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ASYMPTOTIC STUDY OF LATTICE STRUCTURES WITH DAMPING *

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Abstract: This paper considers a time-dependent system with damping and general (non-zero) initial conditions on a perforated domain and presents a careful derivation of the homogenized system.

1 – Introduction

Applications of partial differential equations often involve equations of the form

$$\mathcal{A}_\epsilon u^\epsilon = f \quad \text{in} \;\; \Omega \tag{1}$$

where $\mathcal{A}_\epsilon$ is a family of operators depending on the small parameter ϵ and Ω is a bounded open subset of $\mathbb{R}^N$. For example, the coefficients in $\mathcal{A}_\epsilon$ may be periodic functions with period ϵ (due, e.g., to a periodic mixing of two different materials). For small ϵ it can be quite difficult to obtain an accurate approximation to u^ϵ by standard numerical methods. Thus we seek a simpler problem

$$\mathcal{A} u = f \quad \text{in} \;\; \Omega \tag{2}$$

with the property that for sufficiently small ϵ, the solution u of (2) accurately approximates the solution u^ϵ of (1). Unfortunately, one cannot obtain system (2) simply by averaging the coefficients in (1) over one period. Instead, one can use the homogenization techniques discussed in [5] and [14].

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One can also use these techniques for problems of the form of (1) defined on a perforated domain Ω_ϵ obtained by periodically removing material from Ω (see [7], [8]) In this case, the goal is to obtain a limit problem defined on all of Ω (which may be simply connected) rather than on the original domain with many, possibly thousands, of small holes. In this paper we consider time dependent systems defined on a perforated domain $\Omega_{\epsilon\mu}$ depending on two small parameters ϵ and μ. The domain $\Omega_{\epsilon\mu}$, which is that part of Ω covered by material, is obtained as follows. Set $Y = \prod_{i=1}^{N}[0, l_i]$ and let $T_\mu \subset Y$ be such that the boundary ∂T_μ of T_μ does not meet the boundary ∂Y of Y. Let $\chi_{\cup Y_\mu^*}$ denote the characteristic function of Y_μ^* extended by periodicity to all of $\mathbb{R}^N$. Then we define $\Omega_{\epsilon\mu}$ as

$$\Omega_{\epsilon\mu} = \left\{ x \in \Omega \,|\, \chi_{\cup Y_\mu^*}\left(\frac{x}{\epsilon}\right) = 1 \right\} .$$

We shall assume throughout our presentation that the holes do not meet the boundary $\partial\Omega$. This assumption restricts the geometry of Ω (e.g., Ω can be a finite union of rectangular cells homothetic to the representative cell Y) and the values taken by ϵ (e.g., $\epsilon \in \{n^{-1}\}$ or $\epsilon \in \{2^{-n}\}$). Physically, this assumption means that the material is distributed along the faces of Y rather than along the edges. This assumption is needed for the construction of the extension operators given by Lemmas 2.1 and 2.2.

An example of such a domain is the grid $\Omega_{\epsilon\mu}$ depicted in Fig. la. A natural period of the grid is depicted in Fig. lb. This grid is typical of actual engineering structures.

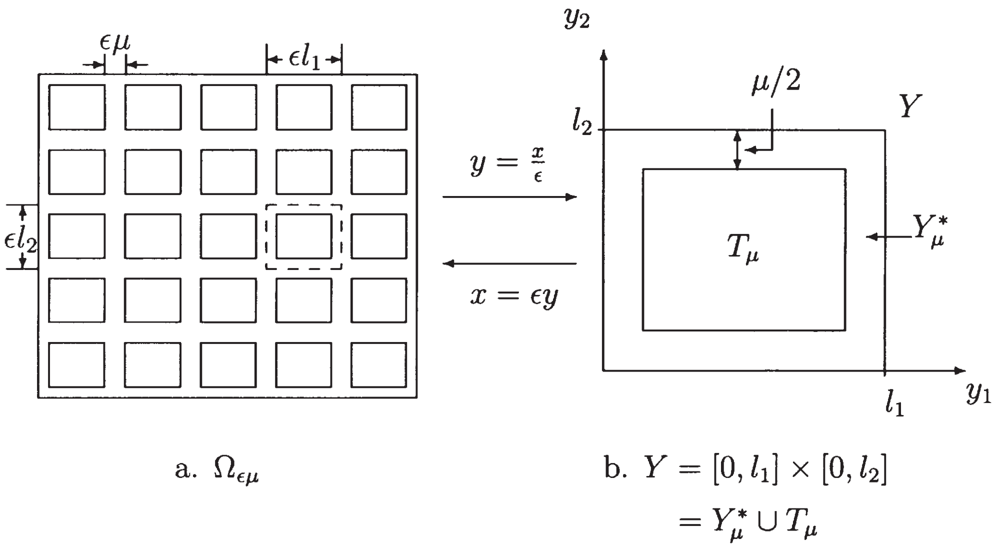

Fig. 1 – The grid $\Omega_{\epsilon\mu}$ and the representative cell Y.

Such structures are characterized by a periodic distribution of holes and a relatively small amount of material (i.e., μ is small compared to l_1 and l_2). We note that $\Omega_{\epsilon\mu}$ is composed of layers of thickness $\epsilon\mu$ with holes of dimension $\epsilon(l_1-\mu)\times\epsilon(l_2-\mu)$. If we take the transformation $y=\frac{x}{\epsilon}$, we are in $\mathbb{R}^2$ covered periodically by cells of Y-type. We denote by $T_{\epsilon\mu}$ the union of all the holes of $\Omega_{\epsilon\mu}$.

Consider the following problem (here and throughout the paper we adopt the summation convention on repeated indices unless explicitly stated otherwise):

$$
(3)\qquad
\begin{aligned}
\rho\,\frac{\partial^2 u^{\epsilon\mu}}{\partial t^2}-\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\epsilon}\right)\frac{\partial u^{\epsilon\mu}}{\partial x_j}\right)&=f \quad \text{in } \Omega_{\epsilon\mu}\ ,\\
u^{\epsilon\mu}&=0 \quad \text{on } \partial\Omega\\
a_{ij}\,\frac{\partial u^{\epsilon\mu}}{\partial x_j}\,n_i&=0 \quad \text{on } \partial T_{\epsilon\mu}\ ,\\
u^{\epsilon\mu}(0)=u_0^{\epsilon\mu}\ ,\qquad u_t^{\epsilon\mu}(0)&=v_0^{\epsilon\mu}\ ,
\end{aligned}
$$

where $\mathbf{n}=(n_i)$ is the outward unit normal. We make the following assumptions:

1.I. The nonhomogeneous forcing function f satisfies $f\in L^2(0,T,L^2(\Omega))$;

1.II. The coefficients satisfy $a_{ij}(\cdot)\in L^\infty(\mathbb{R}^2)$;

1.III. There exists $A>0$ such that $a_{ij}\,\xi_i\,\xi_j\geq A\,\xi_i\,\xi_i$ for all $\xi\in\mathbb{R}^2$;

1.IV. The coefficients a_{ij} are Y-periodic;

1.V. The initial conditions satisfy $u_0^{\epsilon\mu}\in H^1(\Omega_{\epsilon\mu})$, $u_0^{\epsilon\mu}=0$ on $\partial\Omega$ and $v_0^{\epsilon\mu}\in L_2(\Omega_{\epsilon\mu})$.

By Theorem 29.1 in [17], we have that (3) has a unique solution $u^{\epsilon\mu}$. Our goal is to study the dependence of $u^{\epsilon\mu}$ on the parameters ϵ and μ. The equation we obtain by passing to the limit as $\epsilon\to 0$ is called the "homogenized" equation and is defined on all of Ω. The techniques used here to take the limit are similar to those used in problems involving composite media (see [5] and [14]). The homogenized coefficients are expressed in terms of functions defined on Y_μ^*, hence the homogenized system depends on μ. For the static case, it is proved in [1], [8] and [13] that when letting $\mu\to 0$ in the homogenized system, one recovers the simplicity of the original coefficients. We prove here that a similar result holds for systems with damping. We remark that Francfort et al. (see [9] and [10]) have studied similar problems for non-perforated domains.

We close this section with a summary of notation and conventions adopted throughout the paper. In Sections 2 and 3 we consider our problem with μ fixed and take the limit as $\epsilon \to 0$. Hence, for now we suppress the μ in order to simplify the notation.

We will make use of the following function spaces throughout the paper: Let

$$V_{Y^*} = \left\{ \varphi \in H^1(Y^*) \,|\, \varphi \text{ is } Y\text{-periodic} \right\} ,$$

where "Y-periodic" means that the function has equal values on opposite edges of Y. We define

$$V_\epsilon = \left\{ \varphi \in H^1(\Omega_\epsilon) \,|\, \varphi = 0 \text{ on } \partial\Omega \right\}, \qquad H_\epsilon = L^2(\Omega_\epsilon) ,$$

and

$$V = H_0^1(\Omega) , \qquad H = L^2(\Omega) .$$

Note that $\varphi \in V_\epsilon$ vanishes only on the *external* boundary of Ω_ϵ. The embeddings $V_\epsilon \hookrightarrow H_\epsilon \hookrightarrow V_\epsilon^*$ and $V \hookrightarrow H \hookrightarrow V^*$ define Gelfand triples. We denote by $\langle \cdot, \cdot \rangle_{V_\epsilon^*, V_\epsilon}$ and $\langle \cdot, \cdot \rangle_{V^*, V}$, respectively, the corresponding duality pairings. The inner products on the spaces H and H_ϵ will be denoted $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_{H_\epsilon}$. Also, for a function $\varphi \in V$ or $\varphi \in V_\epsilon$, we use the symbol $\partial_i \varphi$ to denote $\frac{\partial \varphi}{\partial x_i}$.

For any function $g \in L^2(\Omega_\epsilon)$ we will denote by $\widetilde{g}$ the extension by zero of g to the whole domain Ω. For any measurable set E, $|E|$ denotes the measure of E, and χ_E denotes the characteristic function of E; i.e.,

$$\chi_E(z) = \begin{cases} 1 & \text{for } z \in E, \\ 0 & \text{for } z \in \mathbb{R}^2 \backslash E . \end{cases}$$

If $f \in L^1(E)$, we denote the mean value of f by $\mathcal{M}_E(f)$:

$$\mathcal{M}_E(f) = \frac{1}{|E|} \int_E f(x)\, dx .$$

The symbol C will be used interchangeably for different constants which are independent of ϵ.

We shall make frequent use of the following lemma (see [14, p. 57]).

Lemma 1.1. *Suppose the Y-periodic function $f \in L^2(Y)$ is extended periodically to all of $\mathbb{R}^2$. If we define $F_\epsilon(x) = f(\frac{x}{\epsilon})$, then as $\epsilon \to 0$, $F_\epsilon \to \mathcal{M}_Y(f)$ in $L^2(\Omega)$ weakly.*

Remark 1.1. Let $\chi_{\cup Y^*}$ denote the extension by periodicity of the characteristic function of Y^* to all of $\mathbb{R}^2$. Then $\chi_{\Omega_\epsilon}(x) = \chi_{\cup Y^*}(\frac{x}{\epsilon})$, so by Lemma 1.1,

$$\chi_{\Omega_\epsilon} \to \mathcal{M}_Y(\chi_{Y^*}) = \theta \quad \text{in } L^2(\Omega) \text{ weakly} .$$

Remark 1.2. If $f \in L^\infty(Y)$, then as $\epsilon \to 0$, $F_\epsilon \to \mathcal{M}_Y(f)$ in $L^\infty(\Omega)$ weak*.

2 – Homogenization of time-dependent systems

Set $a^\epsilon_{ij}(x) = a_{ij}(x/\epsilon)$. The weak form of (3) is

$$\begin{gathered} \langle \rho\, u^\epsilon_{tt}(t), \varphi\rangle_{V^*_\epsilon, V_\epsilon} + \sigma^\epsilon_1(u^\epsilon(t), \varphi) = \langle f(t), \varphi\rangle_{H_\epsilon} \quad \text{for all} \;\; \varphi \in V_\epsilon \;, \\ u^\epsilon(0) = u^\epsilon_0 \,, \quad u^\epsilon_t(0) = v^\epsilon_0 \;, \end{gathered} \tag{4}$$

where the sesquilinear form $\sigma^\epsilon_1(\cdot,\cdot)$ is defined by

$$\sigma^\epsilon_1(\varphi,\psi) = \int_{\Omega_\epsilon} a^\epsilon_{ij}\, \partial_j\varphi\, \partial_i\psi\, dx \quad \text{for} \;\; \varphi,\psi \in V_\epsilon \;.$$

We assume that $u^\epsilon_0 \in V_\epsilon$ and $v^\epsilon_0 \in H_\epsilon$ and moreover that

$$\|u^\epsilon_0\|_{V_\epsilon} \le C \quad \text{and} \quad \|v^\epsilon_0\|_{H_\epsilon} \le C \;, \tag{5}$$

where C is independent of ϵ. By Theorem 29.1 in [17], we have that (4) has a unique solution u^ϵ in the extended V^*_ϵ sense with $u^\epsilon \in L^2(0,T;V_\epsilon)$, $u^\epsilon_t \in L^2(0,T;H_\epsilon)$ and $u^\epsilon_{tt} \in L^2(0,T;V^*_\epsilon)$. A careful examination of the proof of this theorem reveals that in fact $\|u^\epsilon(t)\|_{V_\epsilon} \le C$ and $\|u^\epsilon_t(t)\|_{H_\epsilon} \le C$ for almost every $t \in [0,T]$ where C is independent of ϵ (see [3] and [11, p. 268]). Hence, we in fact obtain $u^\epsilon \in L^\infty(0,T;V_\epsilon)$ and $u^\epsilon_t \in L^\infty(0,T;H_\epsilon)$. We make use of the following extension results (see [6], [7]).

Lemma 2.1. *There exists an extension operator*

$$\mathcal{P}^\epsilon \in \mathcal{L}(V_\epsilon, V)$$

such that

$$\|\mathcal{P}^\epsilon u^\epsilon\|_V \le C\, \|u^\epsilon\|_{V_\epsilon}$$

for all $u^\epsilon \in V_\epsilon$.

Remark 2.1. The above lemma can be generalized to produce extension-operators $\mathcal{P}^\epsilon_\ell \in \mathcal{L}(H^\ell(\Omega_\epsilon), H^\ell(\Omega))$ (where $\Omega_\epsilon, \Omega \subset \mathbb{R}^N$ with $\ell, N \in \mathbb{N}$) which preserve derivative bounds independently of ϵ. For details, see [12].

Lemma 2.2. *There exists an extension operator*

$$\mathcal{Q}^\epsilon \in \mathcal{L}\Big(L^\infty(0,T;V_\epsilon), L^\infty(0,T;V)\Big)$$

such that

$$\sum_{|\alpha|=1} \|D^\alpha \mathcal{Q}^\epsilon u\|_{L^\infty(0,T;L^2(\Omega))} \le C \sum_{|\alpha|=1} \|D^\alpha u\|_{L^\infty(0,T;L^2(\Omega_\epsilon))} \ ,$$

and, if $u_t \in L^\infty(0,T;V_\epsilon)$, *then* $\mathcal{Q}^\epsilon u_t = (\mathcal{Q}^\epsilon u)_t$ *in* $\Omega \times [0,T]$.

Remark 2.2. If $u \in L^\infty(0,T;V_\epsilon)$, then from the construction of $\mathcal{P}^\epsilon$ and $\mathcal{Q}^\epsilon$, we have that $\mathcal{Q}^\epsilon u(t,x) = [\mathcal{P}^\epsilon u(t,\cdot)](x)$ (see [2], [12]).

Thus, we have the existence of a constant C independent of ϵ such that

$$\|\mathcal{Q}^\epsilon u^\epsilon\|_{L^\infty(0,T;V)} \le C \ ;$$

hence, there exists a subsequence of $\{\epsilon\}$ and a function $u \in L^\infty(0,T;V)$ such that

$$\mathcal{Q}^\epsilon u^\epsilon \to u \quad \text{in} \ \ L^\infty(0,T;V) \ \text{weak*} \ . \tag{6}$$

We also need the following lemma.

Lemma 2.3. *Suppose* $\{\varphi^\epsilon\} \subset V_\epsilon$ *with* $\|\varphi^\epsilon\|_{V_\epsilon} \le C$ *where* C *is independent of* ϵ. *Then there exists a subsequence, still denoted* $\{\varphi^\epsilon\}$, *such that* $\widetilde{\varphi}^\epsilon \to \varphi$ *weakly in* H *as* $\epsilon \to 0$. *Moreover,* $\varphi \in V$.

Proof: First of all, notice that $\|\widetilde{\varphi}^\epsilon\|_H = \|\varphi^\epsilon\|_{H_\epsilon} \le \|\varphi^\epsilon\|_{V_\epsilon}$ so there exists $\varphi \in H$ and a subsequence such that $\widetilde{\varphi}^\epsilon \to \varphi$ weakly in H; i.e.,

$$\int_\Omega \widetilde{\varphi}^\epsilon\, \psi\, dx \to \int_\Omega \varphi\, \psi\, dx \qquad \text{for all} \ \ \psi \in H \ .$$

But $\|\mathcal{P}^\epsilon \varphi^\epsilon\|_V \le C$, so there exists $\overline{\varphi} \in V$ such that $\mathcal{P}^\epsilon \varphi^\epsilon \to \overline{\varphi}$ weakly in V; hence, $\mathcal{P}^\epsilon \varphi^\epsilon \to \overline{\varphi}$ strongly in H. Thus, for all $\psi \in H$

$$\int_\Omega \widetilde{\varphi}^\epsilon\, \psi\, dx = \int_\Omega \chi_{\Omega_\epsilon}\, \mathcal{P}^\epsilon\, \varphi^\epsilon\, \psi\, dx \int_\Omega \chi_{\Omega_\epsilon} \to \int_\Omega \theta\, \overline{\varphi}\, \psi\, dx \ ,$$

so $\varphi = \theta \overline{\varphi} \in V$.

Remark 2.3. Observe that φ, hence $\overline{\varphi}$, is independent of $\mathcal{P}^\epsilon$.

By the assumption that the initial data are bounded independently of ϵ, (see (5)) there exist $u_0 \in V$ and $v_0 \in H$ such that

$$\left.\begin{array}{l} \widetilde{u}_0^\epsilon \to u_0 \\ \widetilde{v}_0^\epsilon \to v_0 \end{array}\right\} \quad \text{weakly in} \ \ L^2(\Omega) \ .$$

Consider the equation

$$\tag{7} \begin{gathered} \langle \rho\,\theta\, u_{tt}, \varphi\rangle_{V^*,V} + \int_\Omega q_{ij}\,\partial_j u\,\partial_i\varphi\,dx = \langle \theta f, \varphi\rangle_H \quad \text{for all} \quad \varphi \in V\ , \\ u(0) = u_0/\theta\,, \quad u_t(0) = v_0/\theta\ , \end{gathered}$$

where the q_{ij} are the homogenized coefficients obtained for the static Neumann problem (see [7]) and are given by

$$q_{ij} = \frac{1}{|Y|}\int_{Y^*}\left(a_{ij} - a_{il}\,\frac{\partial\chi^j}{\partial y_l}\right)dy$$

with χ^j the Y-periodic solutions of

$$\int_{Y^*} a_{lk}\,\frac{\partial(\chi^j - y_j)}{\partial y_k}\,\frac{\partial\varphi}{\partial y_l}\,dy = 0 \quad \text{for all} \quad \varphi \in V_{Y^*}\ .$$

The q_{ij} satisfy the ellipticity condition 1.III, so by standard results (see [17]), Eq. (7) also has a unique solution u^h. We are now ready to prove the following theorem.

Theorem 2.4. *The limit u in (6) is the unique solution of (7).*

Proof: We must show that $u = u^h$. First we extend all functions defined on $[0,T]$ by zero for $t > T$. Taking the Laplace transform of (4), we obtain

$$\tag{8} \langle \rho\, s^2\,\widehat{u}^\epsilon(s), \varphi\rangle_{H_\epsilon} + \int_{\Omega_\epsilon} a^\epsilon_{ij}\,\partial_j\widehat{u}^\epsilon(s)\,\partial_i\varphi\,dx = \langle \widehat{f}(s), \varphi\rangle_{H^\epsilon} + \langle \rho(s\,u^\epsilon_0 + v^\epsilon_0), \varphi\rangle_{H_\epsilon}$$

for all $\varphi \in V_\epsilon$, where $\widehat{u}^\epsilon(s) = \mathcal{L}[u^\epsilon](s) = \int_0^\infty e^{-st}\,u^\epsilon(t)\,dt$ is the Laplace transform of $u^\epsilon(t)$. For fixed real $s > 0$, this equation has a unique solution $\widehat{u}^\epsilon(s)$ satisfying $\|\widehat{u}^\epsilon(s)\|_{V_\epsilon} \le C$. By Lemma 2.3, there exists $\overline{u}(s) \in V$ and a subsequence such that

$$\tag{9} \mathcal{P}^\epsilon\widehat{u}^\epsilon(s) \to \overline{u}(s) \quad \text{weakly in} \quad V\ ,$$

and $\widetilde{\widehat{u}}^\epsilon(s) \to \theta\,\overline{u}(s)$ weakly in H. Set $\xi^\epsilon_i(s) = a^\epsilon_{ij}\,\partial_j\widehat{u}^\epsilon(s)$. Then $\|\xi^\epsilon_i(s)\|_{H_\epsilon} \le C$, so there exists $\xi^*_i(s) \in H$ such that $\widetilde{\xi}^\epsilon_i(s) \to \xi^*_i(s)$ weakly in H. We extend equation (8) to all of Ω, obtaining

$$\langle \rho\, s^2\,\widetilde{\widehat{u}}^\epsilon(s), \varphi\rangle_H + \int_\Omega \widetilde{\xi}^\epsilon_i(s)\,\partial_i\varphi\,dx = \langle \chi_{\Omega_\epsilon}\,\widehat{f}(s), \varphi\rangle_H + \langle \rho(s\,\widetilde{u}^\epsilon_0 + \widetilde{v}^\epsilon_0), \varphi\rangle_H$$

for all $\varphi \in V$. Letting $\epsilon \to 0$ we find

$$\langle \rho\, s^2\,\theta\,\overline{u}(s), \varphi\rangle_H + \int_\Omega \xi^*_i(s)\,\partial_i\varphi\,dx = \langle \theta\,\widehat{f}(s), \varphi\rangle_H + \langle \rho(s\,u_0 + v_0), \varphi\rangle_H\ .$$

The energy method of Tartar gives now $\xi_i^*(s) = q_{ij}\,\partial_j \overline{u}(s)$ (for details see for instance [2], [7] and [16]). Thus, $\overline{u}(s)$ satisfies

$$(10)\quad \langle \rho\, s^2\, \theta\, \overline{u}(s),\, \varphi\rangle_H + \int_\Omega q_{ij}\,\partial_j \overline{u}(s)\,\partial_i\varphi\,dx = \langle \theta\,\widehat{f}(s), \varphi\rangle_H + \langle \rho(s\,u_0 + v_0), \varphi\rangle_H$$

for all $\varphi \in V$. Again, by the ellipticity of the q_{ij}, this equation has a unique solution for fixed $s > 0$. Next, taking the Laplace transform of equation (7), we see that $\widehat{u}^h(s)$ satisfies for all $\varphi \in V$

$$\langle \rho\, s^2\, \theta\, \widehat{u}^h(s),\, \varphi\rangle_H + \int_\Omega q_{ij}\,\partial_j \widehat{u}^h(s)\,\partial_i\varphi\,dx = \langle \theta\,\widehat{f}(s), \varphi\rangle_H + \langle \rho(s\,u_0 + v_0), \varphi\rangle_H\ .$$

Since this is exactly the same as equation (10), $\overline{u}(s) = \widehat{u}^h(s)$ for real $s > 0$. Now taking $e^{-st}\varphi$ as a test function in (6), we see that

$$(11)\qquad \widehat{\mathcal{Q}^\epsilon u^\epsilon}(s) \to \widehat{u}(s) \quad \text{in } V \text{ weakly}$$

for all $s > 0$. Finally, observe that for $s \in \mathbb{C}^+$ (i.e., $\operatorname{Re} s > 0$)

$$\begin{aligned}\widehat{\mathcal{Q}^\epsilon u^\epsilon}(s)(x) &= \int_0^\infty e^{-st}\,\mathcal{Q}^\epsilon u^\epsilon(t,x)\,dt\\ &= \int_0^\infty e^{-st}[\mathcal{P}^\epsilon u^\epsilon(t,\cdot)](x)\,dt = \mathcal{P}^\epsilon\Big[\int_0^\infty e^{-st}\,u^\epsilon(t)\,dt\Big](x) = \mathcal{P}^\epsilon \widehat{u}^\epsilon(s)(x)\ .\end{aligned}$$

Thus, combining (9) and (11), we see that $\widehat{u}(s) = \overline{u}(s) = \widehat{u}^h(s)$ for all real $s > 0$, hence for all $s \in \mathbb{C}^+$ since the Laplace transform is an analytic function of s. Therefore, by the uniqueness of the inverse Laplace transform, $u = u^h$.

3 – Homogenization of systems with damping

We now extend equation (4) to include a damping term. We assume that $b_{ij} \in L^\infty(\mathbb{R}^2)$ are Y-periodic and that there exists $B > 0$ such that $b_{ij}\,\xi_i\,\xi_j \geq B\,\xi_i\,\xi_i$ for all $\xi \in \mathbb{R}^2$; i.e., we assume that 1.II–1.IV hold for the b_{ij}. We also assume that $b_{ij} = b_{ji}$. Define b_{ij}^ϵ by $b_{ij}^\epsilon(x) = b_{ij}(\frac{x}{\epsilon})$, and the sesquilinear form $\sigma_2^\epsilon(\cdot,\cdot)$ by

$$\sigma_2^\epsilon(\varphi,\psi) = \int_{\Omega_\epsilon} b_{ij}^\epsilon\,\partial_j\varphi\,\partial_i\psi\,dx \quad \text{for all } \varphi,\psi \in V_\epsilon\ .$$

We consider the problem

$$(12)\quad \begin{aligned}&\langle \rho\, u_{tt}^\epsilon(t), \varphi\rangle_{V_\epsilon^*,V_\epsilon} + \sigma_1^\epsilon(u^\epsilon(t),\varphi) + \sigma_2^\epsilon(u_t^\epsilon(t),\varphi) = \langle f(t),\varphi\rangle_{V_\epsilon^*,V_\epsilon} \ \text{ for all } \varphi \in V_\epsilon\ ,\\ &u^\epsilon(0) = u_0^\epsilon \in V_\epsilon\ , \quad u_t^\epsilon(0) = v_0^\epsilon \in H_\epsilon\ ,\end{aligned}$$

with $f \in L^2(0,T;V_\epsilon^*)$ and u_0^ϵ, v_0^ϵ satisfying bounds as in (5). Again, a unique solution u^ϵ exists with $u^\epsilon \in L^2(0,T;V_\epsilon)$, $u_t^\epsilon \in L^2(0,T;V_\epsilon)$ and $u_{tt}^\epsilon \in L^2(0,T;V_\epsilon^*)$ (e.g., see [3] for details). Moreover, we still have the bounds $\|u^\epsilon(t)\|_{V_\epsilon} \le C$ and $\|u_t^\epsilon(t)\|_{H_\epsilon} \le C$ for almost every $t \in [0,T]$ for a constant C independent of ϵ. Hence, just as before, there exists a subsequence of $\{\epsilon\}$ and a function $u \in L^\infty(0,T;V)$ (this time with $u_t \in L^\infty(0,T;V)$) such that

$$\mathcal{Q}^\epsilon u^\epsilon \to u \quad \text{in } \; L^\infty(0,T;V) \text{ weak*}$$

and

$$\mathcal{Q}^\epsilon u_t^\epsilon \to u_t \quad \text{in } \; L^\infty(0,T;H) \text{ weak* } .$$

In order to obtain the homogenized equation satisfied by u, we again use Laplace transforms and apply standard results. Taking the Laplace transform of (12) we obtain

$$\begin{aligned}\langle \rho\, s^2\, \widehat{u}^\epsilon(s), \varphi\rangle_{H_\epsilon} + \sigma_1^\epsilon(\widehat{u}^\epsilon(s),\varphi) + \sigma_2^\epsilon(s\,\widehat{u}^\epsilon(s),\varphi) &= \\ = \langle \widehat{f}(s),\varphi\rangle_{V_\epsilon^*,V_\epsilon} + \langle \rho(s\,u_0^\epsilon + v_0^\epsilon),\varphi\rangle_{H_\epsilon} + \sigma_2^\epsilon(u_0^\epsilon,\varphi) &\quad \text{for all } \; \varphi \in V_\epsilon \; .\end{aligned}$$

We rewrite this equation as

$$\begin{aligned}\langle \rho\, s^2\, \widehat{u}^\epsilon(s), \varphi\rangle_{H_\epsilon} + \sigma^\epsilon(s)(\widehat{u}^\epsilon(s),\varphi) &= \\ &= \langle \widehat{f}(s),\varphi\rangle_{V_\epsilon^*,V_\epsilon} + \langle \rho(s\,u_0^\epsilon + v_0^\epsilon),\varphi\rangle_{H_\epsilon} + \sigma_2^\epsilon(u_0^\epsilon,\varphi)\end{aligned} \tag{13}$$

for all $\varphi \in V_\epsilon$, where $\sigma^\epsilon(s)$ is defined by

$$\sigma^\epsilon(s)(\varphi,\psi) = \int_{\Omega_\epsilon} (a_{ij}^\epsilon + s\, b_{ij}^\epsilon)\, \partial_j\varphi\, \partial_i\psi\, dx \quad \text{ for all } \; \varphi,\psi \in V_\epsilon \; .$$

Set $c_{ij}^\epsilon(s) = a_{ij}^\epsilon + s\, b_{ij}^\epsilon$. For fixed $s > 0$ (real), $c_{ij}^\epsilon(s) \in L^\infty(\mathbb{R}^2)$ and $c_{ij}^\epsilon(s)\,\xi_i\,\xi_j \ge (A + sB)\,\xi_i\,\xi_i$. Thus, (13) has a unique solution $\widehat{u}^\epsilon(s) \in V_\epsilon$ with $\|\widehat{u}^\epsilon(s)\|_{V_\epsilon} \le C$. Just as in the proof of Theorem 2.4, we can extend (13) to all of Ω and let $\epsilon \to 0$, applying the standard homogenization formulas to the sesquilinear forms $\sigma^\epsilon(s)$ and σ_2^ϵ. If we define

$$\xi_i^\epsilon(s) = c_{ij}^\epsilon(s)\, \partial_j \widehat{u}^\epsilon(s) \quad \text{ and } \quad \zeta_i^\epsilon = b_{ij}^\epsilon\, \partial_j u_0^\epsilon \; ,$$

then when we extend (13) to Ω, the two terms involving these forms become

$$\int_\Omega \widetilde{\xi}_i^\epsilon(s)\, \partial_i\varphi\, dx \quad \text{ and } \quad \int_\Omega \widetilde{\zeta}_i^\epsilon\, \partial_i\varphi\, dx \; ,$$

respectively. There exist $\overline{u}(s) \in V$ and $\overline{u}_0 \in V$ such that

$$\mathcal{P}^\epsilon \widehat{u}^\epsilon(s) \to \overline{u}(s) \quad \text{ and } \quad \mathcal{P}^\epsilon u_0^\epsilon \to \overline{u}_0$$

(both convergences are weak in V); hence, applying the energy method, these terms become, in the limit,

$$\int_\Omega \overline{q}_{ij}(s)\,\partial_j\overline{u}(s)\,\partial_i\varphi\,dx \quad \text{and} \quad \int_\Omega q^b_{ij}\,\partial_j\overline{u}_0\,\partial_i\varphi\,dx\ ,$$

respectively. The coefficients $\overline{q}_{ij}(s)$ are defined by

$$\overline{q}_{ij}(s) = \frac{1}{|Y|}\int_{Y^*}\left[a_{ij} + s\,b_{ij} - (a_{il} + s\,b_{il})\,\frac{\partial\chi^j(s)}{\partial y_l}\right]dy$$

where the functions $\chi^j(s)$ are the Y-periodic solutions of

$$\int_{Y^*}(a_{kl} + s\,b_{kl})\,\frac{\partial(\chi^j(s) - y_j)}{\partial y_l}\,\frac{\partial\varphi}{\partial y_k}\,dy = 0 \quad \text{for all} \quad \varphi \in V_{Y^*}\ ,$$

and the q^b_{ij} are given by

$$q^b_{ij} = \frac{1}{|Y|}\int_{Y^*}\left(b_{ij} - b_{il}\,\frac{\partial\chi^j_b}{\partial y_l}\right)dy$$

where the χ^j_b are the Y-periodic solutions of

$$\int_{Y^*} b_{kl}\,\frac{\partial(\chi^j_b - y_j)}{\partial y_l}\,\frac{\partial\varphi}{\partial y_k}\,dy = 0 \quad \text{for all} \quad \varphi \in V_{Y^*}\ . \tag{14}$$

From Lemma 2.3 we also have $\widetilde{u}^\epsilon_0 \to u_0$ weakly in H where $u_0 = \theta\,\overline{u}_0$. Thus, replacing $\overline{u}_0$ by $\frac{1}{\theta}\,u_0$, we obtain

$$\begin{aligned}\langle\rho\,s^2\,\theta\,\overline{u}(s),\varphi\rangle_H + \int_\Omega \overline{q}_{ij}(s)\,\partial_j\overline{u}(s)\,\partial_i\varphi\,dx =\\ = \langle\theta\,\widehat{f}(s),\varphi\rangle_{V^*,V} + \langle\rho(s\,u_0 + v_0),\varphi\rangle_H + \frac{1}{\theta}\int_\Omega q^b_{ij}\,\partial_j u_0\,\partial_i\varphi\,dx\ .\end{aligned} \tag{15}$$

Using standard arguments we can show that the coefficients $\overline{q}_{ij}(s)$ satisfy an ellipticity condition, so that equation (15) has a unique solution $\overline{u}(s)$. Just as in the proof of Theorem 2.4, we can show that $\widehat{u}(s) = \overline{u}(s)$.

Rather than taking the inverse Laplace transform of equation (15), we derive the homogenized equation for u using the multiple-scale method. First we write equation (12) in strong form as follows:

$$\begin{aligned}&\rho\,u^\epsilon_{tt}(t,x) - \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\epsilon}\right)\frac{\partial u^\epsilon}{\partial x_j}(t,x)\right) - \frac{\partial}{\partial x_i}\left(b_{ij}\left(\frac{x}{\epsilon}\right)\frac{\partial}{\partial t}\,\frac{\partial u^\epsilon}{\partial x_j}(t,x)\right) = f(t,x) \quad \text{in } \Omega_\epsilon\ ,\\ &\left(a^\epsilon\,\frac{\partial u^\epsilon}{\partial x_j} + b^\epsilon_{ij}\,\frac{\partial}{\partial t}\,\frac{\partial u^\epsilon}{\partial x_j}\right)n_i = 0 \quad \text{on} \quad \partial T_\epsilon\ ,\\ &u^\epsilon(0,x) = u^\epsilon_0\,, \quad u^\epsilon_t(0,x) = v^\epsilon_0\ .\end{aligned} \tag{16}$$

Now we seek $u^\epsilon(t,x)$ of the form

$$u^\epsilon(t,x) = u^0(t,x,y) + \epsilon\, u^1(t,x,y) + \epsilon^2\, u^2(t,x,y) + \ldots$$

where $y = x/\epsilon$, and each u^i is defined for all $x \in \Omega$ and all $y \in Y^*$ and is Y-periodic in y. We also need to specify initial conditions. Based on our experience in Section 2, we take for u^0:

$$u^0(0,x,y) = u_0(x)/\theta \quad \text{(independent of } y\text{)}$$

$$u^0_t(0,x,y) = v_0(x)/\theta \ .$$

We will specify other initial conditions later as needed. Let A_ϵ be defined by

$$A_\epsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\epsilon}\right)\frac{\partial}{\partial x_j}\right) \ .$$

Since a_{ij} depends only on $x/\epsilon = y$, we see that A_ϵ can be written as

$$A_\epsilon = \epsilon^{-2} A_0 + \epsilon^{-1} A_1 + A_2$$

where

$$A_0 = -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\,\frac{\partial}{\partial y_j}\right) \ ,$$

$$A_1 = -a_{ij}(y)\,\frac{\partial^2}{\partial x_i\,\partial y_j} - \frac{\partial}{\partial y_i}\left(a_{ij}(y)\,\frac{\partial}{\partial x_j}\right) \ ,$$

$$A_2 = -a_{ij}(y)\,\frac{\partial^2}{\partial x_i\,\partial x_j} \ .$$

Similarly define $B_\epsilon = \epsilon^{-2} B_0 + \epsilon^{-1} B_1 + B_2$ where

$$B_0 = -\frac{\partial}{\partial y_i}\left(b_{ij}(y)\,\frac{\partial}{\partial y_j}\right) \ ,$$

$$B_1 = -b_{ij}(y)\,\frac{\partial^2}{\partial x_i\,\partial y_j} - \frac{\partial}{\partial y_i}\left(b_{ij}(y)\,\frac{\partial}{\partial x_j}\right) \ ,$$

$$B_2 = -b_{ij}(y)\,\frac{\partial^2}{\partial x_i\,\partial x_j} \ .$$

Substituting into equation (16) and matching powers of ϵ, we obtain first of all

$$A_0\, u^0 + B_0\,\frac{\partial u^0}{\partial t} = 0 \quad \text{in } \ Y^* \ ,$$

$$\left(a_{ij}\,\frac{\partial u^0}{\partial y_j} + b_{ij}\,\frac{\partial}{\partial t}\,\frac{\partial u^0}{\partial y_j}\right) n_i = 0 \quad \text{on } \ \partial T \ ,$$

or, in weak form, $u^0 \in V_{Y^*}$, and

$$\int_{Y^*} \left(a_{ij} + b_{ij} \frac{\partial}{\partial t} \right) \frac{\partial u^0}{\partial y_j} \frac{\partial \varphi}{\partial y_i} \, dy = 0 \quad \text{for all } \varphi \in V_{Y^*} \ . \tag{17}$$

Take $\varphi = u^0(t)$ in equation (17). We have

$$\int_{Y^*} a_{ij} \frac{\partial u^0(t)}{\partial y_j} \frac{\partial u^0(t)}{\partial y_i} \, dy + \frac{1}{2} \frac{\partial}{\partial t} \int_{Y^*} b_{ij} \frac{\partial u^0(t)}{\partial y_j} \frac{\partial u^0(t)}{\partial y_i} \, dy = 0$$

by the symmetry of the b_{ij}. Integrating from 0 to t we find

$$\int_0^t \int_{Y^*} a_{ij} \frac{\partial u^0(\tau)}{\partial y_j} \frac{\partial u^0(\tau)}{\partial y_i} \, dy \, d\tau + \\ + \frac{1}{2} \int_{Y^*} b_{ij} \frac{\partial u^0(t)}{\partial y_j} \frac{\partial u^0(t)}{\partial y_i} \, dy - \frac{1}{2\theta^2} \int_{Y^*} b_{ij} \frac{\partial u_0}{\partial y_j} \frac{\partial u_0}{\partial y_i} \, dy = 0 \ .$$

But u_0 is independent of y, so the last term vanishes. Hence, by the coercivity of the a_{ij} and b_{ij} we have

$$0 \le A \int_0^t \int_{Y^*} \frac{\partial u^0(\tau)}{\partial y_i} \frac{\partial u^0(\tau)}{\partial y_i} \, dy \, d\tau + \frac{B}{2} \int_{Y^*} \frac{\partial u^0(t)}{\partial y_i} \frac{\partial u^0(t)}{\partial y_i} \, dy \le 0 \ .$$

Thus, $\frac{\partial u^0(t)}{\partial y_i} = 0$ for all t, $i = 1, 2$.

Remark 3.1. Sanchez–Palencia assumes a priori that u^0 is independent of y (see [14, p. 99]).

Matching the next powers of ϵ we obtain

$$A_0 \, u^1 + A_1 \, u^0 + B_0 \frac{\partial u^1}{\partial t} + B_1 \frac{\partial u^0}{\partial t} = 0 \quad \text{in } \ \Omega \times Y^* \ ,$$

$$\left(a_{ij} + b_{ij} \frac{\partial}{\partial t} \right) \frac{\partial u^1}{\partial y_j} n_i = - \left(a_{ij} + b_{ij} \frac{\partial}{\partial t} \right) \frac{\partial u^0}{\partial x_j} n_i \quad \text{on } \ \Omega \times \partial T \ .$$

Since u^0 is independent of y, we can write the above equation as

$$- \frac{\partial}{\partial y_i} \left[\left(a_{ij} + b_{ij} \frac{\partial}{\partial t} \right) \frac{\partial u^1}{\partial y_j} \right] - \frac{\partial}{\partial y_i} \left(a_{ij} + b_{ij} \frac{\partial}{\partial t} \right) \frac{\partial u^0}{\partial x_j} = 0 \ ,$$

or in weak form as

$$\int_{Y^*} \left[\left(a_{ij} + b_{ij} \frac{\partial}{\partial t} \right) \frac{\partial u^1}{\partial y_j} + \left(a_{ij} + b_{ij} \frac{\partial}{\partial t} \right) \frac{\partial u^0}{\partial x_j} \right] \frac{\partial \varphi}{\partial y_i} \, dy = 0 \tag{18}$$

for all $\varphi \in V_{Y^*}$. We want to express u^1 in terms of u^0. In order to make u^1 unique, we introduce the space

$$\overline{V}_{Y^*} = \Big\{ \varphi \in V_{Y^*} \mid \mathcal{M}_{Y^*}(\varphi) = 0 \Big\}$$

with inner product

$$\langle \varphi, \psi \rangle_{\overline{V}_{Y^*}} = \int_{Y^*} b_{ij} \, \frac{\partial \varphi}{\partial y_j} \, \frac{\partial \psi}{\partial y_i} \, dy \ .$$

This inner product induces a norm $\| \cdot \|_{\overline{V}_{Y^*}}$ on $\overline{V}_{Y^*}$ equivalent to the usual norm $\| \cdot \|_{H^1(Y^*)}$. Now we define $\mathcal{A} \in \mathcal{L}(\overline{V}_{Y^*}, \overline{V}_{Y^*})$ by

$$\langle \mathcal{A}\varphi, \psi \rangle_{\overline{V}_{Y^*}} = \int_{Y^*} a_{ij} \, \frac{\partial \varphi}{\partial y_j} \, \frac{\partial \psi}{\partial y_i} \, dy \ . \tag{19}$$

The right side of equation (19) is a bounded coercive sesquilinear form on $\overline{V}_{Y^*} \times \overline{V}_{Y^*}$, so $\mathcal{A}$ is defined uniquely and is bijective and bicontinuous (by the Lax–Milgram theorem — see [17, p. 272]).

Now define $F_j^a \colon \overline{V}_{Y^*} \to \mathbb{C}$ by

$$F_j^a(\varphi) = \int_{Y^*} a_{ij} \, \frac{\partial \varphi}{\partial y_i} \, dy \ .$$

Since F_j^a is a bounded linear functional on $\overline{V}_{Y^*}$, there exists a unique $f_j^a \in \overline{V}_{Y^*}$ such that

$$\langle f_j^a, \varphi \rangle_{\overline{V}_{Y^*}} = F_j^a(\varphi) = \int_{Y^*} a_{ij} \, \frac{\partial \varphi}{\partial y_i} \, dy \ . \tag{20}$$

Similarly, we define $f_j^b \in \overline{V}_{Y^*}$ by

$$\langle f_j^b, \varphi \rangle_{\overline{V}_{Y^*}} = \int_{Y^*} b_{ij} \, \frac{\partial \varphi}{\partial y_i} \, dy \ . \tag{21}$$

Remark 3.2. If we take χ_b^j defined in (14) so that $\mathcal{M}_{Y^*}(\chi_b^j) = 0$, then $\chi_b^j = f_j^b$.

Using $\mathcal{A}$, f_j^a and f_j^b we can rewrite equation (18) as

$$\left\langle \mathcal{A}u^1 + \frac{\partial u^1}{\partial t} + \left(f_j^a \, \frac{\partial u^0}{\partial x_j} + f_j^b \, \frac{\partial}{\partial t} \, \frac{\partial u^0}{\partial x_j} \right), \ \varphi \right\rangle_{\overline{V}_{Y^*}} = 0 \quad \text{for all} \ \ \varphi \in \overline{V}_{Y^*}$$

which implies that

$$\frac{\partial u^1}{\partial t} + \mathcal{A}u^1 = -f_j^a \, \frac{\partial u^0}{\partial x_j} - f_j^b \, \frac{\partial}{\partial t} \, \frac{\partial u^0}{\partial x_j} \ . \tag{22}$$

Set $w = u^1 + f_j^b \frac{\partial u^0}{\partial x_j}$. Then

$$\frac{\partial w}{\partial t} = \frac{\partial u^1}{\partial t} + f_j^b \frac{\partial}{\partial t} \frac{\partial u^0}{\partial x_j} .$$

If we define $f_j = \mathcal{A} f_j^b - f_j^a$, then we can rewrite equation (22) as

$$\frac{\partial w}{\partial t} + \mathcal{A} w = f_j \frac{\partial u^0}{\partial x_j} . \tag{23}$$

Since $\mathcal{A}$ is bounded it generates a uniformly continuous semigroup (in fact a group) of operators. Thus, once we specify an initial value for w, we can solve equation (23) to obtain w, hence u^1, in terms of u^0. It will be convenient to take $w(0) = 0$. Hence, we take as the initial condition on u^1:

$$u^1(0) = -\frac{1}{\theta} f_j^b \frac{\partial u_0}{\partial x_j} .$$

The unique solution to equation (23) with $w(0) = 0$ is

$$w(t) = \int_0^t e^{-(t-\sigma)\mathcal{A}} f_j \frac{\partial u^0(\sigma)}{\partial x_j} d\sigma .$$

Hence u^1 is given by

$$u^1(t) = \int_0^t e^{-(t-\sigma)\mathcal{A}} f_j \frac{\partial u^0(\sigma)}{\partial x_j} d\sigma - f_j^b \frac{\partial u^0(t)}{\partial x_j} . \tag{24}$$

We will need the following lemma to prove the main result of this section.

Lemma 3.1. *If we take $\chi^j(s)$ as defined above to be in $\overline{V}_{Y^*}$, then*

$$\chi^j(s) = f_j^b - (sI + \mathcal{A})^{-1} f_j .$$

Proof: The $\chi^j(s)$ for which $\mathcal{M}_{Y^*}(\chi^j(s)) = 0$ is the unique solution of

$$\int_{Y^*} \left[(a_{kl} + s\, b_{kl}) \frac{\partial \chi^j(s)}{\partial y_l} - (a_{kj} + s\, b_{kj}) \right] \frac{\partial \varphi}{\partial y_k} dy = 0 \quad \text{for all } \varphi \in \overline{V}_{Y^*} .$$

We can rewrite this equation as

$$\Big\langle (\mathcal{A} + sI)\, \chi^j(s) - (f_j^a + s\, f_j^b),\ \varphi \Big\rangle_{\overline{V}_{Y^*}} = 0 ,$$

which implies that

$$\begin{aligned}(sI+\mathcal{A})\,\chi^j(s) &= f_j^a + s\,f_j^b \\ &= (sI+\mathcal{A})\,f_j^b - \mathcal{A}\,f_j^b + f_j^a \\ &= (sI+\mathcal{A})\,f_j^b - f_j\ .\end{aligned}$$

Hence, $\chi^j(s) = f_j^b - (sI+\mathcal{A})^{-1} f_j$.

We obtain the homogenized equation by matching the next powers of ϵ, integrating over Y^* and substituting in the expression for u^1 in terms of u^0 (equation (24)). We find

$$\begin{aligned}(25)\quad &\rho\,\theta\,\frac{\partial^2 u^0(t)}{\partial t^2} - \frac{1}{|Y|}\int_{Y^*}\left(a_{ij} - a_{il}\,\frac{\partial f_j^b}{\partial y_l} + b_{il}\,\frac{\partial f_j}{\partial y_l}\right)dy\;\frac{\partial^2 u^0(t)}{\partial x_i\,\partial x_j} - \\ &- \frac{1}{|Y|}\int_{Y^*}\left(b_{ij} - b_{il}\,\frac{\partial f_j^b}{\partial y_l}\right)dy\;\frac{\partial}{\partial t}\,\frac{\partial^2 u^0(t)}{\partial x_i\,\partial x_j} \\ &- \frac{1}{|Y|}\int_{Y^*}\int_0^t\left[a_{il}\,\frac{\partial}{\partial y_l}(e^{-(t-\sigma)\mathcal{A}} f_j) - b_{il}\,\frac{\partial}{\partial y_l}(\mathcal{A}\,e^{-(t-\sigma)\mathcal{A}} f_j)\right]\frac{\partial^2 u^0(\sigma)}{\partial x_i\,\partial x_j}\,d\sigma\,dy = \theta\,f(t).\end{aligned}$$

The following theorem is the main result of this section.

Theorem 3.2. *Equation (25) along with initial values $u^0(0) = u_0/\theta$, $u_t^0(0) = v_0/\theta$ is the homogenized system satisfied by u.*

Proof: Define the coefficients

$$\begin{aligned}\alpha_{ij} &= \frac{1}{|Y|}\int_{Y^*}\left(a_{ij} - a_{il}\,\frac{\partial f_j^b}{\partial y_l} + b_{il}\,\frac{\partial f_j}{\partial y_l}\right)dy\ , \\ \beta_{ij} &= \frac{1}{|Y|}\int_{Y^*}\left(b_{ij} - b_{il}\,\frac{\partial f_j^b}{\partial y_l}\right)dy \quad (= q_{ij}^b)\ , \\ \gamma_{ij}(t) &= \frac{1}{|Y|}\int_{Y^*}\left[a_{il}\,\frac{\partial}{\partial y_l}(e^{-t\mathcal{A}} f_j) - b_{il}\,\frac{\partial}{\partial y_l}(\mathcal{A}\,e^{-t\mathcal{A}} f_j)\right]dy\ .\end{aligned}$$

With these definitions, we can rewrite equation (25) as

$$\rho\,\theta\,u_{tt}^0(t) - \alpha_{ij}\,\frac{\partial^2 u^0(t)}{\partial x_i\,\partial x_j} - \beta_{ij}\,\frac{\partial}{\partial t}\,\frac{\partial^2 u^0(t)}{\partial x_i\,\partial x_j} - \int_0^t \gamma_{ij}(t-\sigma)\,\frac{\partial^2 u^0(\sigma)}{\partial x_i\,\partial x_j}\,d\sigma = \theta\,f(t)\ .$$

We then multiply by $\varphi \in V$ and integrate over Ω to obtain

$$\begin{aligned}(26)\quad &\langle \rho\,\theta\,u_{tt}^0(t), \varphi\rangle_{V^*,V} + \int_\Omega \alpha_{ij}\,\frac{\partial u^0(t)}{\partial x_j}\,\frac{\partial \varphi}{\partial x_i}\,dx + \\ &+ \int_\Omega \beta_{ij}\,\frac{\partial}{\partial t}\,\frac{\partial u^0(t)}{\partial x_j}\,\frac{\partial \varphi}{\partial x_i}\,dx + \int_\Omega\left(\gamma_{ij} * \frac{\partial u^0}{\partial x_j}\right)(t)\,\frac{\partial \varphi}{\partial x_i}\,dx = \langle \theta\,f(t), \varphi\rangle_{V^*,V}\ .\end{aligned}$$

Taking the Laplace transform of (26) we find

$$\langle \rho\,\theta\, s^2\,\widehat{u}^0(s),\,\varphi\rangle_H + \int_\Omega (\alpha_{ij} + s\,\beta_{ij} + \widehat{\gamma}_{ij}(s))\,\frac{\partial \widehat{u}^0(s)}{\partial x_j}\,\frac{\partial \varphi}{\partial x_i}\,dx =$$
$$= \langle \theta\,\widehat{f}(s), \varphi\rangle_{V^*,V} + \Big\langle \rho\,\theta(s\,u_0/\theta + v_0/\theta),\,\varphi\Big\rangle_H + \frac{1}{\theta}\int_\Omega \beta_{ij}\,\frac{\partial u_0}{\partial x_j}\,\frac{\partial \varphi}{\partial x_i}\,dx\ .$$

Since $\beta_{ij} = q^b_{ij}$, this equation will be the same as (15) if we can show that

$$\overline{q}_{ij}(s) = \alpha_{ij} + s\,\beta_{ij} + \widehat{\gamma}_{ij}(s)\ . \tag{27}$$

Observe that

$$\begin{aligned}\widehat{\gamma}_{ij}(s) &= \frac{1}{|Y|}\int_{Y^*}\int_0^\infty e^{-st}\Big[a_{il}\,\frac{\partial}{\partial y_l}(e^{-t\mathcal{A}} f_j) - b_{il}\,\frac{\partial}{\partial y_l}(\mathcal{A}\,e^{-t\mathcal{A}} f_j)\Big]\,dt\,dy\\ &= \frac{1}{|Y|}\int_{Y^*}\Big\{a_{il}\,\frac{\partial}{\partial y_l}\Big[(sI+\mathcal{A})^{-1} f_j\Big] - b_{il}\,\frac{\partial}{\partial y_l}\Big[\mathcal{A}(sI+\mathcal{A})^{-1} f_j\Big]\Big\}\,dy\\ &= \frac{1}{|Y|}\int_{Y^*}\Big\{a_{il}\,\frac{\partial}{\partial y_l}\Big[(sI+\mathcal{A})^{-1} f_j\Big] + b_{il}\,\frac{\partial}{\partial y_l}\Big[s(sI+\mathcal{A})^{-1} f_j - f_j\Big]\Big\}\,dy\ .\end{aligned}$$

Thus, by Lemma 3.1,

$$\begin{aligned}\alpha_{ij} + s\beta_{ij} + \widehat{\gamma}_{ij}(s) &= \frac{1}{|Y|}\int_{Y^*}\Big\{(a_{ij}+sb_{ij}) - (a_{il}+sb_{il})\,\frac{\partial}{\partial y_l}\Big[f^b_j - (sI+\mathcal{A})^{-1} f_j\Big]\Big\}\,dy\\ &= \frac{1}{|Y|}\int_{Y^*}\Big[(a_{ij}+sb_{ij}) - (a_{il}+sb_{il})\,\frac{\partial \chi^j(s)}{\partial y_l}\Big]\,dy\ .\end{aligned}$$

Hence, we have established equation (27), and so, by the uniqueness of the inverse Laplace transform, equation (25) (equivalently (26)) is the equation satisfied by u.

An important special case is when $a_{ij} = \kappa\,b_{ij}$ for all i, j where $\kappa \neq 0$ is a constant (this is the case for Kelvin–Voigt damping investigated in [2]). In this case

$$\langle \mathcal{A}\varphi, \psi\rangle_{\overline{V}_{Y^*}} = \kappa\langle \varphi, \psi\rangle_{\overline{V}_{Y^*}} \qquad \text{for all}\ \ \varphi, \psi \in \overline{V}_{Y^*}\ ,$$

which implies that $\mathcal{A} = \kappa\,I$. We also have

$$\langle f^a_j, \varphi\rangle_{\overline{V}_{Y^*}} = \kappa\langle f^b_j, \varphi\rangle_{\overline{V}_{Y^*}} \qquad \text{for all}\ \ \varphi \in \overline{V}_{Y^*}\ ,$$

so $f^a_j = \kappa\,f^b_j$. Hence, $f_j = 0$ which implies that $\alpha_{ij} = \kappa\,\beta_{ij}$ and $\gamma_{ij}(t) = 0$ for all t. Thus, the form of equation (25) simplifies to

$$\rho\,\theta\,u^0_{tt}(t) - \beta_{ij}\Big(\kappa + \frac{\partial}{\partial t}\Big)\,\frac{\partial^2 u^0(t)}{\partial x_i\,\partial x_j} = \theta\,f(t)\ .$$

4 – The limit in μ for constant coefficients

In this section we shall investigate the dependence of the homogenized coefficients on the parameter μ. Investigations of this nature were first carried out in [1], [7] and [13]. We began with a simple equation (system (3)) defined on the domain $\Omega_{\epsilon\mu}$ which has a complicated geometry. Taking the limit as $\epsilon \to 0$, we obtained a homogenized equation on all of Ω, but in order to obtain the coefficients, we must solve a differential equation on the representative cell. Since the parameter μ is small (compared to the dimensions of Ω), we next let $\mu \to 0$ and obtain a limit problem. We shall see that in this limit problem, we retrieve the simplicity in the coefficients, and in fact we shall compute them explicitly. In order to simplify the computations, we assume that the coefficients a_{ij} and b_{ij} are constant and that $l_1 = l_2 = 1$ (hence $\theta = 2\mu(1-\mu/2)$). We now display the dependence on μ. In particular, we write $\mathcal{A}_\mu$ for $\mathcal{A}$, α^μ_{ij} for α_{ij}, etc.

From systems (20) and (21) one can derive the a priori estimates

$$\|\nabla f^{a\mu}_j\|_{[L^2(Y^*_\mu)]^2} \leq C\,\mu^{1/2} \quad \text{and} \quad \|\nabla f^{b\mu}_j\|_{[L^2(Y^*_\mu)]^2} \leq C\,\mu^{1/2} \;.$$

Using the fact that $\mathcal{A}_\mu$ is bounded independently of μ, we also obtain

$$\|\nabla f^{\mu}_j\|_{[L^2(Y^*_\mu)]^2} \leq C\,\mu^{1/2} \;.$$

Thus, $\alpha^\mu_{ij} \leq C\,\mu$, $\beta^\mu_{ij} \leq C\,\mu$ and

$$\gamma^\mu_{ij}(0) = \frac{1}{|Y|}\int_{Y^*_\mu}\left[a_{il}\,\frac{\partial f^\mu_j}{\partial y_l} - b_{il}\,\frac{\partial(\mathcal{A}_\mu\, f^\mu_j)}{\partial y_l}\right] dy \leq C\,\mu \;.$$

Since $\mathcal{A}_\mu$ is positive, γ^μ_{ij} decays exponentially as $t \to \infty$, so $\gamma^\mu_{ij}(t) \leq C\,\mu$ for all t. Thus, as $\mu \to 0$,

$$\mu^{-1}\,\alpha^\mu_{ij} \to \alpha^*_{ij} \;,$$

$$\mu^{-1}\,\beta^\mu_{ij} \to \beta^*_{ij} \;,$$

and

$$\mu^{-1}\,\gamma^\mu_{ij}(t) \to \gamma^*_{ij}(t) \;,$$

with the last convergence being uniform in t on bounded intervals. Rather than seeking the limits α^*_{ij}, β^*_{ij} and $\gamma^*_{ij}(t)$ however, we treat the transformed equation (15) so that we can apply the results of [2] to obtain that

$$\mu^{-1}\,\overline{q}^\mu_{ij}(s) \to \overline{q}^*_{ij}(s) = 2(a_{ij} + s\,b_{ij}) - \sum_{l=1}^{2}\frac{(a_{il} + s\,b_{il})\,(a_{lj} + s\,b_{lj})}{a_{ll} + s\,b_{ll}}$$

and

$$\mu^{-1}\, q_{ij}^{b\mu} \to q_{ij}^{b*} = 2\, b_{ij} - \sum_{l=1}^{2} \frac{b_{il}\, b_{lj}}{b_{ll}} \ .$$

Observe that $\overline{q}_{ij}^{*}(s) = q_{ij}^{b*} = 0$ if $i \neq j$. Thus, multiplying equation (15) by μ^{-1} and letting $\mu \to 0$, we obtain

$$\langle 2\,\rho\, s^2\, \widehat{u}(s), \varphi\rangle_H + \int_\Omega \overline{q}_{ii}^{*}(s)\, \partial_i \widehat{u}(s)\, \partial_i\varphi\, dx = \\ = \langle 2\widehat{f}(s), \varphi\rangle_{V^*,V} + \langle \rho(s\, u_0 + v_0), \varphi\rangle_H + \frac{1}{2}\int_\Omega q_{ii}^{b*}\, \partial_i u_0\, \partial_i \varphi\, dx \ .$$

Since the coefficients appearing in this equation are rational functions in s, it is straightforward to invert the Laplace transform. The limit equation is

$$\begin{aligned} (28)\quad & \langle 2\,\rho\, u_{tt}(t), \varphi\rangle_{V^*,V} + \left(2\, a_{ii} - \sum_{l=1}^{2} \frac{a_{il}\, b_{li} + b_{il}\, a_{li}}{b_{ll}} + \sum_{l=1}^{2} \frac{a_{ll}\, b_{il}\, b_{li}}{b_{ll}^2}\right) \int_\Omega \partial_i u(t)\, \partial_i \varphi\, dx - \\ & - \sum_{l=1}^{2} \left\{ \left[\frac{a_{il}\, a_{li}}{b_{ll}} - \frac{a_{ll}(a_{il}\, b_{li} + b_{il}\, a_{li})}{b_{ll}^2} + \frac{a_{ll}^2\, b_{il}\, b_{li}}{b_{ll}^3} \right] \cdot \right. \\ & \left. \cdot \int_\Omega \int_0^t e^{a_{ll}(t-\sigma)/b_{ll}}\, \partial_i u(\sigma)\, \partial_i \varphi\, d\sigma\, dx \right\} + \\ & + \left(2\, b_{ii} - \sum_{l=1}^{2} \frac{b_{il}\, b_{li}}{b_{ll}}\right) \int_\Omega \partial_i u_t(t)\, \partial_i \varphi\, dx = \langle 2 f(t), \varphi\rangle_{V^*,V} \end{aligned}$$

with initial data $u(0) = u_0/2$ and $u_t(0) = v_0/2$.

For the special case that $a_{ij} = \kappa\, b_{ij}$, the convolution term vanishes, and the equation simplifies to

$$\langle 2\,\rho\, u_{tt}(t), \varphi\rangle_{V^*,V} + \left(2\, b_{ii} - \sum_{l=1}^{2} \frac{b_{il}\, b_{li}}{b_{ll}}\right) \left(\kappa + \frac{\partial}{\partial t}\right) \int_\Omega \partial_i u(t)\, \partial_i \varphi\, dx = \langle 2 f(t), \varphi\rangle_{V^*,V} \ .$$

REFERENCES

[**1**] Bakhvalov, N.S. and Panasenko, G.P. – *Averaged Processes in Periodic Media*, Nauka, Moscow, 1984.

[**2**] Banks, H.T., Cioranescu, D. and Rebnord, D.A. – Homogenization models for 2-D grid structures, *Asymptotic Analysis,* to appear.

[**3**] Banks, H.T., Ito, K. and Wang, Y. – Well-posedness and approximation for damped second order systems with unbounded input operators, *Diff. Int. Eqns.,* 8 (1995), 587–606.

[4] Bender, C.M. and Orszag, S.A. – *Advanced Mathematical Methods for Scientists and Engineers*, New York, McGraw-Hill, 1978.

[5] Bensoussan, A., Lions, J.L. and Papanicolaou, G. – *Asymptotic Analysis for Periodic Structures*, Amsterdam, North Holland, 1978.

[6] Cioranescu, D. and Donato, P. – Exact internal controllability in perforated domains, *J. Math. Pures et Appl.,* 68 (1989), 185–213.

[7] Cioranescu, D. and Saint Jean Paulin, J. – Homogenization in open sets with holes, *J. Math. Anal. Appl.,* 71(2) (1979), 590–607.

[8] Cioranescu, D. and Saint Jean Paulin, J. – Reinforced and honey-comb structures, *J. Math Pures et Appl.,* 65, (1986), 403–422.

[9] Francfort, G., Leguillon, D. and Suquet, P. – Homogénéisation de milieux viscoélastiques linéaires de Kelvin–Voigt, *C.R.A.S. Paris, Série I,* 296 (1983), 287–290.

[10] Francfort, G. and Suquet, P. – Homogenization and mechanical dissipation in thermoviscoelasticity, *Arch. Rat. Mech. Anal.,* 96 (1983), 285–293.

[11] Lions, J.L. and Magenes, E. – *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, New York, Springer-Verlag, 1972.

[12] Miller, R. – Extension theorems for homogenization on lattice structures, *Appl. Math. Lett.,* 5 (1992), 73–78.

[13] Panasenko, C.P. – The principle of splitting an averaged operator for a nonlinear system of equations in periodic and random skeletal structures, *Soviet Math. Dokl.,* 25 (1982), 290–295.

[14] Sanchez-Palencia, E. – *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, 127, Berlin, Springer-Verlag, 1980.

[15] Sanchez-Palencia, E. and Zaoui, A., eds. – *Homogenization Techniques for Composite Media*, Lecture Notes in Physics, 272, Berlin, Springer-Verlag, 1987.

[16] Tartar, L. – *Quelques Remarques sur l'Homogeneisation*, Functional Analysis and Numerical Analysis, Proc. Japan-France Seminar 1976 (Fujita, ed.), Japanese Soc. for the Promotion of Science, 468–482.

[17] Wloka, J. – *Partial Differential Equations*, Cambridge University Press, 1987.

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