A $P$-THEOREM FOR INVERSE SEMIGROUPS WITH ZERO

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Introduction

The result known as McAlister’s $P$-theorem stands as one of the most significant achievements in inverse semigroup theory since Vagner [17, 18] and Preston [12, 13, 14] initiated the theory in the fifties. See the papers by McAlister [4, 5], Munn [9], Schein [15], or the accounts by Petrich [11] and Howie [3]. The theorem refers to what have come to be called $E$-unitary inverse semigroups, and gives a description of such semigroups in terms of a group acting by order-automorphisms on a partially ordered set.

An inverse semigroup with zero cannot be $E$-unitary unless every element is idempotent, but, as noted by Szendrei [16], it is possible to modify the definition and to consider what we shall call $E^*$-unitary semigroups instead.

One of the cornerstones of the McAlister theory is the minimum group congruence

$$\sigma = \{(a, b) \in S \times S : (\exists e \in S) \ e^2 = e, \ ea = eb\}$$

on an inverse semigroup $S$, first considered by Munn [7] in 1961. Again, $\sigma$ is of little interest if $S$ has a zero element, since it must then be the universal congruence. However, in 1964 Munn [8] showed that, for certain inverse semigroups $S$ with zero, the closely analogous relation

$$\beta = \{(a, b) \in S \times S : (\exists e \in S) \ 0 \neq e = e^2, \ ea = eb \neq 0\} \cup \{(0, 0)\}$$

is the minimum Brandt semigroup congruence on $S$.

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In this paper we show how to obtain a result closely analogous to the McAlister theorem for a certain class of inverse semigroups with zero, based on the idea of a Brandt semigroup acting by partial order-isomorphisms on a partially ordered set.

The main ‘building blocks’ of the McAlister structure theory for an $E$-unitary inverse semigroup $S$ are a group $G$ and a partially ordered set $\mathcal{X}$. The source of the group $G$ has always been fairly obvious—it is the maximum group homomorphic image of $S$—but the connection of $\mathcal{X}$ to the semigroup was harder to clarify, and none of the early accounts [4, 5, 9, 15] was entirely satisfactory in this respect. The approach by Margolis and Pin [6] involved the use of $S$ to construct a category, and certainly made $\mathcal{X}$ seem more natural. Here we copy their approach by constructing a carrier semigroup associated with $S$.

A more general situation, in which $S$ is an inverse semigroup and $\rho$ is an idempotent-pure congruence, is dealt with in [2]. See also [10] and [1] for other more general ideas emerging from the McAlister theory. However, by specializing to the case where $S/\rho$ is a Brandt semigroup, we obtain a much more explicit structure theorem than is possible in a general situation, and to underline that point we devote the final section of the paper to an isomorphism theorem.

1 – Preliminaries

For undefined terms see [3]. A congruence $\rho$ on a semigroup $S$ with zero will be called proper if $0\rho = \{0\}$. We shall routinely denote by $E_S$ (or just by $E$ if the context allows) the set of idempotents of the semigroup $S$. For any set $A$ containing 0 we shall denote the set $A\setminus\{0\}$ by $A^*$.

A Brandt semigroup $B$, defined as a completely 0-simple inverse semigroup, can be described in terms of a group $G$ and a non-empty set $I$. More precisely,

$$B = (I \times G \times I) \cup \{0\},$$

and

$$(i, a, j)(k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases}$$

$$(i, a, j)0 = 00 = 00 = 0.$$ 

This is of course a special case of a Rees matrix semigroup: $B = \mathcal{M}^0[G; I, I; P]$, where $P$ is the $I \times I$ matrix $\Delta = (\delta_{ij})$, with

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}.$$
The following easily verified properties will be of use throughout the paper.

**Theorem 1.1.** Let $B$ be a Brandt semigroup.

(i) For all $b, c$ in $B^*$, $bc \neq 0$ if and only if $b^{-1}b = cc^{-1}$.

(ii) In particular, for all $e, f$ in $E_B^*$, $ef \neq 0$ if and only if $e = f$.

(iii) For all $e$ in $E_B^*$ and $b$ in $B^*$,

$$eb \neq 0 \Rightarrow eb = b$$

$$be \neq 0 \Rightarrow be = b .$$

(iv) For all $b, c$ in $B^*$

$$bc = b \Rightarrow c = b^{-1}b$$

$$cb = b \Rightarrow c = bb^{-1} .$$

(v) For all $e \neq f$ in $E_B$, $eB \cap fB = Be \cap Bf = \{0\} .$

Munn [8] considered an inverse semigroup $S$ with zero having the two properties:

(C1) for all $a, b, c$ in $S$,

$$abc = 0 \Rightarrow ab = 0 \text{ or } bc = 0 ;$$

(C2) for all non-zero ideals $M$ and $N$ of $S$, $M \cap N \neq \{0\}$.

Let us call $S$ strongly categorical if it has both these properties. Munn showed that for a strongly categorical inverse semigroup the relation

$$\beta = \left\{ (a, b) \in S \times S : (\exists e \in S) \ 0 \neq e = e^2, \ ea = eb \neq 0 \right\} \cup \{(0, 0)\}$$

is a proper congruence on $S$ such that:

(i) $S/\beta$ is a Brandt semigroup;

(ii) if $\gamma$ is a proper congruence on $S$ such that $S/\gamma$ is a Brandt semigroup, then $\beta \subseteq \gamma$.

We shall refer to $\beta$ as the minimum Brandt congruence on $S$, and to $S/\beta$ as the maximum Brandt homomorphic image of $S$.

An inverse semigroup $S$ with zero is called $E^*$-unitary if, for all $e, s$ in $S^*$,

$$e, es \in E^* \Rightarrow s \in E^* .$$

In fact, as remarked in [3, Section 5.9], the dual implication

$$e, se \in E^* \Rightarrow s \in E^*$$
is a consequence of this property. By analogy with Proposition 5.9.1 in [3], we have

**Theorem 1.2.** Let $S$ be a strongly categorical inverse semigroup. Then the following statements are equivalent:

(a) $S$ is $E^*$-unitary;

(b) the congruence $\beta$ is idempotent-pure;

(c) $\beta \cap R = 1_S$.

**Proof:** For (b) $\equiv$ (c), see [11, III.4.2].

(a) $\Rightarrow$ (b). Let $a \beta f$, where $f \in E^*$. Then there exists $e$ in $E^*$ such that $ea = ef \neq 0$. Since $S$ is by assumption $E^*$-unitary, it now follows from $e, ea \in E^*$ that $a \in E^*$. The $\bar{\beta}$-class $f \bar{\beta}$ consists entirely of idempotents, which is what we mean when we say that $\beta$ is idempotent-pure.

(b) $\Rightarrow$ (a). Suppose that $\beta$ is idempotent-pure. Let $e, es \in E^*$. Then $e(es) = es \neq 0$, and so $es \beta s$. Since $es \in E^*$ we may deduce by the idempotent-pure property that $s \in E^*$.

2 – The carrier semigroup

Let $S$ be a strongly categorical $E^*$-unitary inverse semigroup. We shall define an inverse semigroup $C_S$ called the *carrier semigroup* of $S$.

Denote the maximum Brandt homomorphic image $S/\beta$ of $S$ by $B$, and for each $s$ in $S$ denote the $\bar{\beta}$-class $s \bar{\beta}$ by $[s]$. Let

$$C_S = \left\{(a, s, b) \in B^* \times S^* \times B^* : a[s] = b \right\} \cup \{0\}.$$ 

Notice that for every $(a, s, b)$ in $C_S$ we also have $b[s]^{-1} = a[s][s]^{-1} = a$ (by Theorem 1.1); hence $a R^B b$.

We define a binary operation $\circ$ on $C_S$ as follows:

$$(a, s, b) \circ (c, t, d) = \begin{cases} (a, st, d) & \text{if } b = c, \\ 0 & \text{otherwise} \end{cases},$$

$$0 \circ (a, s, b) = (a, s, b) \circ 0 = 0 \circ 0 = 0.$$ 

Notice that if $b = c$ then $a[st] = d$, and so in particular $st \neq 0$. It is a routine matter to verify that this operation is associative. It is easy also to see that
\((C_S, \circ)\) is an inverse semigroup: the inverse of \((a, s, b)\) is \((b, s^{-1}, a)\), and the non-zero idempotents are of the form \((a, i, a)\), where \(i \in E_S^\ast\). For each \((a, b)\) in \(R^B\), with \(a, b \in B^\ast\), we write \(C(a, b)\) for the set (necessarily non-empty) of all elements \((a, s, b)\) in \(C_S\). Notice that \(C(a, b) \circ C(c, d) \neq \{0\}\) if and only if \(b = c\).

**Lemma 2.1.** For each idempotent \(e\) in \(B\), \(C(e, e)\) consists entirely of idempotents of \(C_S\).

**Proof:** If \((e, s, e) \in C_S\), where \(e \in B^\ast\), then \(e[s] = e\). By Theorem 1.1 this is possible only if \([s] = e\) in \(B\). Then, since \(\beta\) is idempotent-pure (Theorem 1.2), it follows that \(s\) is idempotent in \(S\). 

There is a natural left action of \(B\) on \(C_S\): for all \(c\) in \(B\) and all \((a, s, b)\) in \(C(a, b) \subseteq C^*_S\)

\[
c(a, s, b) = \begin{cases} (ca, s, cb) & \text{if } ca \neq 0 \text{ in } B, \\ 0 & \text{otherwise} . \end{cases}
\]

Also,

\[
c0 = 0 \quad \text{for all } c \text{ in } B .
\]

Notice that since \(a \mathcal{R} b\) in \(B\) we have \(aa^{-1} = bb^{-1}\), and so, by Theorem 1.1,

\[
ca \neq 0 \iff c^{-1}c = aa^{-1} \iff c^{-1}c = bb^{-1} \iff cb \neq 0 .
\]

Also, the action is well-defined, for if \(a[s] = b\) then it is immediate that \((ca)[s] = cb\).

**Lemma 2.2.** For all \(c, d\) in \(B\) and all \(p, q\) in \(C_S\),

\[
c(dp) = (cd)p , \quad c(p \circ q) = (cp) \circ (cq) , \quad c(p^{-1}) = (cp)^{-1} .
\]

**Proof:** The first equality is clear if \(p = 0\). Suppose now that \(p = (a, s, b)\). If \(dp = 0\) then \(da = 0\) in \(B\), and it is then clear that \(c(dp) = (cd)p = 0\). Suppose next that \(dp \neq 0\) and that \(cd = 0\) in \(B\). Then \((cd)p = 0\) in \(C_S\), and

\[
c(dp) = c(da, s, db) = 0 ,
\]

since \(c(da) = (cd)a = 0\) in \(B\). Finally, suppose that \(dp \neq 0\), \(cd \neq 0\). Then, recalling our assumption that \(S\) is strongly categorical, we deduce by the property (C1) that \(cda \neq 0\) in \(B\), and so

\[
(cd)p = c(dp) = (cda, s, cdb) .
\]
The second equality is clear if \( p = 0 \) or \( q = 0 \) or \( c = 0 \). So suppose that \( p = (a, s, b) \), \( q = (b', t, d) \) and \( c \) are all non-zero, and suppose first that \( b' \neq b \). Then \( p \circ q = 0 \), and so certainly \( c(p \circ q) = 0 \). If \( cb = 0 \) then \( cp = 0 \) and so \( (cp) \circ (cq) = 0 \). Similarly, if \( cb' = 0 \) then \( (cp) \circ (cq) = (cp) \circ 0 = 0 \). If \( cb \) and \( cb' \) are both non-zero then \( cb \neq cb' \), for \( cb = cb' \) would imply that
\[
b = c^{-1}cb = c^{-1}cb' = b' ,
\]
contrary to hypothesis. Hence \( (cp) \circ (cq) = 0 \) in this case also.

Suppose finally that \( b' = b = 0 \): thus \( p = (a, s, b) \), \( q = (b, t, d) \), and \( ca, cb \) (and \( cd \)) are non-zero. Then
\[
(cp) \circ (cq) = (ca, s, cb)(cb, t, cd) = (ca, st, cd) = c(a, st, d) = c(p \circ q) ,
\]
as required.

The third equality follows in much the same way.

As in \([3]\), for each \( p \) in \( C_S \), let us denote by \( J(p) \) the principal two-sided ideal generated by \( p \). It is clear that, for each \( p \),
\[
J(p) = J(p \circ p^{-1}) = J(p^{-1} \circ p) = J(p^{-1}) .
\]

**Lemma 2.3.** Let \( p \in C(bb^{-1}, b) \), \( q \in C(cc^{-1}, c) \), where \( b, c \in B^* \). Then
\[
J(p) \cap J(bq) = J(p \circ bq) .
\]

**Proof:** Suppose first that \( p \circ bq = 0 \). Thus \( p = (bb^{-1}, s, b) \), \( q = (cc^{-1}, j, c) \), with \( b \neq bcccc^{-1} \). Now \( bc 
eq 0 \) if and only if \( bcc^{-1} = b \), and so \( p \circ bq = 0 \) happens precisely when \( bc = 0 \). In this case \( bq = 0 \), giving
\[
J(p) \cap J(bq) = J(p) \cap \{0\} = \{0\} .
\]

Suppose now that \( bc \neq 0 \). Since \( J(p \circ bq) \subseteq J(p) \) and \( J(p \circ bq) \subseteq J(bq) \), it is clear that \( J(p \circ bq) \subseteq J(p) \cap J(bq) \). To show the reverse inclusion, suppose that \( r \neq 0 \) and that
\[
r = x_1 \circ p \circ y_1 = x_2 \circ bq^{-1} \circ y_2 \in J(p) \cap J(bq^{-1}) = J(p) \cap J((bq)^{-1}) = J(p) \cap J(bq) .
\]
Here, since \( r \neq 0 \), we must have \( x_1 \in C(a, bb^{-1}), y_1 \in C(b, d), x_2 \in C(a, bc), y_2 \in C(b, d) \), for some \( a, d \) in \( B^* \). Hence
\[
r = r \circ r^{-1} \circ r = x_1 \circ p \circ y_1 \circ y_2^{-1} \circ bq \circ bq^{-1} \circ bq \circ x_2^{-1} \circ r = x_1 \circ p \circ b \left( b^{-1}(y_1 \circ y_2^{-1}) \circ (q \circ q^{-1}) \right) \circ bq \circ x_2^{-1} \circ r.
\]

Now \( b^{-1}(y_1 \circ y_2^{-1}) \in C(b^{-1}b, b^{-1}b) \), and so by Lemma 2.1 is an idempotent in \( C_S \). Commuting idempotents, we obtain
\[
r = x_1 \circ p \circ b \left( b^{-1}(y_1 \circ y_2^{-1}) \right) \circ bq \circ x_2^{-1} \circ r = x_1 \circ (p \circ bq) \circ bq^{-1} \circ y_1 \circ y_2^{-1} \circ bq \circ x_2^{-1} \circ r,
\]
and so \( r \in J(p \circ bq) \) as required. ■

Since \( e = ee^{-1} \) for every idempotent \( e \) in an inverse semigroup, we have the following easy consequence of Lemma 2.3:

**Corollary 2.4.** Let \( p \in C(e, e), q \in C(f, f) \), where \( e, f \) are idempotents in \( B^* \). Then \( J(p \circ q) = J(p) \cap J(q) \). In particular, \( J(p) \cap J(q) = \{0\} \) if \( e \neq f \). ■

Let \( \overline{C}_S = \{(p, b) : p \in C(bb^{-1}, b), b \in B^* \} \cup \{0\} \), and define a multiplication on \( \overline{C}_S \) by
\[
(p, b)(q, c) = \left\{ \begin{array}{ll} (p \circ bq, bc) & \text{if } bc \neq 0, \\
0 & \text{otherwise}, \end{array} \right.
\]
\[
(p, b)0 = 0(p, b) = 00 = 0.
\]
This operation is well-defined. If \( p = (bb^{-1}, s, b), q = (cc^{-1}, t, c) \) and \( bc \neq 0 \), then \( bcc^{-1} = b \) by Theorem 1.1. Hence \( (bc)(bc)^{-1} = bcc^{-1}b^{-1} = bb^{-1} \), and so
\[
p \circ bq = (bb^{-1}, s, b) \circ (b, t, bc) = (bb^{-1}, st, bc)
\]
\[
= \left( (bc)(bc)^{-1}, st, bc \right) \in C \left( (bc)(bc)^{-1}, bc \right),
\]
as required. The verification that the operation is associative is routine.

Now consider the map \( \psi : S \to \overline{C}_S \) given by
\[
s \psi = \left( ([ss^{-1}], s, [s]), [s] \right) \quad (s \in S^*),
\]
\[
0 \psi = 0.
\]
Then $\psi$ is clearly one-one. It is also onto, since for each $((bb^{-1}, s, b), b)$ in $C_S^*$, we deduce from $bb^{-1}[s] = b$ that $[s] = b$ and hence that $((bb^{-1}, s, b), b) = s\psi$.

The map $\psi$ is indeed even an isomorphism. Let $s, t \in S^*$, and suppose first that $[st] = 0$. Then $(st)\psi = 0$, and from the fact that $[s][t] = 0$ in $B$ we deduce that

$$
(s\psi)(t\psi) = \left(\left([ss^{-1}], s, [s]\right), \left([tt^{-1}], t, [t]\right)\right) = 0
$$

in $C_S$. Suppose now that $[st] \neq 0$. Then

$$
(s\psi)(t\psi) = \left(\left([ss^{-1}], s, [s]\right), \left([tt^{-1}], t, [t]\right)\right)
= \left(\left([ss^{-1}], st, [st]\right), \left([tt^{-1}], st, [st]\right)\right) \quad \text{(since } [s] = [s][t][t]^{-1})
= (st)\psi.
$$

We have shown

**Lemma 2.5.** $C_S$ is isomorphic to $S$. 

In a sense we have in this section gone round in a circle, starting with $S$, moving to $C_S$, and returning to $S$ via $C_S$ and the isomorphism $\psi$. We shall see, however, that the set of principal ideals of $C_S$ is the key to our main theorem.

### 3 – The main theorem

We begin with some observations concerning representations of Brandt semigroups. Let $\mathcal{X} = (\mathcal{X}, \leq)$ be a partially ordered set containing a least element 0, and let $B$ be a Brandt semigroup. For each $b$ in $B$, let $\lambda_b$ be a partial order-isomorphism of $\mathcal{X}$, whose domain is an order-ideal of $\mathcal{X}$, and such that the map $b \mapsto \lambda_b$ is a faithful representation (see [3] for a definition of ‘faithful’) of $B$ by partial one-one maps of $\mathcal{X}$. We shall find it convenient to regard each $\lambda_b$ as acting on $\mathcal{X}$ on the left, writing $\lambda_b(\mathcal{X})$ rather than $X\lambda_b$. Notice that each $\text{im} \lambda_b$ is an order-ideal also, since $\text{im} \lambda_b = \text{dom} \lambda_b$.

Suppose that $\text{dom} \lambda_0 = \text{im} \lambda_0 = \{0\}$; then, by the faithful property we deduce that if $b \neq 0$ in $B$ then $\text{dom} \lambda_b$ and $\text{im} \lambda_b$ are both non-zero order-ideals. The order-preserving property implies that $\lambda_b(0) = 0$ for every $b$ in $B$. For each $e$ in $E_B^*$, let $\Delta_e = \text{dom} \lambda_e = \text{im} \lambda_e$. Since $\lambda_e$ is an idempotent in the symmetric inverse semigroup $I_\mathcal{X}$, it is the identity map on its domain. If $e, f$ are distinct
idempotents in $B^*$,
\[ \Delta_e \cap \Delta_f = \{0\} , \]
since $ef = 0$ in $B$ whenever $e \neq f$.

Let us suppose also that the representation is effective, by which we mean that every $X$ in $\mathcal{X}$ lies in the domain of at least one $\lambda_b$. Equivalently, we have
\[ \mathcal{X} = \bigcup \{ \Delta_e : e \in E_B^* \} . \]

Notice that (since we are writing mapping symbols on the left) for all $b$ in $B^*$,
\[ \operatorname{dom} \lambda_b = \operatorname{dom} \lambda_{b^{-1}b} = \Delta_{b^{-1}b} , \quad \operatorname{im} \lambda_b = \operatorname{im} \lambda_{bb^{-1}} = \Delta_{bb^{-1}} . \]

Also, by Theorem 1.1, for all $b, c$ in $B^*$,
\[ bc \neq 0 \quad \text{if and only if} \quad \operatorname{im} \lambda_c = \operatorname{dom} \lambda_b , \]
\[ bc = 0 \quad \text{if and only if} \quad \operatorname{im} \lambda_c \cap \operatorname{dom} \lambda_b = \{0\} . \]

Now let $(\mathcal{X}, \leq)$ be a partially ordered set with a least element 0, and let $\mathcal{Y}$ be a subset of $\mathcal{X}$ such that

(P1) $\mathcal{Y}$ is a lower semilattice with respect to $\leq$, in the sense that for every $J$ and $K$ in $\mathcal{Y}$ there is a greatest lower bound $J \wedge K$, also in $\mathcal{Y}$;

(P2) $\mathcal{Y}$ is an order ideal, in the sense that for all $A, X$ in $\mathcal{X}$,
\[ A \in \mathcal{Y} \quad \text{and} \quad X \leq A \quad \implies \quad X \in \mathcal{Y} . \]

Let $B$ be a Brandt semigroup, and suppose that $b \mapsto \lambda_b$ is an effective, faithful representation of $B$, as described above. Thus each $\lambda_b$ is a partial order-isomorphism of $\mathcal{X}$, acting on the left, and $\operatorname{dom} \lambda_b$ is an order-ideal of $\mathcal{X}$. In practice we shall write $bX$ rather than $\lambda_b(X)$, and so in effect, for each $b$ in $B^*$, we are supposing that there is a partial one-one map $X \mapsto bX$ ($X \in \mathcal{X}$), with the property that, for all $X, Y$ in $\mathcal{X}$,
\[ X \leq Y \quad \implies \quad bX \leq bY . \]

Suppose now that the triple $(B, \mathcal{X}, \mathcal{Y})$ has the following property:

(P3) For all $e \in E_B^*$, and for all $P, Q \in \Delta_e \cap \mathcal{Y}^*$, where $\Delta_e = \operatorname{dom} \lambda_e$,
\[ P \wedge Q \neq 0 . \]
If \( X, Y \in \mathcal{X} \) and if \( X \land Y \) exists, then, for all \( b \) in \( B \) for which \( X \) and \( Y \) belong to \( \text{dom} \, b \), the element \( bX \land bY \) exists, and

\[
bX \land bY = b(X \land Y) .
\]

To see this, notice first that \( X \land Y \in \text{dom} \, b \), since \( \text{dom} \, b \) is an order-ideal, and that \( b(X \land Y) \leq bX, b(X \land Y) \leq bY \). Suppose next that \( Z \leq bX, Z \leq bY \). Then \( Z \in \text{im} \, b \), since \( \text{im} \, b \) is an order-ideal, and so \( Z = bT \) for some \( T \) in \( \text{dom} \, b \). Then

\[
T = b^{-1}Z \leq b^{-1}(bX) = X ,
\]

and similarly \( T \leq Y \). Hence \( T \leq X \land Y \), and so

\[
Z = bT \leq b(X \land Y) ,
\]
as required.

The triple \((B, \mathcal{X}, \mathcal{Y})\) is said to be a Brandt triple if it has the properties (P1), (P2) and (P3) together with the additional properties:

- (P4) \( BY = \mathcal{X} \);
- (P5) for all \( b \) in \( B^* \), \( b\mathcal{Y}^* \cap \mathcal{Y}^* \neq \emptyset \).

Now let

\[
S = \mathcal{M}(B, \mathcal{X}, \mathcal{Y}) = \left\{ (P, b) \in \mathcal{Y}^* \times B^* : b^{-1}P \in \mathcal{Y}^* \right\} \cup \{0\} ,
\]

where \((B, \mathcal{X}, \mathcal{Y})\) is a Brandt triple. We define multiplication on \( S \) by the rule that

\[
(P, b)(Q, c) = \begin{cases} 
(P \land bQ, bc) & \text{if } bc \neq 0, \\
0 & \text{otherwise ,}
\end{cases}
\]

\[
(P, b)0 = 0(P, b) = 00 = 0 .
\]

To verify that \( S \) is closed with respect to this operation, notice first that \( bQ \) is defined, for the tacit assumption that \( c^{-1}Q \) is defined and the assumption that \( bc \neq 0 \) in \( B \) implies that

\[
Q \in \text{dom} \, c^{-1} = \text{im} \, c = \text{dom} \, b .
\]

Next, notice that \( b^{-1}P \land Q \) exists, since both \( b^{-1}P \) and \( Q \) are in \( \mathcal{Y} \). Moreover, \( b^{-1}P \land Q \in \mathcal{Y}^* \), since

\[
b^{-1}P \in \text{im}(b^{-1}) = \Delta_{b^{-1}b}, \quad Q \in \text{dom} \, b = \Delta_{b^{-1}b} ,
\]
and so $b^{-1}P \land Q \neq 0$ by (P3). Also $b^{-1}P \cap Q \in \text{dom } b$, since $Q \in \text{dom } b$ and dom $b$ is an order ideal. Hence $b(b^{-1}P \land Q) = P \land bQ$ exists, and is in $\mathcal{Y}^*$, since $P \land bQ \leq P \in \mathcal{Y}^*$. Moreover, if $bc \neq 0$, then

$$(bc)^{-1}(P \land bQ) = c^{-1}b^{-1}P \land c^{-1}Q \leq c^{-1}Q \in \mathcal{Y}^*,$$

and so $(bc)^{-1}(P \land bQ) \in \mathcal{Y}^*$.

Next, the operation is associative. The Brandt semigroup $B$ satisfies the `categorical' condition

$$bcd = 0 \implies bc = 0 \text{ or } cd = 0;$$

hence either both $[(P, b)(Q, c)](R, d)$ and $(P, b)[(Q, c)(R, d)]$ are zero, or both are equal to $(P \land bQ \land bcR, bcd)$.

Thus $S$ is a semigroup with zero. It is even a regular semigroup, for if $(P, b)$ is a non-zero element of $S$ then $(b^{-1}P, b^{-1}) \in S$, and

$$(P, b)(b^{-1}P, b^{-1})(P, b) = (P, bb^{-1})(P, b) = (P, b),$$

$$(b^{-1}P, b^{-1})(P, b)(b^{-1}P, b^{-1}) = (b^{-1}P, b^{-1})(P, bb^{-1}) = (b^{-1}P, b^{-1}).$$

It is, moreover, clear that a non-zero element $(P, b)$ is idempotent if and only if $b$ is idempotent in $B$ and $bP = P$ (which is equivalent to saying that $bP$ is defined). If $(P, e), (Q, f)$ are idempotents in $S$, then either $e \neq f$, in which case $ef = 0$ and $(P, e)(Q, f) = (Q, f)(P, e) = 0$, or $e = f$, in which case


Thus $S$ is an inverse semigroup, and the unique inverse of $(P, b)$ is $(b^{-1}P, b^{-1})$.

The natural order relation in $S^*$ is given by

$$(P, b) \leq (Q, c) \iff bb^{-1}c \neq 0 \text{ and } (P, b) = (P, bb^{-1})(Q, c) = (P \land Q, bb^{-1}c).$$

That is, since $bb^{-1}c = c$ in such a case,

(2) $$(P, b) \leq (Q, c) \iff b = c \text{ and } P \leq Q.$$

It follows that $S$ is $E^*$-unitary, for if $(P, e) \in E^*$ and $(Q, c) \in S^*$, then $(P, e) \leq (Q, c)$ if and only if $c = e$ and $P \leq Q$, and so in particular $(Q, c)$ is idempotent.

Notice too that $S$ is categorical, for the product $(P, b)(Q, c)(R, d)$ can equal zero only if $bcd = 0$, and the categorical property of $B$ then implies that either $(P, b)(Q, c) = 0$ or $(Q, c)(R, d) = 0$. Indeed $S$ is strongly categorical. That this is
so follows by the work of Munn [8], for it is clear that the relation $\gamma$ on $S$ defined by

$$(3) \quad \gamma = \{( (P, b), (Q, c) ) \in S \times S : b = c \} \cup \{(0, 0)\}$$

is a proper congruence on $S$ and that $S/\gamma$ is isomorphic to the Brandt semigroup $B$.

The congruence $\gamma$ defined by (3) is in fact the minimum Brandt congruence on $S$. Suppose that $((P, b), (Q, c)) \in \gamma$. Then $b^{-1}P, c^{-1}Q \in \mathcal{Y}$, and so $bb^{-1}P = P$, $cc^{-1}Q = Q$. Hence $bb^{-1}(P \wedge Q) = P \wedge Q$, and $P \wedge Q \neq 0$ by (P3). Hence $(P \wedge Q, bb^{-1}) \in E_S^*$. It now follows that

$$(P \wedge Q, bb^{-1}) (P, b) = (P \wedge Q, bb^{-1}) (Q, b) = (P \wedge Q, b) \neq 0.$$ 

Hence, recalling Munn’s characterization (1) of the minimum Brandt congruence, we conclude that $\gamma \subseteq \beta$, the minimum Brandt congruence on $S$. Since $\gamma$ is, as observed before, a Brandt congruence, we deduce that $\gamma = \beta$.

It is useful also at this stage to note the following result:

**Lemma 3.1.** The semilattice of idempotents of $\mathcal{M}(B, \mathcal{X}, \mathcal{Y})$ is isomorphic to $\mathcal{Y}$.

**Proof:** We have seen that the non-zero idempotents of $S = \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$ are of the form $(P, e)$, where $P \in \mathcal{Y}^*$, $e \in E_B^*$ and $eP = P$. The statement that $eP = P$ is equivalent to saying in our previous notation that $P \in \mathcal{D}_e$, and since the order ideals $\Delta_e$ and $\Delta_f$ (with $e \neq f$) have zero intersection, there is for each $P$ in $\mathcal{Y}^*$ at most one $e$ such that $(P, e) \in E_S^*$.

In fact for each $P$ in $\mathcal{Y}^*$ there is exactly one $e$ in $E_B^*$ such that $(P, e) \in E_S^*$; for by our assumption that the representation $b \mapsto \lambda_b$ is effective we can assert that $P \in \text{dom } b$ for some $b$ in $B^*$, and then $(P, b^{-1}b) \in E_S^*$. The conclusion is that for each $P$ in $\mathcal{Y}^*$ there is a unique $e_P$ in $B^*$ such that $(P, e_P) \in E_S^*$. We have a bijection $P \mapsto (P, e_P)$ from $\mathcal{Y}^*$ onto $E_S^*$. If $P \wedge Q \neq 0$, then $e_P = e_Q = e$ (say), and

$$(P, e) (Q, e) = (P \wedge Q, e).$$

If $P \wedge Q = 0$, then $e_P \neq e_Q$ by (P3), and so $(P, e_P)(Q, e_Q) = 0$. We deduce that the bijection $P \mapsto (P, e_P)$, $0 \mapsto 0$ is an isomorphism from $\mathcal{Y}$ onto $E_S$.

We have in fact proved half of the following theorem:

**Theorem 3.2.** Let $(B, \mathcal{X}, \mathcal{Y})$ be a Brandt triple. Then $\mathcal{M}(B, \mathcal{X}, \mathcal{Y})$ is a strongly categorical $E^*$-unitary inverse semigroup. Conversely, every strongly categorical $E^*$-unitary inverse semigroup is isomorphic to one of this kind.
Hence $b$, $q$ are idempotents in $S$ such that $b\circ q = 0$ and so $J$ is a semilattice with respect to the inclusion order inherited from $X$. Thus $J$ is an order ideal of $X$.

Proof: To prove the converse part, let $S$ be a strongly categorical $E^*$-unitary inverse semigroup. Let $X$ be the set of principal two-sided ideals of the carrier semigroup $C_S$:

$X = \left\{ J(p) : p \in C_S \right\}.$

The set $X$ is partially ordered by inclusion, with a minimum element 0 (strictly the zero ideal $\{0\}$). Let $Y$ be the subset of $X$ consisting of 0 together with all principal ideals $J(p)$ for which $p \in C(e,e)$ for some idempotent $e$ of $B^*$. Let $J(p), J(q) \in Y$. Then by Corollary 2.4 we have either $J(p) \cap J(q) = 0 \in Y$, or $e = f$, $p \circ q \in C(e,e)$ and

$J(p) \cap J(q) = J(p \circ q) \in Y.$

Thus $Y$ is a semilattice with respect to the inclusion order inherited from $X$. This is the property (P1).

To show the property (P2), suppose that $J(p) \subseteq J(q)$, where $J(q) \in Y^*$. Thus we may assume that $q = (e, i, e) \in C(e,e)$ for some idempotents $e$ in $B$ and $i$ in $S$, such that $[i] = e$. We may suppose that $p$ is idempotent in $C_S$. (If not we replace it by $p \circ p^{-1}$, observing that $J(p \circ p^{-1}) = J(p)$.) Hence there exist $r$, $s$ in $C_S$ such that

$p = r \circ q \circ s.$

Let $n = s \circ p \circ s^{-1}$. Then $n \in C(e,e)$, and clearly $J(n) \subseteq J(p)$. Also

$p = p^3 = (r \circ q \circ s) \circ p \circ (s^{-1} \circ q^{-1} \circ r^{-1})$

$= r \circ q \circ n \circ q^{-1} \circ r^{-1} \in J(n),$

and so $J(p) = J(n) \in Y$. Thus $Y$ is an order ideal of $X$.

Now we define a representation $b \mapsto \lambda_b$ of the Brandt semigroup $B = S/\beta$ by partial order-isomorphisms of $X$. Let $\lambda_0 = \{(0,0)\}$. For each $b$ in $B^*$, let

$\lambda_b = \left\{ (J(p), J(bp)) : p, bp \neq 0 \right\} \cup \{(0,0)\}.$

That is to say, we define dom $\lambda_b = \{ J(p) : p, bp \neq 0 \} \cup \{0\}$, and define $\lambda_b(J(p)) = J(bp)$, $\lambda_b(0) = 0$.

The domain of $\lambda_b$ is in fact an order ideal of $X$. For suppose that $0 \neq J(q) \subseteq J(p)$, where $p = (a, s, c)$ is such that $bp \neq 0$ and $q = (d, t, e)$. Then there exist elements $(d, u, a)$, $(c, v, e)$ in $C_S$ such that

$q = (d, u, a) \circ (a, s, c) \circ (c, v, e) = (d, usv, e).$

Now $d[u] = a$, and so if $bd = 0$ it follows that $ba = 0$, contrary to hypothesis. Hence $bq \neq 0$, and so $J(q) \in \text{dom } \lambda_b$. 


Notice now that
\[ \text{im } \lambda_b = \{ J(bp) : p, bp \neq 0 \} = \{ J(q) : q, b^{-1}q \neq 0 \} = \text{dom } \lambda_{b^{-1}} , \]
and that \( \lambda_{b^{-1}} \lambda_b \) and \( \lambda_b \lambda_{b^{-1}} \) are the identity maps of \( \text{dom } \lambda_b, \text{im } \lambda_b \), respectively. Since \( J(p) \subseteq J(q) \Rightarrow J(bp) \subseteq J(bq) \), each \( \lambda_b \) is a partial order-isomorphism of \( X \). Next, notice that if \( bc = 0 \) then \( \lambda_b\lambda_c = \lambda_0 \), the trivial map whose domain and image are both 0; for otherwise there exists \( q \neq 0 \) in \( C_S \) such that \( J(q) \in \text{dom}(\lambda_b\lambda_c) \), from which it follows that \( (bc)q = (bc) \neq 0 \), a contradiction.

Suppose now that \( bc \neq 0 \). Then \( \text{dom}(\lambda_b\lambda_c) = \text{dom } \lambda_c \), since the conditions \( p \neq 0 \), \( cp \neq 0 \), \( b(cp) \neq 0 \) for \( J(p) \) to be in \( \text{dom}(\lambda_b\lambda_c) \) are equivalent to the conditions \( p \neq 0 \), \( (bc)p \neq 0 \) for \( J(p) \) to be in \( \text{dom } \lambda_c \). Moreover, for all \( p \) in the common domain,
\[ (\lambda_b\lambda_c)(J(p)) = \lambda_b(J(cp)) = J(b(cp)) = J((bc)p) = \lambda_c(J(p)). \]

Thus \( \lambda_b\lambda_c = \lambda_c \) in all cases, and so \( b \mapsto \lambda_b \) is a representation of \( B \) by partial order-isomorphisms of \( X \). We can regard \( B \) as acting on \( X \) on the left, and write \( B J(p) \) rather than \( \lambda_b(J(p)) \). Notice that \( b J(p) = J(bp) \) provided \( bp \neq 0 \).

To show that the representation is faithful, suppose that \( \lambda_b = \lambda_c \), where \( b, c \in B^* \), and let \( p = (a, s, d) \) in \( C_S \) be such that \( bp \neq 0 \). Then \( cp \neq 0 \), and so
\[ b = b aa^{-1} = caa^{-1} = c. \]

To show that the representation is effective, let \( p = (a, s, d) \) be an arbitrary element of \( C_S^* \). Then \( aa^{-1}p \neq 0 \) and so \( J(p) \in \text{dom } \lambda_{aa^{-1}} \).

To verify (P3), let \( e \in E_B^* \), and let \( J(p), J(q) \in Y^* \cap \Delta_e \). Thus \( p = (f, i, f), q = (g, j, g) \), where \( f, g \in E_B^*, i, j \in E_S^* \) and \( f[i] = f, g[j] = g \). Since \( cp \) and \( eq \) are non-zero, we must in fact have \( f = g = e \). Thus \( p \circ q = (e, ij, e) \neq 0 \) and so, using Corollary 2.4, we see that
\[ J(p) \cap J(q) = J(p \circ q) \neq 0 . \]

To show the property (P4), consider a non-zero element \( J(p) \) of \( X \), where \( p = (a, s, b) \). Then \( J(p) = J(p \circ p^{-1}) \), with \( p \circ p^{-1} = (a, ss^{-1}, a) \), and \( a[ss^{-1}] = a \). Let \( q \) be the element \( (a^{-1}a, ss^{-1}, a^{-1}a) \) of \( C_S \). Then \( J(q) \in Y \), and
\[ aq = (a, ss^{-1}, a) = p \circ p^{-1} . \]
It follows that \( J(p) = a J(q) \in a Y \), and so \( X = BY \), as required.
To show (P5), let $a \in B^*$, let $q = (a, x, aa^{-1})$, where $[x] = a^{-1}$, and $p = (a^{-1}a, xx^{-1}, a^{-1}a)$. Then $J(p) \in \mathcal{Y}$. Also $ap = (a, xx^{-1}, a)$. If we define $r = (aa^{-1}, x^{-1}x, aa^{-1})$, then we easily verify that

$$q^{-1} \circ (ap) \circ q = r, \quad q \circ r \circ q^{-1} = ap.$$  

Hence $aJ(p) = J(q) \in a\mathcal{Y}^* \cap \mathcal{Y}^*$, as required.

It remains to show that $S \simeq \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$. We show in fact that $C \subseteq \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$, which by virtue of Lemma 2.5 is enough. Let $\phi: C \rightarrow \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$ be given by

$$(p, b)\phi = (J(p), b) \quad (p \in C(bb^{-1}, b), \ b \in B^*)$$

$$0 \phi = 0.$$  

Since $J(p) = J(p \circ p^{-1})$ and $p \circ p^{-1} \in C(bb^{-1}, bb^{-1})$, we deduce that $J(p) \in \mathcal{Y}^*$. Also, $b^{-1}p \neq 0$,

$$b^{-1}J(p) = J(b^{-1}p) = J\left((b^{-1}p)^{-1} \circ (b^{-1}p)\right),$$  

and

$$(b^{-1}p)^{-1} \circ (b^{-1}p) \in C(bb^{-1}, b^{-1}) \circ C(b^{-1}, bb^{-1}) \subseteq C(bb^{-1}, bb^{-1});$$

hence $b^{-1}J(p) \in \mathcal{Y}^*$. Thus $(J(p), b) \in \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$.

To show that $\phi$ is one-one, suppose that $(J(p), b) = (J(q), c)$, where $p \in C(bb^{-1}, b), q \in C(cc^{-1}, c)$. Then certainly $b = c$. If we now write $p = (bb^{-1}, s, b)$ and $q = (bb^{-1}, t, b)$, we have that

$$p \circ q^{-1} = (bb^{-1}, st^{-1}, bb^{-1})$$

and so $st^{-1} \in E^*_S$. Hence

$$st^{-1} = (st^{-1})^{-1}st^{-1} = ts^{-1}st^{-1}.$$  

Next, since $p^{-1} \circ p \in J(q^{-1} \circ q)$, there exist elements $(b, u, b)$, $(b, v, b)$ in $C^*_S$ such that $p^{-1} \circ p = (b, s^{-1}s, b) = (b, u, b)(b, t^{-1}t, b)(b, v, b)$; hence

$$s^{-1}s = ut^{-1}tv.$$  

Now, from $b[u] = b$ we deduce that $[u]$ is idempotent in $B$, and hence (since $\beta$ is idempotent-pure) that $u$ is idempotent in $S$. The same argument applies to $v$,
and so from (5) we conclude that \( s^{-1}s \leq t^{-1}t \). The opposite inequality can be proved in just the same way, and so \( s^{-1}s = t^{-1}t \).

It now easily follows from this and from (4) that
\[
s = ss^{-1}s = st^{-1}t = ts^{-1}st^{-1}t = tt^{-1}tt^{-1}t = t.
\]
Hence \((p, b) = (q, c)\) as required.

To show that \( \phi \) is onto, suppose that \((J(p), b)\) is a non-zero element of \( \mathcal{M}(B, X, \mathcal{Y}) \). Then we may assume that \( p = (e, i, e) \), where \( e \in E_B^\circ \), \( i \in E_S^\circ \) and \([i] = e \). Also, \( J(b^{-1}p) \in \mathcal{Y}^\circ \), and so, since \( b^{-1}e \neq 0 \), we deduce that \( e = bb^{-1} \).

We have \( J(b^{-1}p) = J(q) \) for some \( q = (f, j, f) \), with \( f \in E_B^\circ \), \( j \in E_S^\circ \) and \([j] = f \).

Hence there exist \((b^{-1}, u, f)\) and \((f, v, b^{-1})\) such that
\[
b^{-1}p = (b^{-1}, i, b^{-1}) = (b^{-1}, u, f) \circ (f, j, f) \circ (f, v, b^{-1}).
\]
It follows that
\[
(6) \quad p = b(b^{-1}p) = (bb^{-1}, i, bb^{-1}) = (bb^{-1}, u, b,f) \circ (bf, j, bf) \circ (bf, v, bb^{-1}).
\]
Since \(bf \neq 0\) we deduce that \(bf = b\) and \(f = b^{-1}b\).

Now, since \( J(q) = J(b^{-1}p) \), there exist elements \((b^{-1}b, x, b^{-1})\) and \((b^{-1}, y, b^{-1}b)\) such that
\[
q = (b^{-1}b, j, b^{-1}b) = (b^{-1}b, x, b^{-1}) \circ (b^{-1}, i, b^{-1}) \circ (b^{-1}, y, b^{-1}b).
\]
Hence
\[
(7) \quad bq = (b, j, b) = (b, x, bb^{-1}) \circ p \circ (bb^{-1}, y, b).
\]
We may rewrite (6) as
\[
p = (bb^{-1}, u, b) \circ (b, j, b) \circ (b, v, bb^{-1}) = r \circ (b, v, bb^{-1}),
\]
where \(r = (bb^{-1}, uj, b)\), and we immediately deduce that \( J(p) \subseteq J(r) \). Also, from (7) it follows that
\[
r = (bb^{-1}, u, b) \circ (b, j, b) \in J(p),
\]
and so \( J(r) = J(p) \). It now follows that \((r, b) \in \overline{C}_S \) and that \((J(p), b) = (r, b)\phi \).

Thus \( \phi \) is onto.

Finally, we show that \( \phi \) is a homomorphism. Let \((p, b), (q, c) \in \overline{C}_S^\circ \). If \( bc = 0 \) in \( B \) then both \([J(p)(q, c)]\) and \([(p, b)\phi][q, c] \phi \) are zero. Otherwise we use Lemma 2.3 and observe that
\[
[(p, b)(q, c)] \phi = (p \circ bq, bc) \phi = (J(p \circ bq), bc)
\]
\[
= (J(p) \cap bJ(q), bc) = (J(p), b)J(q), c) = [(p, b)\phi][q, c] \phi.
\]
This completes the proof of Theorem 3.2.  

**Example:** Let \( T = \mathcal{M}(G, \mathcal{X}, \mathcal{Y}) \) be an \( E \)-unitary inverse semigroup (without zero), let \( I \) be a set, and let \( S = (I \times T \times I) \cup \{0\} \), and define multiplication in \( S \) by 

\[
(i, a, j) (k, b, l) = \begin{cases} 
(i, ab, l) & \text{if } j = k, \\
0 & \text{otherwise},
\end{cases}
\]

\[
0 (i, a, j) = (i, a, j), \quad 0 \cdot 0 = 0.
\]

Then it is not hard to check that \( S \) is a strongly categorical \( E^* \)-unitary inverse semigroup. Its maximum Brandt image is \( B = (I \times G \times I) \cup \{0\} \), where \( G \) is the maximum group image of \( T \).

For each \( i \) in \( I \), let \( \mathcal{X}_i \) be a copy of \( \mathcal{X} \), and suppose that \( X \rightarrow X_i \) (\( X \in \mathcal{X} \)) is an order-isomorphism. Let \( \mathcal{Y}_i \) correspond to \( \mathcal{Y} \) in this isomorphism. Suppose that the sets \( \mathcal{X}_i \) are pairwise disjoint, and form an ordered set \( \mathcal{X}' \) as the union of all the sets \( \mathcal{X}_i \) together with an extra minimum element \( 0 \). The order on \( \mathcal{X}' \) coincides with the order on \( \mathcal{X}_i \) within \( \mathcal{X}_i \), and \( 0 \leq X' \) for all \( X' \) in \( \mathcal{X}' \). Define \( \mathcal{Y}' = \bigcup \{ \mathcal{Y}_i : i \in I \} \cup \{0\} \).

The action of \( B \) on \( \mathcal{X}' \) is given as follows. If \( b = (i, a, j) \in B \), then the domain of \( \lambda_b \) is \( \mathcal{X}_j \cup \{0\} \), and the action of \( b \) on the elements of its domain is given by 

\[
(i, a, j) X_j = (aX)_i \quad (X \in \mathcal{X})
\]

\[
(i, a, j) 0 = 0.
\]

(Trivially, if \( b = 0 \), then the domain of \( \lambda_0 \) is \( \{0\} \), and the action of \( b \) simply sends \( 0 \) to \( 0 \).)

Then \( (B, \mathcal{X}', \mathcal{Y}') \) is a Brandt triple, and \( S \simeq \mathcal{M}(B, \mathcal{X}', \mathcal{Y}') \).

### 4 - An isomorphism theorem

Given two semigroups \( S = \mathcal{M}(B, \mathcal{X}, \mathcal{Y}) \) and \( S' = \mathcal{M}(B', \mathcal{X}', \mathcal{Y}') \), it is now important to describe the conditions under which \( S' \simeq S \). In a sense it is clear from the last section that the ‘building blocks’ of \( S \) are intrinsic: \( B \) is the maximum Brandt homomorphic image of \( S \), \( \mathcal{X} \) is the partially ordered set of principal ideals of the carrier semigroup \( C_S \), and \( \mathcal{Y} \) is in effect the semilattice of idempotents of \( S \). It is, however, conceivable that two non-isomorphic semigroups \( S \) and \( S' \) might have isomorphic maximum Brandt images, isomorphic semilattices of idempotents, and might be such that \( C_S \) and \( C_{S'} \) have order-isomorphic sets of principal ideals, and so we must prove a formal isomorphism theorem.
Theorem 4.1. Let \( S = \mathcal{M}(B, X, Y) \), \( S' = \mathcal{M}(B', X', Y') \), and suppose that 
\( \phi: S \rightarrow S' \) is an isomorphism. Then 

(i) there exists an isomorphism \( \omega: B \rightarrow B' \); 

(ii) there exists an order isomorphism \( \theta: X \rightarrow X' \) whose restriction to \( Y \) is a semilattice isomorphism from \( Y \) onto \( Y' \); 

(iii) for all \( b \) in \( B \) and \( X \) in \( X \), 

\[
(bX) \theta = (b\omega)(X\theta) \; ;
\]

(iv) for all \( (P,b) \) in \( S^* \), 

\[
(P,b) \phi = (P\theta,b\omega) \; .
\]

Conversely, if there exist \( \omega \) and \( \theta \) with the properties (i), (ii) and (iii), then (9), together with \( 0 \phi = 0 \), defines an isomorphism from \( S \) onto \( S' \).

Proof: Notice that (8) is to be interpreted as including the information that \( bX \) is defined if and only if \( (b\omega)(X\theta) \) is defined.

We begin by proving the converse part. So, for each \( (P,b) \) in \( S^* \), define 

\[
(P,b)\phi = (P\theta,b\omega),
\]

in accordance with (9). Notice first that this does define a map from \( S \) into \( S' \), for \( P\theta \in (Y')^* \) and by (8) we also have 

\[
(b\omega)^{-1}(P\theta) = (b^{-1}\omega)(P\theta) = (b^{-1}P)\theta \in (Y')^* .
\]

(The first equality follows from (i), and \( (b')^{-1}P' \in (Y')^* \) is a consequence of (ii).)

Next, the map \( \phi \) defined by (9) is a bijection. If \( (P',b') \in (S')^* \), then there exist a unique \( P \) in \( Y \) such that \( P\theta = P' \) and a unique \( b \) in \( B \) such that \( b\omega = b' \). Moreover, 

\[
(b^{-1}P)\theta = (b\omega)^{-1}(P\theta) = (b')^{-1}P' \in (Y')^* .
\]

Hence \( (P,b) \in S \), and is the unique element of \( S \) mapping to \( (P',b') \) by \( \phi \).

Finally, \( \phi \) is a homomorphism. Given \( (P,b),(Q,c) \) in \( S^* \) such that \( bc \neq 0 \), we have that 

\[
[(P,b)(Q,c)]\phi = (P \land bQ, bc)\phi = ((P \land bQ)\theta, (bc)\omega) =
\]
If $bc = 0$ then $(b!)(c!) = 0$, and so both $[(P, b)(Q, c)]\phi$ and $[(P, b)\phi][Q, c]\phi$ are equal to zero.

Conversely, suppose that $\phi : S \to S'$ is an isomorphism. Let $\beta, \beta'$ be the minimum Brandt congruences on $S, S'$, respectively. As we saw in the last section, $S/\beta \simeq B$ and $S'/\beta' \simeq B'$. In fact we have an isomorphism $\omega : B \to B'$ such that the diagram

\[
\begin{array}{ccc}
S & \phi & S' \\
\gamma & \downarrow & \gamma' \\
B & \omega & B'
\end{array}
\]

commutes. Here $\gamma$ and $\gamma'$ are the projections $(P, b) \mapsto b, (P', b') \mapsto b'$ respectively.

Now let $(P', b') = $ the image under $\phi$ of $(P, b)$. Then

$$b' = (P', b') \gamma' = (P, b) \phi \gamma' = (P, b) \gamma \omega = b \omega,$$

and so $(P, b)\phi = (P', b\omega)$, where $P' \in Y'$ and is such that $(b\omega)^{-1}P' \in Y'$.

We now have a lemma

**Lemma 4.2.** Let $(P, b), (P, c) \in S^*$, and suppose that $(P, b)\phi = (P', b\omega)$. Then $(P, c)\phi = (P', c\omega)$.

**Proof:** Suppose that $(P, c)\phi = (P', c\omega)$. Both $(P, bb^{-1}) = (P, b)(P, b)^{-1}$ and $(P, cc^{-1}) = (P, c)(P, c)^{-1}$ belong to $S^*$, and so, by the argument in the proof of Lemma 3.1, we must have $bb^{-1} = cc^{-1}$. Hence

\[
(P', (bb^{-1})\omega) = (P', b\omega) (P', b\omega)^{-1} = [(P, b)\phi][P, b^{-1}]\phi
\]

\[
= (P, bb^{-1})\phi = (P, cc^{-1})\phi = [(P, c)\phi][P, c^{-1}]\phi
\]

\[
= (P', c\omega)(P', c\omega)^{-1} = (P', (cc^{-1})\omega)
\]
and so $P'' = P'$. ■

From this lemma it follows that we can define a map $\theta : \mathcal{Y} \rightarrow \mathcal{Y}'$ such that, for all $(P, b)$ in $S$,

$$(P, b) \circ \phi = (P \theta, b \omega) .$$

The domain of $\theta$ is in fact the whole of $\mathcal{Y}$, since, by the effectiveness of the representation $b \mapsto \lambda_b$, there exists for every $P$ in $\mathcal{Y}'$ an element $b$ in $B'$ such that $(P, b^{-1}b) \in S$.

**Lemma 4.3.** The map $\theta : \mathcal{Y} \rightarrow \mathcal{Y}'$ is an order-isomorphism.

**Proof:** That $\theta$ is a bijection follows from the observation that we can do for the inverse isomorphism $\phi^{-1} : S' \rightarrow S$ exactly what we have just done for $\phi$, obtaining maps $\omega' : B' \rightarrow B$ and $\theta' : \mathcal{Y}' \rightarrow \mathcal{Y}$ such that $(P', b')\circ \phi^{-1} = (P'\theta', b'\omega')$.

Then, from the inverse property of $\phi^{-1}$, we deduce that $\omega'$ and $\theta'$ are two-sided inverses of $\omega$ and $\theta$ respectively. Let $P \leq Q$ in $\mathcal{Y}$, and let $b$ be such that $Q \in \text{dom } b$. Then, since $\text{dom } b$ is an order-ideal, $P \in \text{dom } b$ also, and so, by (2), $(P, b^{-1}b) \leq (Q, b^{-1}b)$ in $S$. Applying $\phi$, we deduce that $(P\theta, (b^{-1}b)\omega) \leq (Q\theta, (b^{-1}b)\omega)$ in $S'$, and so $P\theta \leq Q\theta$. ■

**Lemma 4.4.** Let $P \in \mathcal{Y}'$, and let $b$ in $B'$ be such that $bP \in \mathcal{Y}'$. Then $(bP)\theta = (b\omega)(P\theta)$.

**Proof:** The elements $(bP, b)$ and $(P, b^{-1})$ are both in $S$, and are mutually inverse. By applying $\phi$ to both sides of the equality

$$(bP, b) \circ (P, b^{-1}) = (bP, bb^{-1}) ,$$

we deduce that

$$((bP)\theta, b\omega) \circ (P\theta, b^{-1}\omega) = ((bP)\theta, (b^{-1}b)\omega) ,$$

and hence that

$$(bP)\theta \circ (b\omega)(P\theta) = (bP)\theta .$$

It follows that $(bP)\theta \leq (b\omega)(P\theta)$.

Similarly, by applying $\phi$ to both sides of the equality

$$(P, b^{-1}) \circ (bP, b) = (P, b^{-1}b) ,$$

we obtain

$$P\theta \circ (b^{-1}\omega) \circ ((bP)\theta) = P\theta ,$$
and from this it follows that \((b \omega)(P \theta) \leq (bP) \theta\).

To extend the map \(\theta\) to \(X\) we use (P4) to express an arbitrary \(X\) in \(X^*\) in the form \(bP\), where \(b \in B^*\) and \(P \in \mathcal{Y}^*\), and define \(X\theta\) to be \((b \omega)(P \theta)\). To show that this defines \(X\theta\) uniquely, we must show that \(bP = cQ\) implies that \((b \omega)(P \theta) = (c \omega)(Q \theta)\). In fact we shall deduce this from the result that 
\[bP \leq cQ \implies (b \omega)(P \theta) \leq (c \omega)(Q \theta),\]
and so obtain also the information that \(\theta\) is order-preserving on \(X\). So suppose that \(bP \leq cQ\). Then \(bP \in \text{dom}(c^{-1})\), and so we may deduce that \(c^{-1}bP \leq Q\) in \(\mathcal{Y}\). From Lemmas 4.3 and 4.4 we deduce that \(((c^{-1}b) \omega)(P \theta) \leq Q \theta\), which immediately gives the required inequality.

It is now easy to see that \(\theta\): \(X \to X'\) is a bijection. To show that it is one-one, suppose that \(X \theta = Y \theta\), where \(X = bP\) and \(Y = cQ\), with \(b, c \in B^*\) and \(P, Q\) in \(\mathcal{Y}^*\). Then \((b \omega)(P \theta) = (c \omega)(Q \theta)\), and so, in \(\mathcal{Y}'\),
\[c^{-1}bP \theta = ((c^{-1}b \omega)^{-1}(b \omega))(P \theta) = Q \theta.\]
Hence \(c^{-1}bP = Q\), and from this it is immediate that \(X = Y\). To show that \(\theta\) is onto, consider an element \(X' = b'P'\) in \(X'\), where \(b' \in (B')^*\) and \(P' \in (\mathcal{Y}')^*\). Then there exist \(b\) in \(B\) and \(P\) in \(\mathcal{Y}\) such that \(b \omega = b'\) and \(P \theta = P'\), and so \((b) \theta = b'P' = X'\).

Finally, we show that the equality (8) holds for all \(b\) in \(B^*\) and all \(X\) in \(X^*\). Let \(X = cP\), where \(c \in B^*\) and \(P \in \mathcal{Y}^*\). Then
\[(bX) \theta = (b(cP)) \theta = ((bc)P) \theta = ((bc \omega)(P \theta) = (b \omega)((c \omega)(P \theta)) = (b \omega)(X \theta) .\]

This completes the proof of Theorem 4.1.

**REFERENCES**


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