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A CRITERION OF IRRATIONALITY

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Abstract: We generalize P. Gordan's proof of the transcendence of e ([3]; [5], p. 170), and obtain a criterion of irrationality (Theorem 1 below). Using this criterion, we can prove the irrationality of $f(z) = 1 + \sum_{n=1}^{+\infty} \frac{z^n}{v_1\, v_2 \cdots v_n\, q^{n(n+1)/2}}$, when z, q and v_n satisfy suitable hypotheses (see Theorem 2).

Résumé: Nous généralisons la démonstration de la transcendance de e par P. Gordan ([3]; [5], p. 170), pour obtenir un critère d'irrationalité (Théorème 1 ci-après). Nous en donnons une application en prouvant l'irrationalité de $f(z) = 1 + \sum_{n=1}^{+\infty} \frac{z^n}{v_1\, v_2 \cdots v_n\, q^{n(n+1)/2}}$, lorsque z, q et v_n vérifient des hypothèses convenables (voir le Théorème 2).

1 – Notations

Let $U = \{u_1, u_2, ..., u_n, ...\}$ be a sequence of non-zero complex numbers. We put $\mathcal{U}^0 = 1$ and:

$$\forall\, n \in \mathbb{N} - \{0\} \colon \mathcal{U}^n = u_1 \cdot u_2 \cdots u_n \qquad \mathcal{U}^{-n} = [\mathcal{U}^n]^{-1} \tag{1}$$

Consider the complex function f defined by

$$\boxed{f(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\mathcal{U}^n}}\,. \tag{2}$$

We assume this series to fulfil d'Alembert's criterion.

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A straightforward computation shows that

$$\limsup_{k\in\mathbb{N}}\Big(\sum_{i=k+1}^{+\infty}|u_{k+1}^{-1}|\cdots|u_i^{-1}|\,|z|^{i-k}\Big)<\infty\ ,$$

and we put

$$\boxed{Mf(z)=\sup_{k\in\mathbb{N}}\sum_{i=k+1}^{+\infty}|u_{k+1}^{-1}|\cdots|u_i^{-1}|\,|z|^{i-k}}\ . \tag{3}$$

The sequence U being given, we define the U-Newton's binomial, for complex variables X and Y, by

$$\boxed{(X\oplus Y)^n=\sum_{k=0}^{n}U_n^k\,X^k\,Y^{n-k}} \tag{4}$$

where

$$\boxed{U_n^k=\mathcal{U}^n\,\mathcal{U}^{-k}\,\mathcal{U}^{k-n}}\ . \tag{5}$$

Now let P be a polynomial with complex coefficients

$$P(X)=\sum_{p=0}^{n}a_p\,X^p\ .$$

We put

$$\boxed{P(\mathcal{U})=\sum_{p=0}^{n}a_p\,\mathcal{U}^p}\ , \tag{6}$$

$$\boxed{P(X\oplus Y)=\sum_{p=0}^{n}a_p(X\oplus Y)^p}\ , \tag{7}$$

$$\boxed{P(\mathcal{U}\oplus z)=\sum_{p=0}^{n}a_p(\mathcal{U}\oplus z)^p=\sum_{p=0}^{n}a_p\ \sum_{k=0}^{p}\mathcal{U}^p\,\mathcal{U}^{k-p}\,z^{p-k}}\ , \tag{8}$$

$$\boxed{|P|(X)=\sum_{p=0}^{n}|a_p|\,X^p}\ . \tag{9}$$

One sees that, in fact, a number or a variable can be identified with a constant sequence, and that the ordinary exponentiation is a special case of (1).

2 – Criterion of irrationality

Theorem 1. *Let $K = \mathbb{Q}$ or $\mathbb{Q}[i\sqrt{d}]$. Let A be the ring of the integers of K. Let $f(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\mathcal{U}^n}$ and $a, b \in K$. Assume that there exists $P \in K[X]$, such that*

$$a\,P(\mathcal{U}) + b\,P(\mathcal{U} \oplus z) \;\in\; A - \{0\}\ , \tag{10}$$

$$|b| \cdot Mf(z) \cdot |P|(|z|) < 1\ . \tag{11}$$

Then $a + b\,f(z) \neq 0$.

Proof: An easy computation shows that

$$\mathcal{U}^k\,f(z) = (\mathcal{U} \oplus z)^k + \mathcal{U}^k \sum_{i=k+1}^{+\infty} \mathcal{U}^{-i}\,z^i\ . \tag{12}$$

Let $P(X) = \sum_{k=0}^{N} a_k\,X^k$. From (12) we get at once

$$P(\mathcal{U})\,f(z) = P(\mathcal{U} \oplus z) + \sum_{k=0}^{N} a_k\,\mathcal{U}^k \sum_{i=k+1}^{+\infty} \mathcal{U}^{-i}\,z^i\ .$$

Suppose that $a + b\,f(z) = 0$. Then $a\,P(\mathcal{U}) + b\,P(\mathcal{U})\,f(z) = 0$, whence

$$a\,P(\mathcal{U}) + b\,P(\mathcal{U} \oplus z) + b \sum_{k=0}^{N} a_k\,\mathcal{U}^k \sum_{i=k+1}^{+\infty} \mathcal{U}^{-i}\,z^i = 0\ .$$

Therefore

$$\Big|a\,P(\mathcal{U}) + b\,P(\mathcal{U} \oplus z)\Big| \le |b| \sum_{k=0}^{N} |a_k|\,|z|^k \sum_{i=k+1}^{+\infty} |u_{k+1}^{-1}| \cdots |u_i^{-1}|\,|z|^{i-k}\ .$$

Hence, using (3) and (11), we get

$$\Big|a\,P(\mathcal{U}) + b\,P(\mathcal{U} \oplus z)\Big| \le |b| \cdot Mf(z) \cdot |P|(|z|) < 1\ .$$

But this is impossible, because $x \in A$ and $|x| < 1 \Rightarrow x = 0$ ([7], Th. 2-1, p. 46). Contradiction with (10). ∎

3 – U-derivation

Definition 1. Let $f(X) = \sum_{n\geq 0} a_n X^n$ be a formal series with complex coefficients, and let $U = \{u_1, u_2, ..., u_n, ...\}$ be a sequence of complex numbers. The U-derivative of f is the formal series defined by:

$$\partial_U f(X) = \sum_{n\geq 1} a_n\, u_n\, X^{n-1} .$$

Proposition 1. *Let P be a polynomial of degree N. Then:*

$$\boxed{P(X\oplus z) = P(z) + \frac{\partial_U P(z)}{\mathcal{U}^1}\, X + \frac{\partial_U^2 P(z)}{\mathcal{U}^2}\, X^2 + \cdots + \frac{\partial_U^N P(z)}{\mathcal{U}^N}\, X^N} .$$

Proof: Just the same as the usual Taylor's formula (see [2]). ∎

Corollary 1. *Let P be a polynomial of degree $N \geq n$. Then $P(X\oplus z)$ has valuation at least n if, and only if:*

$$\boxed{P(z) = \partial_U P(z) = \cdots = \partial_U^{n-1} P(z) = 0} .$$

Moreover, in that case:

$$\boxed{P(\mathcal{U}\oplus z) = \sum_{k=n}^{N} \partial_U^k P(z)} .$$

4 – An application

Theorem 2. *Let $K = \mathbb{Q}$ or $\mathbb{Q}[i\sqrt{d}]$. Let A be the ring of the integers of K. Let $m \in A$, $|m| > 1$. Let $V = \{v_1, v_2, ..., v_n, ...\}$ be a sequence of elements of A, with the following properties:*

a) *$|v_n| = \exp(o(n))$;*

b) *There exists an infinite subset $P = \{\mathcal{B}_1, \mathcal{B}_2, ...\}$ of the set of the prime ideals of A, and a sequence $N = \{n_1, n_2, ...\}$ of rational integers, such that $v_{n_i} \in \mathcal{B}_i$ for each i, and $v_n \notin \mathcal{B}_i$ if $n < n_i$.*

c) *For every $q \in \mathbb{N}^*$, there exists infinitely many $n_i \in N$ such that $v_n \notin \mathcal{B}_i$ for $n_i < n \leq n_i + q$.*

Let $f(z) = 1 + \sum_{n=1}^{+\infty} \frac{z^n}{v_1\, v_2 \cdots v_n\, m^{\frac{n(n+1)}{2}}}$.
Then, if $z \in K^*$, $f(z) \notin K$.

Remark. By elementary considerations one can prove the irrationality of $f(z)$ in the case where $z \in A - \{0\}$, with $|z| < |m|$ (see [8], Theorem 1).

Corollary 2. *Let* $m \in A$, $|m| > 1$ *and* $h \in \mathbb{N} - \{0\}$. *Then, if* $z \in K^*$,

$$\sum_{n=0}^{+\infty} \frac{z^n}{(n!)^h\, m^{\frac{n(n+1)}{2}}} \notin K \ .$$

Corollary 2 is a well-known result; see [9], [1], [5]. On the other hand, the following result seems to be new:

Corollary 3. *Let* $m \in A$, $|m| > 1$. *Let* $p_1, p_2, ..., p_n, ...$ *be the sequence of the prime numbers in* $\mathbb{N}$. *Then, if* $z \in K^*$,

$$\sum_{n=1}^{+\infty} \frac{z^n}{p_1\, p_2 \cdots p_n\, m^{\frac{n(n+1)}{2}}} \notin K \ .$$

It is likely that, if $z \in K^*$, $\sum_{n=0}^{+\infty} \frac{z^n}{p_1\, p_2 \cdots p_n} \notin K$, but it is surely much more difficult to prove.

The proof of Theorem 2 rests on four lemmas; the proofs of lemmas 1 and 3 are elementary, and omitted.

Lemma 1. *For every* $h \in \mathbb{N}^*$, *let*

$$f_h(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\mathcal{U}_h^n} \ , \qquad z \in A - \{0\} \ ,$$

where $u_{n,h} = u_{n+h}$ *and* $\mathcal{U}_h^n = u_{1,h} \cdot u_{2,h} \cdots u_{n,h}$.

If there exists $a \in A$ *and* $b \in A - \{0\}$ *such that* $a + b\, f(z) = 0$, *then:*

$$a_h + b_h\, f_h(z) = 0 \ , \qquad \forall\, h \in \mathbb{N}^* , \qquad \textit{with:} \tag{13}$$

$$a_h = \mathcal{U}^h \Big(a + b \sum_{n=0}^{h-1} \frac{z^n}{\mathcal{U}^n} \Big) \ \in \ A \ , \tag{14}$$

$$b_h = b\, z^h \ \in \ A - \{0\} \ . \tag{15}$$

Lemma 2. *Suppose that all the u_i's lie in A. Let $\mathcal{B}$ be a prime ideal of A, such that $\mathcal{U}^{h+1} \notin \mathcal{B}$, $b \notin \mathcal{B}$ and $z \notin \mathcal{B}$. Then $a_h \notin \mathcal{B}$, or $a_{h+1} \notin \mathcal{B}$.*

Proof of Lemma 2: If $a_h \in \mathcal{B}$ and $a_{h+1} \in \mathcal{B}$, as $u_{h+1} \notin \mathcal{B}$, we have:

$$\mathcal{U}^h \left(a + b \sum_{n=0}^{h-1} \frac{z^n}{\mathcal{U}^n} \right) \in \mathcal{B} \quad \text{and} \quad \mathcal{U}^h \left(a + b \sum_{n=0}^{h} \frac{z^n}{\mathcal{U}^n} \right) \in \mathcal{B} \ .$$

Subtracting these two numbers, we get $b\, z^h \in \mathcal{B}$, a contradiction. ∎

Lemma 3. *Let $P_n(X) = X^{n-1} \sum_{k=0}^{n} \Gamma_n^k \, z^{n-k} \, X^k$. Then*

$$P_n(z) = \partial_U P_n(z) = ... = \partial_U^{n-1} P_n(z) = 0$$

if, and only if, the Γ_n^k's are solution of the system:

$$\begin{cases} \Gamma_n^0 + \Gamma_n^1 + \cdots + \Gamma_n^n = 0 \\ u_{n-1}\, \Gamma_n^0 + u_n\, \Gamma_n^1 + \cdots + u_{2n-1}\, \Gamma_n^n = 0 \\ \vdots \\ u_{n-1} \cdots u_1\, \Gamma_n^0 + u_n \cdots u_2\, \Gamma_n^1 + \cdots + u_{2n-1} \cdots u_{n+1}\, \Gamma_n^n = 0 \ . \end{cases}$$

Lemma 4. *Let $M = (\alpha_{ij})$, $1 \le i \le n$, $1 \le j \le n+1$, be a matrix with coefficients in A. Then the system*

$$M \cdot \begin{pmatrix} \Gamma_n^0 \\ \Gamma_n^1 \\ \vdots \\ \Gamma_n^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{16}$$

admits a solution $(\Gamma_n^0, \Gamma_n^1, ..., \Gamma_n^n)$ such that:

$$\Gamma_n^i \in A \quad \text{for} \ \ i = 0, 1, ..., n \ . \tag{17}$$

$$0 < \max |\Gamma_n^i| \le n^{\frac{n}{2}} H^n, \quad \text{with} \ \ H = \max |\alpha_{ij}| \ . \tag{18}$$

Moreover, if $\mathcal{B}$ is a prime ideal of A and $\alpha_{ij} \in \mathcal{B}$ for every (i,j) such that $2 \le j \le i$, while $\alpha_{j-1,j} \notin \mathcal{B}$ for every $j \in \{2, ..., n+1\}$, then $\Gamma_n^0 \notin \mathcal{B}$.

Proof of Lemma 4: It is a well-known result of elementary linear algebra, that the system (16) admits for solution

$$\Gamma_n^k = (-1)^k \, \Delta_{n,k} \, , \qquad 0 \le k \le n \ ,$$

where $\Delta_{n,k}$ is the determinant one obtains by canceling the $(k+1)$-th column of M. Hence (17) is trivial, and (18) is Hadamard's upper bound for the module of a determinant [3].

The second part of the lemma results of the fact that we have only zeroes (modulo $\mathcal{B}$) under the diagonal of $\Delta_{n,0}$, while the terms on the diagonal are non zero (modulo $\mathcal{B}$). ∎

Proof of Theorem 2: We can suppose that $z \in A$, as otherwise we may replace z by $Nz \in A$ and v_n by $v_n N$ with a suitable rational integer N. Put $u_n = v_n m^n$ and define Γ_n^k as a solution of the system

$$\begin{cases} \Gamma_n^0 + \Gamma_n^1 + \cdots + \Gamma_n^n = 0 \\ v_{n-1+h}\,\Gamma_n^0 + v_{n+h}\,m\,\Gamma_n^1 + \cdots + v_{2n-1+h}\,m^n\,\Gamma_n^n = 0 \\ \vdots \\ v_{n-1+h}\cdots v_{1+h}\,\Gamma_n^0 + v_{n+h}\cdots v_{2+h}\,m^{n-1}\,\Gamma_n^1 + \cdots + \\ \qquad + \cdots + v_{2n-1+h}\cdots v_{n+1+h}\,m^{(n-1)n}\,\Gamma_n^n = 0\ , \end{cases}$$

with $h > n$ which satisfies

$$|\Gamma_n^k| \le n^{\frac{n}{2}}\,|m|^{n^3}(L_{3h})^{n^2}\ , \tag{19}$$

where

$$L_n = \max_{1\le i\le n} |v_i|\ . \tag{20}$$

The existence of such solutions follows from Lemma 4.

Suppose $a + b\,f(z) = 0$ with $(a,b) \in A^2$, and put

$$P_{h,n}(X) = \frac{X^{n-1}}{m^{(n-1)h}} \sum_{k=0}^{n} \Gamma_n^k\, z^{n-k}\, X^k\ . \tag{21}$$

To be able to apply Theorem 1, we have to obtain an upper bound for $|b_h|\cdot M(f_h)\cdot |P_{h,n}|(|z|)$. It is easy to see that $M(f_h) \le B$, where $B = \sum_{n=0}^{+\infty} |z|^n\,|m|^{-\frac{n(n+1)}{2}}$. Hence, using (19), we get

$$|b_h|\cdot M(f_h)\cdot |P_{h,n}|(|z|) \le |b|\,|z|^h\,B\,\frac{|z|^{2n-1}}{|m|^{(n-1)h}}\,(n+1)\,n^{\frac{n}{2}}\,|m|^{n^3}(L_{3h})^{n^2}\ .$$

But from a) it results that $L_{3h} = \exp(h\,\varepsilon(h))$, with $\lim_{h\to\infty}\varepsilon(h) = 0$, and we get

$$|b_h|\cdot M(f_h)\cdot |P_{h,n}|(|z|) \le |b|\,B|z|^{2n-1}\,(n+1)\,n^{\frac{n}{2}}\,|m|^{n^3}\left[\frac{|z|}{|m|^{(n-1)}}\exp(n^2\,\varepsilon(h))\right]^h\ .$$

Let us choose n such that $\frac{|z|}{|m|^{(n-2)}} \le \frac{1}{2}$.

Such an n being fixed, there exists $h_0 \in \mathbb{N}$ such that

$$h \ge h_0 \implies |b_h| \cdot M(f_h) \cdot |P_{h,n}|(|z|) < 1 \ . \tag{22}$$

We choose h such that $h > n$ and $n + h = n_i$, n_i fulfilling *the two conditions* b) and c) with $q = n$. Therefore, using Lemma 4, we get $\Gamma_n^0 \notin \mathcal{B}_i$. It is clear that we can suppose $z \notin \mathcal{B}_i$, $m \notin \mathcal{B}_i$, and $b \notin \mathcal{B}_i$, by choosing n_i large enough. We can also suppose that $a_h \notin \mathcal{B}_i$ (otherwise we replace n by $n-1$, h by $h+1$, and use Lemma 2). For this choice of n and h condition (11) is fulfilled by (22).

Let us verify condition (10). We have

$$P_{n,h}(\mathcal{U}_h) = \frac{1}{m^{(n-1)h}} \sum_{k=0}^{n} \Gamma_n^k \, z^{n-k} \, \mathcal{U}_h^{n+k-1} \ .$$

But $u_{h,n}$ is always divisible by m^h, whence $P_{n,h}(\mathcal{U}_h) \in A$. Moreover, the term corresponding to $k = 0$ does not lie in $\mathcal{B}_i$ (hypothesis b)), while the other terms lie in $\mathcal{B}_i$ (they contain $v_{n+h} = v_{n_i}$). Therefore $P_{n,h}(\mathcal{U}_h) \notin \mathcal{B}_i$.

Denote by $(X \oplus Y)^n$ the U_h-Newton's binomial, and use Corollary 1. We get

$$P_{n,h}(\mathcal{U}_h \oplus z) = \frac{1}{m^{h(n-1)}} \sum_{k=n}^{2n-1} \partial_{U_h}^k Q_{n,h}(z) \ ,$$

where $Q_{n,h}(X) = X^{n-1} \sum_{j=0}^{n} \Gamma_n^j \, z^{n-j} \, X^j \in A[X]$.

But each $\partial_{U_h}^k Q_{n,h}(z)$ contains the product of at least n consecutive terms of the sequence U_h, including u_{n+h}. Hence $P_{n,h}(\mathcal{U}_h \oplus z) \in \mathcal{B}_i$.

As $a_h \notin \mathcal{B}_i$, $P_{n,h}(\mathcal{U}_h) \notin \mathcal{B}_i$ and $P_{n,h}(\mathcal{U}_h \oplus z) \in \mathcal{B}_i$, we have

$$a_h P_{n,h}(\mathcal{U}_h) + b_h P_{n,h}(\mathcal{U}_h \oplus z) \notin \mathcal{B}_i \ .$$

Therefore condition (10) is fulfilled, and the proof of Theorem 2 is complete. ∎

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