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# SOME RESULTS ON THE SPECTRAL ANALYSIS OF NONSTATIONARY TIME SERIES

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**Abstract:** We present some results regarding the periodogram analysis of nonstationary time series, allowing for the extension of spectral regression methods to cases in which the degree of integration $d$ of a process is not in the stationary range.

## 1 – Introduction

Periodogram analysis has been a standard tool in stationary time series analysis. More recently, with the interest in long-memory fractionally differenced models, periodogram regression methods have been suggested to estimate the degree of integration of a stationary time series.

Let $(\varepsilon_t)$ be a *white noise*, i.e., an uncorrelated zero-mean process: $E\,\varepsilon_t = 0$ and $E\,\varepsilon_t^2 = \sigma_\varepsilon^2$, for all $t$ and $E\,\varepsilon_t\,\varepsilon_{t+h} = 0$ for all $h \neq 0$. Let $B$ represent the backwards shift operator, i.e., $BX_t = X_t$, and let $\nabla = 1 - B$ represent the differencing operator. For $d \in (-.5, .5)$ the process $(X_t)$ is said to be a *fractional noise* if

$$\nabla^d X_t = \varepsilon_t \ , \tag{1}$$

where the operator $\nabla^d$ can be defined trough the binomial expansion of $(1-B)^d$. In this case, the process $(X_t)$ has the spectral density

$$f(\lambda)_X = |1 - e^{-i\lambda}|^{-2d} f_\varepsilon(\lambda) \ , \tag{2}$$

where $f_\varepsilon(\lambda) = \sigma_\varepsilon^2/(2\pi)$ is the spectral density of the noise. See Brockwell and Davis ([1], section 13.2) for details.

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Noting that the behavior of $f_X(\lambda)$ near zero is determined by the value of $d$, Geweke and Porter-Hudak ([4]), among others, have suggested a regression over the finite sample counterpart of the spectrum, the periodogram, in order to estimate the degree of integration $d$ of the process. A variant of this procedure has recently been rigorously developed by Robinson ([6]).

If the process has autoregressive and moving average components, then the spectral regression procedure has nonnegligible biases. Nevertheless, other estimation procedures also have significant drawbacks, and simulation results in Cheung and Diebold ([3]) show that the spectral procedures have performances that are competitive and present significant computational advantages in large samples.

In practical situations, the use of the spectral estimator procedure to estimate the degree of integration $d$ of a time series can yield a nonstationary value. In this situation, we are immediately faced with a major obstacle: the spectrum of a non-stationary model, as the random walk, is not defined. Does an estimate $\hat{d} \approx 1$ suggest that the series is not stationary, being instead generated by an integrated process of order 1, as a random walk? This question is of practical interest, and the spectral theory for stationary processes does not provide an answer. Of course, taking differences of an integrated process would lead to a stationary process on which the existing theory can be applied. But the question at stake is: without rigorous results regarding the nonstationary case, how do we know whether a periodogram indicates that further differencing is needed?

In this paper, we present some partial results in the spectral characterization of nonstationary processes. Our approach is directly focused on the periodogram.

## 2 – Periodogram behavior of random walk type processes

The variance of a nonstationary process is not defined. We will condition on the first observation and assume throughout, without loss of generality, that $X_0 = 0$.

The spectrum of a nonstationary model is also not defined. We will always work with the periodograms. For a finite sample of size $n$, the periodogram $I_{X,n}(\omega_j)$, with Fourier frequencies $\omega_j = 2\pi j/n \in [0, \pi]$, and the finite Fourier transforms $J_X(\lambda)$, with $\lambda \in [0,\pi]$, are always well defined

$$I_{X,n}(\omega_j) := n^{-1}\Big|\sum_{t=1}^{n} X_t\, e^{-i\omega_j t}\Big|^2$$

$$(3) \qquad = \Big(n^{-1/2} \sum_{t=1}^{n} X_t\, e^{-i\omega_j t}\Big)\Big(n^{-1/2} \sum_{t=1}^{n} X_t\, e^{i\omega_j t}\Big)$$

$$:= J_X(-\lambda)\, J_X(\lambda), \qquad \text{with} \;\; \lambda = \omega_j \;.$$

We will find it convenient to extend this definition in order to work with any frequency $\lambda \in (0,\pi)$:

$$(4) \qquad I_{X,n}(\lambda) := I_{X,n}(\omega_j) \qquad \text{with} \;\; \omega_j - \pi/n < \lambda \le \omega_j + \pi/n \;.$$

Our first theorem fixes the spectral frequency and presents an asymptotic result.

**Theorem 1.** *Let $(X_t)$ be a random walk ARIMA, $\nabla X_t = \varepsilon_t$. Consider the realization sample $(X_t)_{t=0}^n$ and assume that $X_0 = 0$. Then the periodograms of $(X_t)_{t=0}^n$, say $I_{X,n}(\lambda)$, and of $(\varepsilon_t)_{t=1}^n$, say $I_{\varepsilon,n}(\lambda)$, are related through the identity*

$$(5) \qquad |1-e^{-i\lambda}|^2\, I_{X,n}(\lambda) = I_{\varepsilon,n}(\lambda) + n^{-1} X_n^2 - R_n(\lambda) \;,$$

*where, for any fixed $\lambda \in (0,\pi)$,*

$$(6) \qquad E\,|R_n(\lambda)| \to 0 \qquad \textit{as} \;\; n \to \infty \;.$$

**Proof:** Since $\varepsilon_t = X_t - X_{t-1}$, we get

$$J_\varepsilon(\lambda) := n^{-1/2} \sum_{t=1}^{n} \varepsilon_t\, e^{-i\lambda t}$$

$$= n^{-1/2}\,(1-e^{-i\lambda}) \sum_{t=1}^{n} X_t\, e^{-i\lambda t} + n^{-1/2}\,(X_n\, e^{-i\lambda(n+1)} - X_0\, e^{-i\lambda}) \;,$$

and

$$(7) \qquad (1-e^{-i\lambda})\, J_X(\lambda) = J_\varepsilon(\lambda) - n^{-1/2}\, e^{-i\lambda(n+1)}\, X_n \;.$$

Multiplying each side of (7) by its conjugate, we get

$$|1-e^{-i\lambda}|^2\, I_{X,n}(\lambda) = I_{\varepsilon,n}(\lambda) + n^{-1} X_n^2 - R_n(\lambda) \;,$$

where

$$R_n(\lambda) := J_\varepsilon(-\lambda)\, n^{-1/2}\, e^{-i\lambda(n+1)}\, X_n + J_\varepsilon(\lambda)\, n^{-1/2}\, e^{i\lambda(n+1)}\, X_n \\ := Q_n(\lambda) + Q_n(-\lambda)\ . \tag{8}$$

Since $X_n = \sum_{t=1}^n \varepsilon_t$, we get

$$\begin{aligned} E\, Q_n(\lambda) &= n^{-1/2}\, e^{-i\lambda(n+1)}\, E\Big[\Big(n^{-1/2} \sum_{t=1}^n \varepsilon_t\, e^{i\lambda t}\Big)\Big(\sum_{t=1}^n \varepsilon_t\Big)\Big] \\ &= n^{-1} \sum_{t=1}^n e^{-i\lambda t}\, \sigma_\varepsilon^2\ . \end{aligned} \tag{9}$$

After similar computations for $Q_n(-\lambda)$ we have

$$E\, R_n(\lambda) = n^{-1}\, \sigma_\varepsilon^2 \sum_{t=1}^n (e^{i\lambda t} + e^{-i\lambda t}) = n^{-1}\, \sigma_\varepsilon^2 \sum_{t=1}^n 2\cos\lambda t\ .$$

For $\lambda \in (0,\pi)$ we have the upper bound,

$$\begin{aligned} E\,|R_n(\lambda)| &= n^{-1}\, \sigma_\varepsilon^2\, 2\Big|\sum_{t=1}^n \cos\lambda t\Big| \\ &= n^{-1}\, \sigma_\varepsilon^2 \left|\frac{\sin(n+1/2)\,\lambda}{\sin(\lambda/2)} - 1\right| \\ &\le n^{-1}\, \sigma_\varepsilon^2 \left(\frac{\pi}{\lambda} + 1\right)\ . \end{aligned} \tag{10}$$

Hence, for any fixed $\lambda > 0$, $E\,|R_n(\lambda)| \to 0$ as $n \to \infty$. ∎

The next theorem presents a result for any $j$-th Fourier frequency of the periodogram. This frequency converges to zero as the number of observations $n$ increases.

**Theorem 2.** *Let $(X_t)$ be a random walk $\nabla X_t = \varepsilon_t$. Consider the realization sample $(X_t)_{t=0}^n$ and assume that $X_0 = 0$. Then the periodograms of $(X_t)_{t=0}^n$ and of $(\varepsilon_t)_{t=1}^n$ are related through the identity*

$$|1 - e^{-i\omega_j}|^2\, I_{X,n}(\omega_j) = I_{\varepsilon,n}(\omega_j) + n^{-1}\, X_n^2 - R_n(\omega_j)\ , \tag{11}$$

*where, for any $j$-th Fourier frequency $\omega_j = 2\pi j/n \in (0,\pi)$,*

$$E\, R_n(\omega_j) = 0\ . \tag{12}$$

**Proof:** If $\lambda$ is a Fourier frequency, then

$$\sum_{t=1}^{n} e^{i\lambda t} = \frac{1-e^{in\lambda}}{1-e^{i\lambda}}\, e^{i\lambda} = 0 \ . \tag{13}$$

Hence, we get directly from (8) and (9) that $E\,R_n(\omega_j) = 0 + 0 = 0$. ∎

As suggested by Künsch ([5]) and Robinson ([6]), long-memory properties of a stationary time series with fractional degree of integration $d \in (0, 1/2)$ can be detected by analysing the periodogram on an interval neighboring zero but *excluding the zero frequency*. To be specific, consider the periodogram ordinates for Fourier frequencies $\omega_j$ such that $n^{1/3} \le \omega_j \le n^{1/2}$, thus satisfying the conditions for the spectral regression in Robinson ([6]). Then, from (5), as $|1-e^{-i\lambda}|^2 \sim \lambda^2$ when $\lambda \to 0^+$, the results above imply that a sufficiently long realization of an ARIMA(0,1,0) will display a singularity of order 2 at these low-order Fourier frequencies.

These results also imply that an ARIMA$(p,1,q)$, having a limiting periodogram, at low-order frequencies, as the one of an ARIMA(0,1,0), should have a spectral singularity of order 2 for a sufficiently large realization. By restricting the analysis to low-order frequencies and sufficiently large time series, the influence of the ARMA parameters on the periodogram can be appropriately reduced. Thus, these results provide a spectral characterization of a certain type of nonstationarity, although they do not provide any finite sample distribution theory.

We now discuss the joint statistical properties of the random variables $I_{\varepsilon,n}(\omega_j)$ and $R_n(\omega_j)$.

**Theorem 3.** *Let $(X_t)$ be a random walk $\nabla X_t = \varepsilon_t$, with $\varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$ and $E\,\varepsilon_t^4 < \infty$. Consider the realization sample $(X_t)_{t=0}^n$ and assume without loss of generality that $X_0 = 0$. Let $R_n(\omega_j)$ be defined as in (8). Then,*

$$E\,R_n(\omega_j)^2 = 2\sigma_\varepsilon^4 + 2n^{-1}(E\,\varepsilon_t^4 - 3\sigma_\varepsilon^4) \ \to \ 2\sigma_\varepsilon^4 \ , \tag{14}$$

*and, if $(\varepsilon_t)$ is Gaussian,*

$$E\,R_n(\omega_j)^2 = 2\sigma_\varepsilon^4 \ . \tag{15}$$

*Moreover, for Fourier frequencies $\omega_k$, $\omega_j$,*

$$E[R_n(\omega_k)\,R_n(\omega_j)] = 0, \quad \text{if } \ \omega_k \ne \omega_j \ , \tag{16}$$

*and*

$$E[I_{\varepsilon,n}(\omega_k)\,R_n(\omega_j)] = 0\,, \quad \forall\, \omega_k, \omega_j \ . \tag{17}$$

**Proof:** We have, following the notation in (8),

$$\begin{aligned}
E\,Q_n(\lambda)^2 &= E\Big|\Big(n^{-1/2}\sum_{t=1}^{n}\varepsilon_t\,e^{-i\lambda t}\Big)\,n^{-1/2}\,e^{i\lambda(n+1)}\Big(\sum_{t=1}^{n}\varepsilon_t\Big)\Big|^2\\
&= n^{-2}\,E\Big|\sum_{t=1}^{n}\varepsilon_t\,e^{-i\lambda t}\sum_{t=1}^{n}\varepsilon_t\,e^{-i\lambda t}\sum_{t=1}^{n}\varepsilon_t\sum_{t=1}^{n}\varepsilon_t\Big|\\
&\le n^{-2}\Big|\sum_{t=1}^{n}e^{-2i\lambda t}\Big|\,E\,\varepsilon_t^4+n^{-2}\Big|\sum_{t=1}^{n}e^{-2i\lambda t}\Big|\,\sigma_\varepsilon^2\sum_{j\neq t}\sigma_\varepsilon^2\\
&\quad+2n^{-2}\Big|\sum_{t=1}^{n}e^{-i\lambda t}\,\sigma_\varepsilon^2\sum_{j\neq t}e^{-i\lambda j}\Big|\,\sigma_\varepsilon^2\;.
\end{aligned}$$

For $\lambda=\omega_j=2\pi j/n\in(0,\pi)$ the first two terms vanish by direct application of (13). The third term also vanishes since

$$\sum_{t=1}^{n}e^{-i\lambda t}\sum_{j\neq t}e^{-i\lambda j}=\sum_{t=1}^{n}e^{-i\lambda t}\sum_{j=1}^{n}e^{-i\lambda j}-\sum_{t=1}^{n}e^{-i\lambda t}\,e^{-i\lambda t}=0-0\;.$$

The same argument applies to $Q_n(-\lambda)$; thus

$$E\,Q_n(\omega_j)^2=E\,Q_n(-\omega_j)^2=0\;.$$

After tedious but similar routine computations we get

$$2E\Big[Q_n(\lambda)\,Q_n(-\lambda)\Big]=2\sigma_\varepsilon^4+2n^{-1}(E\,\varepsilon_t^4-3\sigma_\varepsilon^4)\;. \tag{18}$$

Hence

$$\begin{aligned}
E\,[R_n(\omega_j)]^2 &= E\,[Q_n(\omega_j)]^2+E\,[Q_n(-\omega_j)]^2+2E\Big[Q_n(\omega_j)\,Q_n(-\omega_j)\Big]\\
&= 0+0+2\sigma_\varepsilon^4+2n^{-1}(E\,\varepsilon_t^4-3\sigma_\varepsilon^4)\;,
\end{aligned}$$

and (14) holds. If the noise is Gaussian $E\varepsilon_t^4=3\sigma_\varepsilon^4$, then (15). In order to prove (16) and (17) we apply similar arguments. ∎

## 3 – Extension to general ARIMA and ARFIMA processes

The previous results can be extended to the general ARIMA$(p,1,q)$ case. We were not able, however, to obtain results as strong as the ones obtained before.

**Theorem 4.** *Let $(X_t)$ be an ARIMA$(p,1,q)$ process. Consider the realization sample $(X_t)_{t=0}^n$ and assume that $X_0 = 0$. Then, the periodograms of $(X_t)_{t=0}^n$ and of $(\nabla X_t)_{t=1}^n$ are related through the identity*

$$|1 - e^{-i\omega_j}|^2 \, I_{X,n}(\omega_j) = I_{\nabla X,n}(\omega_j) + n^{-1} X_n^2 - R_n(\omega_j) \ , \tag{19}$$

*where*

$$E\,|R_n(\omega_j)| \le \Big(2\pi\, f_{\nabla X}(\omega_j) \sum_{|k|<n} |\gamma_{\nabla X}(k)|\Big)^{1/2} \longrightarrow \ 2\pi\Big(f_{\nabla X}(\omega_j)\, f_{\nabla X}(0)\Big)^{1/2} \ , \tag{20}$$

*with $\gamma_{\nabla X}$ and $f_{\nabla X}$ representing, respectively, the autocovariance and the spectral density of the stationary process $(\nabla X_t)$.*

**Proof:** The expression (19) is obtained as in Theorem 2. To prove (20) write

$$\|J_{\nabla X}(-\lambda)\| := E^{1/2}\, I_{\nabla X,n}(-\lambda) = \sqrt{f_{\nabla X}(\lambda)\, 2\pi}$$

and

$$\begin{aligned} \|n^{-1/2}\, e^{-\lambda(n+1)}\, X_n\| &:= n^{-1/2}\, E^{1/2}\, X_n^2 \\ &= n^{-1/2}\, \mathrm{Var}^{1/2}\Big[\sum_{t=1}^n \nabla X_t\Big] \\ &= \Big(\sum_{|k|<n} \Big(1 - \frac{k}{n}\Big)\, \gamma_{\nabla X}(k)\Big)^{1/2} \\ &\le \Big(\sum_{|k|<n} |\gamma_{\nabla X}(k)|\Big)^{1/2} \ . \end{aligned}$$

Then, applying the Cauchy–Schwarz inequality we have

$$E\,|R_n(\lambda)| = \Big|\langle J_\varepsilon(-\lambda), n^{-1/2}\, e^{i\lambda(n+1)}\, X_n\rangle\Big| \le \Big(2\pi\, f_{\nabla X}(\lambda) \sum_{|k|<n} |\gamma_{\nabla X}(k)| biggr)^{1/2} \ .$$

Since $(\nabla X_t)$ is an ARMA, it results from Theorem 7.11 of Brockwell and Davis ([1]) that $n^{-1/2}\,\mathrm{Var}^{1/2}[\sum_{t=1}^n \nabla X_t] \to (2\pi\, f_{\nabla X}(0))^{1/2}$. Hence (20). ∎

**Corollary 5.** *For low-order Fourier frequencies $\omega_j$ the bound (20) can be approximated as follows*

$$E\,|R_n(\omega_j)| \le 2\pi\Big(f_{\nabla X}(\omega_j)\, f_{\nabla X}(0)\Big)^{1/2} \simeq 2\pi\, f_{\nabla X}(0) \ . \tag{21}$$

**Proof:** As $(\nabla X_t)$ is an ARMA process, it has bounded continuous rational spectral density function. Then $f_{\nabla X}(\omega_j)$ can be approximated by $f_{\nabla X}(0)$. ∎

If we only consider causal and invertible ARMA processes then $f_{\nabla X}(0)$ is bounded and has a non-zero value. This fact shows that the relation (19) is dominated by $|1-e^{-i\omega_j}|^2$ for sufficiently low-order Fourier frequencies.

## 4 – Concluding remark

The results we have presented show that the behavior of the periodogram of nonstationary integrated processes is dominated by the transfer function of the differencing operator. This suggests the extension of a spectral regression method, as Robinson's ([6]), to nonstationary ARIMA or ARFIMA processes. Simulation results in Crato ([2]) show that tests for stationarity based on such regression methods have quite reasonable properties.

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