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QUASILINEAR ELLIPTIC PROBLEMS WITH NONMONOTONE DISCONTINUITIES AND MEASURE DATA

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Abstract: We prove the existence of weak solutions to the mixed boundary value problem for a quasilinear elliptic second order equation involving nonmonotone discontinuous terms and with measure data in the equation and in the Neumann boundary condition.

1 – Introduction

In this paper, we prove the existence of at least one solution to the following quasilinear elliptic mixed boundary value problem

$$\left\{\begin{aligned} -\operatorname{div}\mathbf{a}(x,u,Du)+\beta(x,u) &\ni \mu, && \text{in } \Omega,\\ \mathbf{a}(x,u,Du)\cdot\mathbf{n}+\gamma(x,u) &\ni \nu, && \text{on } \Gamma,\\ u &= 0, && \text{on } \Gamma_0\ , \end{aligned}\right. \tag{1.1}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, with the relatively open components Γ_0 and Γ, such that $\overline{\Gamma}\cup\overline{\Gamma}_0=\partial\Omega$, $\Gamma\cap\Gamma_0=\emptyset$, and $\operatorname{meas}_{N-1}(\Gamma_0)>0$, and $\mathbf{n}$ denotes the unit vector normal to Γ.

We consider second order quasilinear operators $Au=-\operatorname{div}\mathbf{a}(x,u,Du)$ of Leray–Lions type, coercive in

$$W^{1,p}_{\Gamma_0}(\Omega)=\Big\{v\in W^{1,p}(\Omega)\colon\ v|_{\Gamma_0}=0\Big\} \tag{1.2}$$

for $2-\frac{1}{N}<p\le N$, with given bounded measures $\mu\in M(\Omega)$ and $\nu\in M(\Gamma)$.

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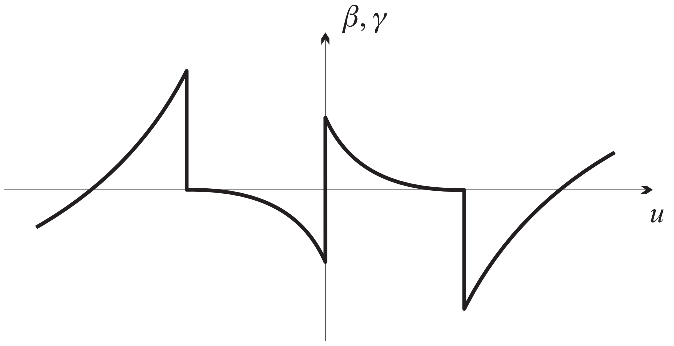
The nonlinear perturbation β and γ may be nonmonotone and discontinuous in u, but they must satisfy at infinity a sign condition as well as a suitable growth condition. We recall that even continuous nonlinearities in equations involving measures may have *no solutions*, in the sense of distributions, if a certain growth at infinity of β is not satisfied (see [6]). We conjecture here that a similar fact holds on the boundary nonlinearity with the growth of γ.

The solution u is found in $W^{1,q}_{\Gamma_0}(\Omega)$ for $\forall\, q < \frac{N}{N-1}(p-1)$ as in the previous works of Boccardo, Gallouët and Rakotoson on quasilinear elliptic Dirichlet problems with measure as data [4], [22] and [5] (i.e. in case $\Gamma = \emptyset$). The novelty here is the extension to the case of nonmonotone discontinuous nonlinearities and also the Neumann-type boundary condition on Γ, covering examples from the applications, where β and/or γ may have graphs of the following type with positive and/or negative jumps including multiple valued nonlinearities (see [19], [20] for instance).



For the case in which the nonlinearities β and γ are monotone nondecreasing, previous results of existence and uniqueness of the solutions with L^1 data were obtained by Brézis and Strauss in [8] for semilinear equations in the framework of accretive operators (see also [13], Chap. 4) and by Bénilan, Crandall and Sacks in [3] for a semilinear Neumann problem, where the continuous dependence of the solutions on the maximal monotone nonlinearities was also shown. Extensions of these results have been also obtained for the Dirichlet problem of a semilinear elliptic equation involving measures and monotone β by Attouch, Bouchitte and Mabrouk [1].

As it is well known, nonlinear discontinuities are strongly related to some free boundary problems arising in specific models of mathematical physics and have been treated by several authors with different methods. We refer the method of smoothing the "filled jumps" [24], used by Rauch, the order method of sub and super-solutions considered by Stuart [26] and Carl [9], the method of fixed points of set valued mappings, developed by Chang [10] or variational methods, also used by this author in [11], and by Ambrosetti and Turner, by means of a dual variational principle, in [2].

In particular, Chang has given in [10] sufficient conditions in order that a solution of inclusions in (1.1) are in fact solutions of the respective equations almost everywhere. Since this property requires a regularity property of the type $D^2u \in L^1_{\text{loc}}(\Omega)$ and $Du \in L^1(\Gamma)$, which cannot be expected in the case of measure data, we need to consider solutions of (1.1) in the generalized sense. We observe that, under certain regularity of the solutions, these problems may be regarded as variational ones, like in [19], [20] or [16], but in the general situation, in particular when measure data are considered, the weak formulation is necessary. However they are well adapted to the numerical approximation, as indicated in [17] for a special case.

We introduce the definition of generalized solution and we state our main theorem in Section 2. Our method of proof is based on the regularization approach and is explained in Section 3. It relies on an extended *a priori* estimate of Du in $L^q(\Omega)$, which is obtained with the truncation technique used in [25], [4] and [14], and also on a compactness lemma adapted from [5]. The details are given in Section 4.

Finally, let us remark that we consider here mainly the case $2 - \frac{1}{N} < p \leq N$, since in the easier case $p > N$, we have $W^{1,p}_{\Gamma_0}(\Omega) \subset C^0(\overline{\Omega})$ and the linear form T, given by

$$\langle T, v\rangle = \int_\Omega v\, d\mu + \int_\Gamma v\, d\nu \tag{1.3}$$

lies in the dual space of $W^{1,p}_{\Gamma_0}(\Omega)$; the variational theory of Leray-Lions can then be applied more directly (see [15], [6] or [13], and Remark on $p > N$ in the next section). On the other hand, the more delicate case $1 < p < 2 - \frac{1}{N}$ requires a new type of sets and of generalized solutions (see [23], for recent results in this direction).

After the completion of this work the papers [12] and [21] have been brought to our attention. However, they deal with different problems, namely in [12], the homogeneous Dirichlet problem is treated with Carathéodory functions, and in [21] the classical nonhomogeneous boundary value problems for A are solved only in the cases $\beta \equiv 0$ and $\gamma \equiv 0$ or $\gamma(u) = \lambda u$, with $\lambda > 0$ and $\Gamma_0 = \emptyset$.

2 – Assumptions and the existence result

We assume in the quasilinear elliptic operator A the vector valued function $\mathbf{a} = \mathbf{a}(x,s,\xi)\colon\ \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e. measurable in x and continuous in (s,ξ)) and has the standard properties of coercivity, strict monotonicity and growth:

$$\text{(H1)}\qquad \begin{cases} \text{i)} & \mathbf{a}(x,s,\xi)\cdot\xi \geq \alpha\,|\xi|^p\ , \\ \text{ii)} & \Big[\mathbf{a}(x,s,\xi)-\mathbf{a}(x,s,\eta)\Big]\cdot(\xi-\eta) > 0\,, \qquad \xi\neq\eta\ , \\ \text{iii)} & |\mathbf{a}(x,s,\xi)| \leq C\Big(h(x)+|s|^{p-1}+|\xi|^{p-1}\Big)\ , \end{cases}$$

where α, C are positive constants, $h \in L^{p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$, $2-\frac{1}{N}<p\leq N$, and (H1) holds for any $s\in\mathbb{R}$, $\xi,\eta\in\mathbb{R}^N$ and for a.e. $x\in\Omega$.

We assume that $\beta=\beta(x,s)\colon\ \Omega\times\mathbb{R}\to\mathbb{R}$, and $\gamma=\gamma(x,s)\colon\ \Gamma\times\mathbb{R}\to\mathbb{R}$, are measurable functions such that, for a.e. x (a.e. in Ω means with respect to the Lebesgue N-measure dx and a.e. in Γ means with respect to the $(N-1)$-surface measure $d\sigma$) they are locally bounded in s and therefore may have discontinuities. For a.e. $x\in\Omega$ and for any $\delta>0$, we let

$$\text{(2.1)}\qquad \overline{\beta}^{\delta}(x,s) = \operatorname*{ess\,sup}_{|\theta-s|\leq\delta}\beta(x,\theta) \quad\text{ and }\quad \underline{\beta}^{\delta}(x,s) = \operatorname*{ess\,inf}_{|\theta-s|\leq\delta}\beta(x,\theta)\ ,$$

and for fixed (x,s), $\overline{\beta}^{\delta}$ is a decreasing function of δ, $\underline{\beta}^{\delta}$ is an increasing function of δ. We let

$$\text{(2.2)}\qquad \overline{\beta}(x,s) = \lim_{\delta\to0}\overline{\beta}^{\delta}(x,s)\,,\qquad \underline{\beta}(x,s) = \lim_{\delta\to0}\underline{\beta}^{\delta}(x,s)\ ,$$

and we define the multivalued map in s (for a.e. $x\in\Omega$),

$$\text{(2.3)}\qquad (x,s)\mapsto\widehat{\beta}(x,s) = [\underline{\beta}(x,s),\overline{\beta}(x,s)]\ .$$

With analogous definitions for γ, for a.e. $x\in\Gamma$, we set

$$\text{(2.4)}\qquad (x,s)\mapsto\widehat{\gamma}(x,s) = [\underline{\gamma}(x,s),\overline{\gamma}(x,s)]\ ,$$

and we note that $\widehat{\beta}$ and $\widehat{\gamma}$ are the graphs of β and γ, respectively, with the "jumps filled in" following [24].

We shall assume on β and γ the following assumptions respectively, a N-measurability condition (according to [10], $\phi(x,s):\ X\times\mathbb{R}\to\mathbb{R}$ is called N-measurable if for every measurable real function $u:\ X\to\mathbb{R}$, the function $\phi(x,u(x))$ is measurable on X), a growth condition and a "ultimately" increasing condition as in [24]:

$$\text{(H2)} \quad \begin{cases} \text{i)} & \text{The functions } \overline{\beta}(x,s) \text{ and } \underline{\beta}(x,s) \text{ are } N\text{-measurable ,} \\ \text{ii)} & |\beta(x,s)| \le e_1\,|s|^{\rho_1} + c_1, \quad \text{for } \; 0 \le \rho_1 < \frac{N(p-1)}{N-p} \;, \\ \text{iii)} & \operatorname*{ess\,sup}_{s\le -t_*} \beta(x,s) \le \operatorname*{ess\,inf}_{s\ge t_*} \beta(x,s), \quad \text{for a.e. } \; x \in \Omega \;, \end{cases}$$

$$\text{(H3)} \quad \begin{cases} \text{i)} & \text{The functions } \overline{\gamma}(x,s) \text{ and } \underline{\gamma}(x,s) \text{ are } N\text{-measurable ,} \\ \text{ii)} & |\gamma(x,s)| \le e_0\,|s|^{\rho_0} + c_0, \quad \text{for } \; 0 \le \rho_0 < \frac{(N-1)\,(p-1)}{N-p} \;, \\ \text{iii)} & \operatorname*{ess\,sup}_{s\le -t_*} \gamma(x,s) \le \operatorname*{ess\,inf}_{s\ge t_*} \gamma(x,s), \quad \text{for a.e. } \; x \in \Gamma \;, \end{cases}$$

for a fixed $t_* > 0$, where c_0, e_0, c_1, e_1 are positive constants. If $p \ge N$, ρ_0 and ρ_1 may be any positive numbers.

We observe that if $\beta(x,s)$ or $\gamma(x,s)$ do not depend on x, as in [24], then the function $\overline{\beta}(x,s)$ and $\overline{\gamma}(x,s)$ (resp. $\underline{\beta}(x,s)$ or $\underline{\gamma}(x,s)$) are upper (resp. lower) semicontinuous and the conditions (H2-i) or (H3-i) are automatically satisfied.

Conjecture and Remark: We observe that the growth condition (H2-ii) should be optimal for measure data, since it is well known that even in very special cases (for instance, if $p = 2$ and $A = -\Delta$, $\beta(u) = |u|^{\sigma-1}\,u$, $\sigma \ge N/(N-2)$, $N \ge 3$ and $\mu = \delta_{x_0}$, $x_0 \in \Omega$, $\Gamma = \emptyset$), there are *no weak solutions* if (H2-ii) is not imposed (see [6]). Analogously, the growth condition at the boundary (H3-ii), by Sobolev inequalities for traces, is also conjectured to be optimal.

Finally, we assume the nonhomogeneous terms given by

$$\text{(H4)} \qquad \mu \in M(\Omega) \quad \text{and} \quad \gamma \in M(\Gamma) \;,$$

where $M(\Omega)$ and $M(\Gamma)$ denote the space of bounded Radon measures on Ω and Γ, respectively.

For any p, $1 \le p \le \infty$, we use the Sobolev subspace $W^{1,p}_{\Gamma_0}(\Omega)$ given by (1.2) in the following definition

Definition. We say that a triple $(u,b,g) \in W^{1,1}_{\Gamma_0}(\Omega) \times L^1(\Omega) \times L^1(\Gamma)$ is a weak solution of the quasilinear mixed boundary value problem (1.1), if $\mathbf{a}(x,u,Du) \in [L^1(\Omega)]^N$, and

$$\text{(2.5)} \quad b(x) \in \widehat{\beta}(x,u(x)), \;\; \text{a.e. in } \Omega\,, \qquad g(x) \in \widehat{\gamma}(x,u(x)) \;\; \text{a.e. on } \Gamma \;,$$

$$\text{(2.6)} \quad \int_\Omega \mathbf{a}(x,u,Du)\cdot Dv\,dx + \int_\Omega b\,v\,dx + \int_\Gamma g\,v\,d\sigma = \\ = \int_\Omega v\,d\mu + \int_\Gamma v\,d\nu, \quad \text{for } \; \forall\, v \in W^{1,\infty}_{\Gamma_0}(\overline{\Omega}) \;.$$

Main Theorem. *Under the preceding assumptions, namely (H1)–(H4), there exists a weak solution u of (1.1), such that*

$$u \in W^{1,q}_{\Gamma_0}(\Omega) \quad \textit{for any} \ \ q < q^* \equiv \frac{N}{N-1}\,(p-1) \ .$$

Remark on p>N. The existence result still holds and the weak solution $u \in W^{1,p}_{\Gamma_0}(\Omega) \subset C^0(\overline{\Omega})$. In this case the growth conditions (H2-ii), (H3-ii) have no restrictions on ρ_0 and ρ_1, which may be any real positive numbers.

Remark on the Neumann problem. For the case $\beta \equiv 0$, $\gamma \equiv 0$ and $\Gamma_0 = \emptyset$ (here $\Gamma = \partial\Omega$), Prignet in [21] has shown the existence of a solution $u \in \bigcap_{q<q^*} W^{1,q}(\Omega)$, $\int_\Omega u = 0$, under the compatibility condition $\int_\Omega d\mu + \int_{\partial\Omega} d\nu = 0$. However, in general if $\beta \neq 0$ or $\gamma \neq 0$, the existence of a solution seems to be an open problem except if we assume an additional assumption of the type

$$\beta(u)\,u \geq \lambda\,|u|^\sigma \quad (\text{or } \gamma(u)\,u \geq \lambda\,|u|^\sigma) \quad \text{ with } \ \lambda > 0 \ \text{ and } \ \sigma \geq 1 \ ,$$

which provides a kind of coerciveness (see, for instance, also [21] for the special case $\gamma(u) = \lambda u$, $\lambda > 0$) even in the case where $\Gamma_0 = \emptyset$.

Remark on an additional nonlinearity in the gradient. Analogously to [12], it is possible to extend the above existence results to the case of nonlinear equations of the type

$$-\mathrm{div}\,\mathbf{a}(x,u,Du) + K(x,Du) + \beta(x,u) \ \ni \ \mu \ ,$$

where $K(x,\xi)\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition $K(x,\xi) \leq k(x)\,|\xi|^{p-1}$, $\forall\,\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$, with $k \in L^p(\Omega)$ with $r > N \geq p$ or $r > p > N$. Actually in [12], it is only considered the case $\Gamma_0 = \partial\Omega$ (hence $W^{1,p}_{\Gamma_0}(\Omega) = W^{1,p}_0(\Omega)$) and β is continuous in u, with $\beta(x,u)\,u \geq 0$ and satisfying also the growth condition (H2-ii).

3 – Solvability by regularization

The proof of the main theorem is obtained by considering the variational solution of the following regularized problem, for each $\epsilon > 0$, $u_\epsilon \in W^{1,p}_{\Gamma_0}(\Omega)$:

(3.1)
$$\int_\Omega \mathbf{a}(x,u_\epsilon,Du_\epsilon)\cdot Dv\,dx + \int_\Omega \beta_\epsilon(u_\epsilon)\,v\,dx + \int_\Gamma \gamma_\epsilon(u_\epsilon)\,v\,d\sigma = \langle T_\epsilon, v\rangle\ , \quad \forall\,v \in W^{1,p}_{\Gamma_0}(\Omega)\ ,$$

where $\beta_\epsilon(u_\epsilon) = \beta_\epsilon(x, u_\epsilon(x))$ and $\gamma_\epsilon(u_\epsilon) = \gamma_\epsilon(x, u_\epsilon(x))$, with β_ϵ and γ_ϵ regularized in the second variable by mollification (e.g., $\beta_\epsilon(x,s) = (\beta(x,\cdot) * j_\epsilon)(s)$ and $\gamma_\epsilon(x,s) = (\gamma(x,\cdot) * j_\epsilon)(s)$ with $j_\epsilon \in C_0^\infty([-\epsilon,\epsilon])$, $j_\epsilon \geq 0$, $\int_{-\infty}^{+\infty} j_\epsilon(s)\,ds = 1$) and

$$
\begin{aligned}
&\langle T_\epsilon, v\rangle = \int_\Omega \mu_\epsilon\, v\, dx + \int_\Gamma \nu_\epsilon\, v\, d\sigma \underset{\epsilon\to 0}{\longrightarrow} \langle T, v\rangle\,, \qquad \forall\, v \in W^{1,\infty}_{\Gamma_0}(\Omega)\ ;\\
&\mu_\epsilon \in L^{p'}(\Omega)\,, \qquad \|\mu_\epsilon\|_{L^1(\Omega)} \leq \|\mu\|_{M(\Omega)}\ ;\\
&\nu_\epsilon \in L^{p'}(\Gamma)\,, \qquad \|\nu_\epsilon\|_{L^1(\Gamma)} \leq \|\nu\|_{M(\Gamma)}\ .
\end{aligned}
$$

Clearly, $T_\epsilon \in [W^{1,p}_{\Gamma_0}(\Omega)]'$, and known results yield the existence of $u_\epsilon \in W^{1,p}_{\Gamma_0}(\Omega)$ (see Theorem 4.3 of page 250 of [13] and [7]).

Proposition 3.1. *There exists $u_\epsilon \in W^{1,p}_{\Gamma_0}(\Omega)$ solving (3.1) such that as $\epsilon \to 0$*

- **i)** *$\{u_\epsilon\}$ is strongly precompact in $W^{1,q}_{\Gamma_0}(\Omega)$, $\forall\, q < q^* = \frac{N}{N-1}\,(p-1)$;*
- **ii)** *$\{\beta_\epsilon(u_\epsilon)\}$ and $\{\gamma_\epsilon(u_\epsilon)\}$ are weakly precompact in $L^1(\Omega)$ and $L^1(\Gamma)$, respectively.*

The proof of this Proposition will be shown in Section 4.

Proof of the Main Theorem: Now, with this Proposition, letting $\epsilon \to 0$, we have for a subsequence,

$$
\beta_\epsilon(u_\epsilon) \to b, \ \ \text{in } L^1(\Omega)\text{-weak}\,, \qquad \gamma_\epsilon(u_\epsilon) \to g, \ \ \text{in } L^1(\Gamma)\text{-weak}\ , \tag{3.2}
$$

and

$$
u_\epsilon \to u \begin{cases} \text{in } W^{1,q}_{\Gamma_0}(\Omega) \text{ strongly}\,, & \forall\, q < q^*;\\ \text{in } L^\kappa(\Omega), \quad \text{where } \ 1 \leq \kappa < \frac{Nq}{N-q};\\ \text{in } L^\tau(\Gamma), \quad \text{where } \ 1 \leq \tau < \frac{(N-1)q}{N-q};\\ \text{a.e. in } \Omega \quad \text{and} \quad \text{a.e. in } \Gamma\ . \end{cases} \tag{3.3}
$$

The compactness result for the solutions of (3.1) yields, in particular, $Du_\epsilon \to Du$ a.e. in Ω, and consequently, also

$$
\mathbf{a}(x, u_\epsilon, Du_\epsilon) \to \mathbf{a}(x, u, Du), \qquad \text{in} \ \ L^1(\Omega)\ .
$$

Since, by construction, we also have $T_\epsilon \rightharpoonup T$ weakly, after $\epsilon \to 0$, we obtain from (3.1)

$$
\int_\Omega \Big(\mathbf{a}(x, u, Du)\cdot Dv + bv\Big)\, dx + \int_\Gamma g\, v\, d\sigma = \langle T, v\rangle\,, \qquad \forall\, v \in W^{1,\infty}_{\Gamma_0}(\Omega)\ .
$$

This is equivalent to (2.6) and then the problem has a weak solution if $b \in \widehat{\beta}(u)$, $g \in \widehat{\gamma}(u)$ almost everywhere, i.e., we only need to show (2.5), which is given by the following

Lemma 3.2. *The limit functions in (3.2) b and g satisfy (2.5), i.e.,*

$$\underline{\beta}(x,u(x)) \le b(x) \le \overline{\beta}(x,u(x)), \qquad \text{a.e. in } \Omega ;$$

$$\underline{\gamma}(x,u(x)) \le g(x) \le \overline{\gamma}(x,u(x)), \qquad \text{a.e. in } \Gamma .$$

Proof: Since the proof for β is similar to the one for γ, we just prove one of them, e.g. for γ. Because u_ϵ converges to u almost everywhere in Γ from (3.3), and u_ϵ, $u \in L^\tau(\Gamma)$, for any $\eta > 0$, there exists a subset θ of Γ such that $\mathrm{meas}_{N-1}(\theta) < \eta$ and $u_\epsilon \to u$ uniformly on $\Gamma\backslash\theta$. Moreover $u \in L^\infty(\Gamma\backslash\theta)$. So, for any $\delta > 0$, let $\epsilon_0 < \frac{\delta}{2}$, be such that for $\epsilon < \epsilon_0$, $|u_\epsilon(x) - u(x)| \le \frac{\delta}{2}$, for all $x \in \Gamma\backslash\theta$. When $\epsilon < \epsilon_0$, and $x \in \Gamma\backslash\theta$, we have

$$\begin{aligned}
\gamma_\epsilon(x,u_\epsilon(x)) &= \int_{u_\epsilon(x)-\epsilon}^{u_\epsilon(x)+\epsilon} j_\epsilon(u_\epsilon(x)-t)\,\gamma(x,t)\,dt \\
&\le \operatorname*{ess\,sup}_{s\in[u_\epsilon(x)-\epsilon,\ u_\epsilon(x)+\epsilon]} \gamma(x,s) \\
&\le \operatorname*{ess\,sup}_{s\in[u(x)-\delta,\ u(x)+\delta]} \gamma(x,s) = \overline{\gamma}^\delta(x,u(x)) \ .
\end{aligned}$$

Analogously,

$$\gamma_\epsilon(x,u_\epsilon(x)) \ge \underline{\gamma}^\delta(x,u(x)) \ ,$$

where $\overline{\gamma}^\delta(x,s)$ and $\underline{\gamma}^\delta(x,s)$ are defined as in (2.1).

Now for any $v \in C^0_{\Gamma_0}(\Gamma)$, $v \ge 0$ in $\Omega\backslash\theta$, and for all $\epsilon < \epsilon_0$ we have

$$\int_{\Gamma\backslash\theta} \underline{\gamma}^\delta(x,u(x))\,v\,d\sigma \le \int_{\Gamma\backslash\theta} \gamma_\epsilon(x,u_\epsilon(x))\,v\,d\sigma \le \int_{\Gamma\backslash\theta} \overline{\gamma}^\delta(x,u(x))\,v\,d\sigma \ .$$

Then let $\epsilon \to 0$, and using the weak convergence (3.2), we have

$$\int_{\Gamma\backslash\theta} \underline{\gamma}^\delta(x,u(x))\,v\,d\sigma \le \int_{\Gamma\backslash\theta} g\,v\,d\sigma \le \int_{\Gamma\backslash\theta} \overline{\gamma}^\delta(x,u(x))\,v\,d\sigma \ .$$

Since $\overline{\gamma}^\delta$ and $\underline{\gamma}^\delta$ are monotone functions with respect to δ when $\delta \to 0$, we have

$$\int_{\Gamma\backslash\theta} \underline{\gamma}(x,u(x))\,v\,d\sigma \le \int_{\Gamma\backslash\theta} g\,v\,d\sigma \le \int_{\Gamma\backslash\theta} \overline{\gamma}(x,u(x))\,v\,d\sigma \ .$$

Since $v \geq 0$ is arbitrary, it follows

$$\underline{\gamma}(x,u(x)) \leq g \leq \overline{\gamma}(x,u(x)), \qquad \text{a.e. } \ x \in \Gamma \backslash \theta \ .$$

Because we can choose η as small as we like, we may conclude

$$\underline{\gamma}(x,u(x)) \leq g \leq \overline{\gamma}(x,u(x)), \qquad \text{a.e. in } \ \Gamma \ . \ \blacksquare$$

4 – A priori estimates

In this section, we obtain easily the Proposition 3.1 as a consequence of the following three lemmas.

Lemma 4.1. *A solution of the problem (3.1) satisfies the estimate*

$$\|u_\epsilon\|_{W^{1,q}(\Omega)} \leq C \ , \tag{4.1}$$

for any $q > 1$ satisfying the inequality $q < \frac{N}{N-1}\,(p-1)$, where the constant C is independent of ϵ.

Proof: Denote

$$u_{\epsilon(m)} = \begin{cases} -1, & \text{if } \ u_\epsilon < -m, \\ u_\epsilon + (m-1), & \text{if } \ -m \leq u_\epsilon < 1{-}m, \\ 0, & \text{if } \ 1{-}m \leq u_\epsilon \leq m{-}1, \\ u_\epsilon - (m-1), & \text{if } \ m{-}1 \leq u_\epsilon \leq m, \\ 1, & \text{if } \ u_\epsilon > m \ , \end{cases} \tag{4.2}$$

for natural numbers m and notice that

$$u_\epsilon = \sum_{m=1}^{\infty} u_{\epsilon(m)} \quad \text{ and } \quad u_{\epsilon(m)} \in W^{1,p}_{\Gamma_0}(\Omega) \ . \tag{4.3}$$

We define the subdomains R_m, G_m and Ω_m as follows:

$$\begin{cases} \Omega_m = \Big\{x \in \Omega,\ m{-}1 \leq |u_\epsilon(x)| < m,\ |Du_\epsilon(x)| {>} 0\Big\}; \\ R_m = \Big\{x \in \Omega,\ |u_\epsilon(x)| > m{-}1\Big\}; \\ G_m = \Big\{x \in \Gamma,\ |u_\epsilon(x)| > m{-}1\Big\} \ . \end{cases} \tag{4.4}$$

For simplicity, and without loss of generality, we discuss the problem under the translated hypothesis (H2-iii) and (H3-iii),

$$\tag{4.5} \begin{cases} \operatorname*{ess\,sup}_{s\leq -t^*} \beta(x,s) \leq 0 \leq \operatorname*{ess\,inf}_{s\geq t^*} \beta(x,s), \\ \operatorname*{ess\,sup}_{s\leq -t^*} \gamma(x,s) \leq 0 \leq \operatorname*{ess\,inf}_{s\geq t^*} \gamma(x,s)\ , \end{cases}$$

for a.e. $x\in\Omega$ and a.e. $x\in\Gamma$, respectively, and for some $t^*>0$ sufficiently large.

With (4.5) and the definitions of β_ϵ and γ_ϵ, we can choose two constants ρ, M large enough, which are independent of ϵ, such that when $m>\rho-1$, we have uniformly

$$\beta_\epsilon(u_\epsilon)\, u_{\epsilon(m)} \geq 0\,, \qquad \gamma_\epsilon(u_\epsilon)\, u_{\epsilon(m)} \geq 0\ ;$$

and

$$\tag{4.7} \begin{cases} \Big|\beta_\epsilon(u_\epsilon)\, \chi(\{x\in\Omega,\ |u_\epsilon(x)|\leq\rho\})\Big| < M, \\ \Big|\gamma_\epsilon(u_\epsilon)\, \chi(\{x\in\Gamma,\ |u_\epsilon(x)|\leq\rho\})\Big| < M\ , \end{cases}$$

where $\chi(G)$ denotes the characteristic function of G.

Taking $u_{\epsilon(m)}$ as a test function in the equation of (3.1), and using (4.6) and (4.7), we have by (H1)

$$\alpha\,\|Du_{\epsilon(m)}\|^p_{L^p(\Omega)} = \alpha\int_{\Omega_m} |Du_{\epsilon(m)}|^p\,dx \leq \int_{\Omega_m} \mathbf{a}(x,u_\epsilon,Du_\epsilon)\cdot Du_{\epsilon(m)}\,dx \leq$$

$$\leq -\int_{R_m} \beta_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,dx - \int_{G_m} \gamma_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,d\sigma + \int_{R_m} \mu_\epsilon\, u_{\epsilon(m)}\,dx + \int_{G_m} \nu_\epsilon\, u_{\epsilon(m)}\,d\sigma$$

$$= -\int_{\{|u_\epsilon|>\rho\}\cap R_m} \beta_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,dx - \int_{\{|u_\epsilon|\leq\rho\}\cap R_m} \beta_\epsilon(s,u_\epsilon)\,u_{\epsilon(m)}\,dx + \int_{R_m} \mu_\epsilon\, u_{\epsilon(m)}\,dx$$

$$-\int_{\{|u_n|>\rho\}\cap G_m} \gamma_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,d\sigma - \int_{\{|u_\epsilon|\leq\rho\}\cap G_m} \gamma_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,d\sigma + \int_{G_m} \nu_\epsilon\, u_{\epsilon(m)}\,d\sigma$$

$$\leq M\,\mathrm{meas}_N(\Omega) + M\,\mathrm{meas}_{N-1}(\Gamma) + \|\mu_\epsilon\|_1 + \|\nu_\epsilon\|_1 \leq C\ .$$

Hence, by exploiting Hölder's inequality and the Sobolev imbedding theorem, using the same procedure as in [4] and [14], for any q such that

$$\tag{4.8} 1\vee(p-1)\vee \frac{N\rho_1}{N+\rho_1} \vee \frac{N\rho_0}{(N-1)+\rho_0} < q < \frac{N}{N-1}\,(p-1)\ .$$

we can sum on m and obtain

$$\tag{4.9} \|Du_\epsilon\|^q_{L^q(\Omega)} \leq C\ ,$$

where C is another constant independent of ϵ. Since $u_\epsilon = 0$ on Γ_0, with $\text{meas}_{N-1}(\Gamma_0) > 0$, by Poincaré inequality, we have (4.1). ∎

Lemma 4.2. *The sequences $\{\beta_\epsilon(u_\epsilon)\}$ and $\{\gamma_\epsilon(u_\epsilon)\}$ in problem (3.1) are uniformly bounded with respect to ϵ in $L^1(\Omega)$ and $L^1(\Gamma)$, respectively. Moreover as $\varepsilon \to 0$ they are weakly precompact in $L^1(\Omega)$ and $L^1(\Gamma)$, respectively.*

Proof: First, since $\|\mu_\epsilon\|_1$ and $\|\nu_\epsilon\|_1$ are uniformly bounded with respect to ϵ, taking $\rho \le m < \rho + 1$ and using $u_{\epsilon(m)}$ as a test function in (3.1), we have

$$\int_\Omega |\beta_\epsilon(x,u_\epsilon)|\,dx + \int_\Gamma |\gamma_\epsilon(x,u_\epsilon)|\,d\sigma =$$

$$\begin{aligned}
&= \int_{\{|u_\epsilon|\le m\}\cap\Omega} |\beta_\epsilon(x,u_\epsilon)|\,dx + \int_{\{|u_\epsilon|>m\}\cap\Omega} |\beta_\epsilon(x,u_\epsilon)|\,dx \\
&\quad + \int_{\{|u_\epsilon|\le m\}\cap\Gamma} |\gamma_\epsilon(x,u_\epsilon)|\,d\sigma + \int_{\{|u_\epsilon|>m\}\cap\Gamma} |\gamma_\epsilon(x,u_\epsilon)|\,d\sigma \\
&\le M\,\text{meas}_N(\Omega) + M\,\text{meas}_{N-1}(\Gamma) + \int_{R_m} \beta_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,dx + \int_{G_m} \gamma_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,d\sigma \\
&\quad - \int_{\{m-1<|u_\epsilon|\le m\}\cap\Omega} \beta_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,dx - \int_{\{m-1<|u_\epsilon|\le m\}\cap\Gamma} \gamma_\epsilon(x,u_\epsilon)\,u_{\epsilon(m)}\,d\sigma \\
&\le M\,\text{meas}_N(\Omega) + M\,\text{meas}_{N-1}(\Gamma) - \alpha\,\|Du_{\epsilon(m)}\|^p_{L^p(\Omega)} + \int_\Omega |\mu_\epsilon|\,dx + \int_\Gamma |\nu_\epsilon|\,d\sigma \\
&\le M\,\text{meas}_N(\Omega) + M\,\text{meas}_{N-1}(\Gamma) + \|\mu_\epsilon\|_1 + \|\nu_\epsilon\|_1 \le C\ ,
\end{aligned}$$

where C is independent of ϵ.

That is, $\{\beta_\epsilon(u_\epsilon)\}$ and $\{\gamma_\epsilon(u_\epsilon)\}$ are uniformly bounded with respect to ϵ in $L^1(\Omega)$ and $L^1(\Gamma)$, respectively.

Secondly, since (4.1) holds uniformly with respect to ϵ, by using Sobolev imbedding theorem, we have

$$\|u_\epsilon\|^\kappa_{L^\kappa(\Omega)} \le C_0\,, \qquad \|u_\epsilon|_\Gamma\|^\tau_{L^\tau(\Gamma)} \le C_0\ , \tag{4.10}$$

where C_0 is independent of ϵ, and $\kappa = \frac{Nq}{N-q}$ and $\tau = \frac{(N-1)q}{N-q}$.

To prove the sequences are precompact in $L^1(\Omega)$ and $L^1(\Gamma)$ respectively, it is enough to show that for each $\eta > 0$ there exists a $\delta > 0$ so that for any $\omega \subset \Omega$ and $\theta \subset \Gamma$, if $\text{meas}_N(\omega) < \delta$ and $\text{meas}_{N-1}(\theta) < \delta$, then $\int_\omega |\beta_\epsilon(x,u_\epsilon)|\,dx + \int_\theta |\gamma_\epsilon(x,u_\epsilon)|\,d\sigma < \eta$.

In fact, with the choice of q in (4.8), $\kappa - \rho_1 > 0$, $\tau - \rho_0 > 0$. By the growth condition in (H2-ii) and (H3-ii) of β_ϵ, γ_ϵ, and (4.10), when $\text{meas}_N(\omega) < \delta$ and

$\operatorname{meas}_{N-1}(\theta) < \delta$, we have, for any ϵ,

$$\int_\omega |\beta_\epsilon(x,u_\epsilon)|\,dx + \int_\theta |\gamma_\epsilon(x,u_\epsilon)|\,d\sigma \le \int_\omega |e_1\,u_\epsilon^{\rho_1} + c_1|\,dx + \int_\theta |e_0\,u_\epsilon^{\rho_0} + c_0|\,d\sigma \le$$

$$\le C\Big(\operatorname{meas}_N^{\frac{\kappa-\rho_1}{\kappa}}(\omega)\Big(\int_\omega |u_\epsilon|^\kappa\,dx\Big)^{\frac{\rho_1}{\kappa}} + \operatorname{meas}_N(\omega)$$

$$+ \operatorname{meas}_{N-1}^{\frac{\tau-\rho_0}{\tau}}(\theta)\Big(\int_\theta |u_\epsilon|^\tau\,d\sigma\Big)^{\frac{\rho_0}{\tau}} + \operatorname{meas}_{N-1}(\theta)\Big)$$

$$\le C(C_0^{\rho_1} + C_0^{\rho_0} + 1)\Big(\operatorname{meas}_N^{\frac{\kappa-\rho_1}{\kappa}}(\omega) + \operatorname{meas}_{N-1}^{\frac{\tau-\rho_0}{\tau}}(\theta) + \operatorname{meas}_N(\omega) + \operatorname{meas}_{N-1}(\theta)\Big)$$

$$< \eta\ ,$$

as long as $\delta < \big(\frac{\eta}{4C(C_0^{\rho_1}+C_0^{\rho_0}+1)}\big)^r$, where $r = \max\{\frac{\kappa}{\kappa-\rho_1}, \frac{\tau}{\tau-\rho_0}\}$. Consequently, the result follows. ∎

Lemma 4.3. *Assume $T_\epsilon = (\mu_\epsilon, \nu_\epsilon)$ is bounded in $L^1(\Omega) \times L^1(\Gamma)$, then the sequence $\{u_\epsilon\}$ of solutions of the approximation problem (3.1) is precompact in $W^{1,q}_{\Gamma_0}(\Omega)$, $\forall\, q < q^*$.*

Proof: The proof of this lemma may be ommited since it is essentially the same as the one in Lemma 1 in [5], with the similar adaptations and corrections done in the Appendix of [12] for the Dirichlet problem, and in "Etapes 2" of Sections 2 and 3 of [21] for Neumann type problems. ∎

Remark 4.4. The elliptic operator may be only weakly coercive, that is, instead of (i) in (H1), $\mathbf{a}$ satisfies $\mathbf{a}(x,s,\xi)\cdot\xi \ge \alpha\,|\xi|^p - g(x)\,|\xi|$, for $g \in L^{p'}(\Omega)$ (see [14]).

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