmatrix} 3 & 1\\ -1 & 0\end{pmatrix}^n =\\
&= \det\begin{pmatrix} F_{2(n+1)} & F_{2n}\\ -F_{2n} & -F_{2(n-1)}\end{pmatrix} = F_{2n}^2 - F_{2(n-1)}\cdot F_{2(n+1)},
\end{aligned}$$

i.e.,

$$F_{2n}^2 - F_{2(n-1)}\cdot F_{2(n+1)} = 1, \quad (\forall)\, n \in \mathbb{N}^*. \tag{4.9}$$

Also, the identity (4.9) is in fact the Catalan's Identity for $m = 2n$ and $r = 2$ (see (4.11)).

Remark IV.

a) Clearly, in (4.8) we utilized the identity

$$U_{n-1}(T_k(x)) = \frac{U_{nk-1}(x)}{U_{k-1}(x)}\,, \quad (\forall)\, n,k \in \mathbb{N}^*, \quad x \in \mathbb{C}\ ; \tag{4.10}$$

b) The Catalan's Identity for Fibonacci numbers is

$$F_{m-r} \cdot F_{m+r} + (-1)^{m-r} \cdot F_r^2 = F_m^2\ , \tag{4.11}$$

where $m, r \in \mathbb{N}$, $m \geq 2$, $m > r$ (see [7]).

5. In [11] and [12] I established the following identities for the Chebyshev polynomials $(T_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$, respectively:

$$\begin{aligned}
&T_n \cdot T_{n+2r} - (x^2-1) \cdot U_{r-1}^2 = T_{n+r}^2\ ,\\
&T_{n+2r} \cdot T_{n+4r} - (x^2-1) \cdot U_{r-1}^2 = T_{n+3r}^2\ ,\\
&T_n \cdot T_{n+4r} - (x^2-1) \cdot U_{2r-1}^2 = T_{n+2r}^2\ ,\\
&4 \cdot T_n \cdot T_{n+r} \cdot T_{n+2r} \cdot T_{n+3r} + (x^2-1)^2 \cdot U_{r-1}^2 \cdot U_{2r-1}^2 =\\
&\qquad = (T_n \cdot T_{n+3r} + T_{n+r} \cdot T_{n+2r})^2\ ,\\
&4 \cdot T_{n+r} \cdot T_{n+2r} \cdot T_{n+3r} \cdot T_{n+4r} + (x^2-1)^2 \cdot U_{r-1}^2 \cdot U_{2r-1}^2 =\\
&\qquad = (T_{n+2r} \cdot T_{n+3r} + T_{n+r} \cdot T_{n+4r})^2\ ,\\
&4 \cdot T_{n+r} \cdot T_{n+2r}^2 \cdot T_{n+3r} + (x^2-1)^2 \cdot U_{r-1}^4 =\\
&\qquad = (T_{n+r} \cdot T_{n+3r} + T_{n+2r}^2)^2\,, \quad n, r \in \mathbb{N}^*\ ;
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&U_m \cdot U_{m+2r} + U_{r-1}^2 = U_{m+r}^2\ ,\\
&U_{m+2r} \cdot U_{m+4r} + U_{r-1}^2 = U_{m+3r}^2\ ,\\
&U_m \cdot U_{m+4r} + U_{2r-1}^2 = U_{m+2r}^2\ ,\\
&4 \cdot U_m \cdot U_{m+r} \cdot U_{m+2r} \cdot U_{m+3r} + U_{r-1}^2 \cdot U_{2r-1}^2 =\\
&\qquad = (U_m \cdot U_{m+3r} + U_{m+r} \cdot U_{m+2r})^2\ ,\\
&4 \cdot U_{m+r} \cdot U_{m+2r} \cdot U_{m+3r} \cdot U_{m+4r} + U_{r-1}^2 \cdot U_{2r-1}^2 =\\
&\qquad = (U_{m+2r} \cdot U_{m+3r} + U_{m+r} \cdot U_{m+4r})^2\ ,\\
&4 \cdot U_{m+r} \cdot U_{m+2r}^2 \cdot U_{m+3r} + U_{r-1}^4 =\\
&\qquad = (U_{m+r} \cdot U_{m+3r} + U_{m+2r}^2)^2\,, \quad m, r \in \mathbb{N}^*\ .
\end{aligned} \tag{5.2}$$

Now, we define $S_k = T_k(A)$ and $R_k = U_k(A)$, $A \in \mathcal{M}_p(\mathbb{C})$, $p, k \in \mathbb{N}$, $p \geq 2$ ($M_p(\mathbb{C})$ is the set of square matrix of order p with complex elements).

From (5.1) and (5.2) we obtain the following theorems:

Theorem (S). *For the sequence $(S_n)_{n\geq 0}$ the product of any two distinct elements of the set*

$$\left\{S_n, S_{n+2r}, S_{n+4r};\ 4 \cdot S_{n+r} \cdot S_{n+2r} \cdot S_{n+3r}\right\}$$

increased by $(-1)^t \cdot (A^2 - I_2)^t \cdot R_{h-1}^2 \cdot R_{k-1}^{2(t-1)}$, where $t = 1$ or 2 depending on whether 2 or 4 factors occur in the product and R_{h-1}, R_{k-1} are suitable elements of the sequences $(R_n)_{n\geq 0}$, is a "perfect square" (in $\mathcal{M}_p(\mathbb{C})$), $(\forall)\, n, r \in \mathbb{N}$, $(\forall)\, \mathbf{A} \in \mathcal{M}_p(\mathbb{C})$, $p \in \mathbb{N}$, $p \geq 2$.

Theorem (R). *For the sequence $(R_n)_{n\geq 0}$ the product of any two distinct elements of the set*

$$\left\{R_n, R_{n+2r}, R_{n+4r};\ 4 \cdot R_{n+r} \cdot R_{n+2r} \cdot R_{n+3r}\right\}$$

increased by $R_{h-1}^2 \cdot R_{k-1}^{2(t-1)}$, where $t = 1$ or 2 depending on whether 2 or 4 factors occur in the product and R_{h-1}, R_{k-1} are suitable elements of the sequence $(R_n)_{n\geq 0}$, is a "perfect square" (in $\mathcal{M}_p(\mathbb{C})$), $(\forall)\, n, r \in \mathbb{N}$, $(\forall)\, A \in \mathcal{M}_p(\mathbb{C})$, $p \in \mathbb{N}$, $p \geq 2$.

Remarks V.

a) The analogues of the Catalan's Identity for the sequences $(S_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ can be considered as the identities

$$S_{n-r} \cdot S_{n+r} - (A^2 - I_2) \cdot R_{r-1}^2 = S_n^2\ , \tag{5.3}$$

respectively

$$R_{n-r} \cdot R_{n+r} + R_{r-1}^2 = R_n^2\ , \tag{5.4}$$

where $m, r \in \mathbb{N}^*$, $m > r$.

b) Since for every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$ we have $A^2 = (a+d) \cdot A - \delta_A \cdot I_2$, it results

$$A^{n+2} = (a+d) \cdot A^{n+1} - \delta_A \cdot A^n\ , \qquad (\forall)\, n \in \mathbb{N}^*\ . \tag{5.5}$$

If we denote $\widetilde{W}_n = A^n$ we obtain the sequence $(\widetilde{W}_n)_{n\geq 0}$:

$$\left\{\begin{array}{l} \widetilde{W}_{n+2} = (a+d)\cdot\widetilde{W}_{n+1} - \delta_A\cdot\widetilde{W}_n\,, \qquad n\in\mathbb{N}\ , \\ \widetilde{W}_0 = I_2\,, \qquad \widetilde{W}_1 = A\ . \end{array}\right. \tag{5.6}$$

For this sequence we have (see [4])

$$E_{\widetilde{W}} = (a+b)\cdot I_2\cdot\mathbf{A} - \delta_A\cdot I_2^2 - A^2 = (a+b)\cdot A - \delta_A\cdot I_2 - A^2 = O_2\ ,$$

i.e.,

$$E_{\widetilde{W}} = O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\ . \tag{5.7}$$

Consequently,

Theorem (O). *The set*

$$\left\{\widetilde{W}_n, \widetilde{W}_{n+2r}, \widetilde{W}_{n+4r};\ 4\cdot\widetilde{W}_{n+r}\cdot\widetilde{W}_{n+2r}\cdot\widetilde{W}_{n+3r}\right\}\ ,$$

$n,r\in\mathbb{N}$, *is a (trivial) solution, in* $\mathcal{M}_2(\mathbb{C})$, *of the Problem of Diophantos–Fermat (see [4], [10]).*

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