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A P-THEOREM FOR INVERSE SEMIGROUPS WITH ZERO

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Introduction

The result known as McAlister's P-theorem stands as one of the most significant achievements in inverse semigroup theory since Vagner [17, 18] and Preston [12, 13, 14] initiated the theory in the fifties. See the papers by McAlister [4, 5], Munn [9], Schein [15], or the accounts by Petrich [11] and Howie [3]. The theorem refers to what have come to be called E-unitary inverse semigroups, and gives a description of such semigroups in terms of a group acting by order-automorphisms on a partially ordered set.

An inverse semigroup with zero cannot be E-unitary unless every element is idempotent, but, as noted by Szendrei [16], it is possible to modify the definition and to consider what we shall call E^*-unitary semigroups instead.

One of the cornerstones of the McAlister theory is the *minimum group congruence*

$$\sigma = \Big\{(a,b) \in S \times S \colon\ (\exists\, e \in S)\ e^2 = e,\ ea = eb\Big\}$$

on an inverse semigroup S, first considered by Munn [7] in 1961. Again, σ is of little interest if S has a zero element, since it must then be the universal congruence. However, in 1964 Munn [8] showed that, for certain inverse semigroups S with zero, the closely analogous relation

$$\beta = \Big\{(a,b) \in S \times S \colon\ (\exists\, e \in S)\ 0 \neq e = e^2,\ ea = eb \neq 0\Big\} \cup \{(0,0)\}$$

is the minimum Brandt semigroup congruence on S.

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In this paper we show how to obtain a result closely analogous to the McAlister theorem for a certain class of inverse semigroups with zero, based on the idea of a Brandt semigroup acting by partial order-isomorphisms on a partially ordered set.

The main 'building blocks' of the McAlister structure theory for an E-unitary inverse semigroup S are a group G and a partially ordered set $\mathcal{X}$. The source of the group G has always been fairly obvious—it is the maximum group homomorphic image of S—but the connection of $\mathcal{X}$ to the semigroup was harder to clarify, and none of the early accounts [4, 5, 9, 15] was entirely satisfactory in this respect. The approach by Margolis and Pin [6] involved the use of S to construct a category, and certainly made $\mathcal{X}$ seem more natural. Here we copy their approach by constructing a *carrier semigroup* associated with S.

A more general situation, in which S is an inverse semigroup and ρ is an idempotent-pure congruence, is dealt with in [2]. See also [10] and [1] for other more general ideas emerging from the McAlister theory. However, by specializing to the case where S/ρ is a Brandt semigroup, we obtain a much more explicit structure theorem than is possible in a general situation, and to underline that point we devote the final section of the paper to an isomorphism theorem.

1 – Preliminaries

For undefined terms see [3]. A congruence ρ on a semigroup S with zero will be called *proper* if $0\rho = \{0\}$. We shall routinely denote by E_S (or just by E if the context allows) the set of idempotents of the semigroup S. For any set A containing 0 we shall denote the set $A\backslash\{0\}$ by A^*.

A *Brandt semigroup* B, defined as a completely 0-simple inverse semigroup, can be described in terms of a group G and a non-empty set I. More precisely,

$$B = (I \times G \times I) \cup \{0\} \ ,$$

and

$$(i,a,j)(k,b,l) = \begin{cases} (i,ab,l) & \text{if } j = k, \\ 0 & \text{otherwise} \ ; \end{cases}$$

$$0(i,a,j) = (i,a,j)0 = 00 = 0 \ .$$

This is of course a special case of a Rees matrix semigroup: $B = \mathcal{M}^0[G; I, I; P]$, where P is the $I \times I$ matrix $\Delta = (\delta_{ij})$, with

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \ . \end{cases}$$

The following easily verified properties will be of use throughout the paper.

Theorem 1.1. *Let B be a Brandt semigroup.*

(**i**) *For all b, c in B^*, $bc \neq 0$ if and only if $b^{-1}b = cc^{-1}$.*

(**ii**) *In particular, for all e, f in E_B^*, $ef \neq 0$ if and only if $e = f$.*

(**iii**) *For all e in E_B^* and b in B^*,*

$$eb \neq 0 \;\Rightarrow\; eb = b\,, \qquad be \neq 0 \;\Rightarrow\; be = b\ .$$

(**iv**) *For all b, c in B^**

$$bc = b \;\Rightarrow\; c = b^{-1}b\,, \qquad cb = b \;\Rightarrow\; c = bb^{-1}\ .$$

(**v**) *For all $e \neq f$ in E_B, $eB \cap fB = Be \cap Bf = \{0\}$.* ∎

Munn [8] considered an inverse semigroup S with zero having the two properties:

(**C1**) for all a, b, c in S,

$$abc = 0 \;\Rightarrow\; ab = 0 \;\text{ or }\; bc = 0\ ;$$

(**C2**) for all non-zero ideals M and N of S, $M \cap N \neq \{0\}$.

Let us call S *strongly categorical* if it has both these properties. Munn showed that for a strongly categorical inverse semigroup the relation

$$\beta = \Big\{(a,b) \in S \times S\colon\ (\exists\, e \in S)\ 0 \neq e = e^2,\ ea = eb \neq 0\Big\} \cup \{(0,0)\} \tag{1}$$

is a proper congruence on S such that:

(**i**) S/β is a Brandt semigroup;

(**ii**) if γ is a proper congruence on S such that S/γ is a Brandt semigroup, then $\beta \subseteq \gamma$.

We shall refer to β as the *minimum Brandt congruence* on S, and to S/β as the *maximum Brandt homomorphic image* of S.

An inverse semigroup S with zero is called *E^*-unitary* if, for all e, s in S^*,

$$e, es \in E^* \;\Rightarrow\; s \in E^*\ .$$

In fact, as remarked in [3, Section 5.9], the dual implication

$$e, se \in E^* \;\Rightarrow\; s \in E^*$$

is a consequence of this property. By analogy with Proposition 5.9.1 in [3], we have

Theorem 1.2. *Let S be a strongly categorical inverse semigroup. Then the following statements are equivalent:*

(**a**) *S is E^*-unitary;*

(**b**) *the congruence β is idempotent-pure;*

(**c**) $\beta \cap \mathcal{R} = 1_S$.

Proof: For (b)≡(c), see [11, III.4.2].

(a)⇒(b). Let $a \;\beta\; f$, where $f \in E^*$. Then there exists e in E^* such that $ea = ef \neq 0$. Since S is by assumption E^*-unitary, it now follows from $e, ea \in E^*$ that $a \in E^*$. The β-class $f\beta$ consists entirely of idempotents, which is what we mean when we say that β is idempotent-pure.

(b)⇒(a). Suppose that β is idempotent-pure. Let $e, es \in E^*$. Then $e(es) = es \neq 0$, and so $es \;\beta\; s$. Since $es \in E^*$ we may deduce by the idempotent-pure property that $s \in E^*$. ∎

2 – The carrier semigroup

Let S be a strongly categorical E^*-unitary inverse semigroup. We shall define an inverse semigroup C_S called the *carrier semigroup* of S.

Denote the maximum Brandt homomorphic image S/β of S by B, and for each s in S denote the β-class $s\beta$ by $[s]$. Let

$$C_S = \Big\{(a, s, b) \in B^* \times S^* \times B^* \colon a[s] = b\Big\} \cup \{0\} .$$

Notice that for every (a, s, b) in C_S we also have $b[s]^{-1} = a[s][s]^{-1} = a$ (by Theorem 1.1); hence $a \,\mathcal{R}^B\, b$.

We define a binary operation $\circ$ on C_S as follows:

$$(a, s, b) \circ (c, t, d) = \begin{cases} (a, st, d) & \text{if } b = c, \\ \quad 0 & \text{otherwise} , \end{cases}$$

$$0 \circ (a, s, b) = (a, s, b) \circ 0 = 0 \circ 0 = 0 .$$

Notice that if $b = c$ then $a[st] = d$, and so in particular $st \neq 0$. It is a routine matter to verify that this operation is associative. It is easy also to see that

$(C_S, \circ)$ is an inverse semigroup: the inverse of (a, s, b) is (b, s^{-1}, a), and the non-zero idempotents are of the form (a, i, a), where $i \in E_S^*$. For each (a, b) in $\mathcal{R}^B$, with a, b in B^*, we write $C(a, b)$ for the set (necessarily non-empty) of all elements (a, s, b) in C_S. Notice that $C(a, b) \circ C(c, d) \neq \{0\}$ if and only if $b = c$.

Lemma 2.1. *For each idempotent e in B, $C(e, e)$ consists entirely of idempotents of C_S.*

Proof: If $(e, s, e) \in C_S$, where $e \in B^*$, then $e[s] = e$. By Theorem 1.1 this is possible only if $[s] = e$ in B. Then, since β is idempotent-pure (Theorem 1.2), it follows that s is idempotent in S. ∎

There is a natural left action of B on C_S: for all c in B and all (a, s, b) in $C(a, b) \subseteq C_S^*$

$$c(a, s, b) = \begin{cases} (ca, s, cb) & \text{if } ca \neq 0 \text{ in } B, \\ 0 & \text{otherwise .} \end{cases}$$

Also,

$$c0 = 0 \quad \text{for all } c \text{ in } B \ .$$

Notice that since $a \, \mathcal{R} \, b$ in B we have $aa^{-1} = bb^{-1}$, and so, by Theorem 1.1,

$$ca \neq 0 \iff c^{-1}c = aa^{-1} \iff c^{-1}c = bb^{-1} \iff cb \neq 0 \ .$$

Also, the action is well-defined, for if $a[s] = b$ then it is immediate that $(ca)[s] = cb$.

Lemma 2.2. *For all c, d in B and all p, q in C_S,*

$$c(dp) = (cd)p\,, \qquad c(p \circ q) = (cp) \circ (cq)\,, \qquad c(p^{-1}) = (cp)^{-1} \ .$$

Proof: The first equality is clear if $p = 0$. Suppose now that $p = (a, s, b)$. If $dp = 0$ then $da = 0$ in B, and it is then clear that $c(dp) = (cd)p = 0$. Suppose next that $dp \neq 0$ and that $cd = 0$ in B. Then $(cd)p = 0$ in C_S, and

$$c(dp) = c(da, s, db) = 0 \ ,$$

since $c(da) = (cd)a = 0$ in B. Finally, suppose that $dp \neq 0$, $cd \neq 0$. Then, recalling our assumption that S is strongly categorical, we deduce by the property (C1) that $cda \neq 0$ in B, and so

$$(cd)p = c(dp) = (cda, s, cdb) \ .$$

The second equality is clear if $p = 0$ or $q = 0$ or $c = 0$. So suppose that $p = (a, s, b)$, $q = (b', t, d)$ and c are all non-zero, and suppose first that $b' \neq b$. Then $p \circ q = 0$, and so certainly $c(p \circ q) = 0$. If $cb = 0$ then $cp = 0$ and so $(cp) \circ (cq) = 0$. Similarly, if $cb' = 0$ then $(cp) \circ (cq) = (cp) \circ 0 = 0$. If cb and cb' are both non-zero then $cb \neq cb'$, for $cb = cb'$ would imply that

$$b = c^{-1}cb = c^{-1}cb' = b' ,$$

contrary to hypothesis. Hence $(cp) \circ (cq) = 0$ in this case also.

Suppose finally that $b' = b \neq 0$: thus $p = (a, s, b)$, $q = (b, t, d)$, and ca, cb (and cd) are non-zero. Then

$$(cp) \circ (cq) = (ca, s, cb)\,(cb, t, cd) = (ca, st, cd) = c(a, st, d) = c(p \circ q) ,$$

as required.

The third equality follows in much the same way. ∎

As in [3], for each p in C_S, let us denote by $J(p)$ the principal two-sided ideal generated by p. It is clear that, for each p,

$$J(p) = J(p \circ p^{-1}) = J(p^{-1} \circ p) = J(p^{-1}) .$$

Lemma 2.3. *Let $p \in C(bb^{-1}, b)$, $q \in C(cc^{-1}, c)$, where $b, c \in B^*$. Then*

$$J(p) \cap J(bq) = J(p \circ bq) .$$

Proof: Suppose first that $p \circ bq = 0$. Thus $p = (bb^{-1}, s, b)$, $q = (cc^{-1}, j, c)$, with $b \neq bcc^{-1}$. Now $bc \neq 0$ if and only if $bcc^{-1} = b$, and so $p \circ bq = 0$ happens precisely when $bc = 0$. In this case $bq = 0$, giving

$$J(p) \cap J(bq) = J(p) \cap \{0\} = \{0\} .$$

Suppose now that $bc \neq 0$. Since $J(p \circ bq) \subseteq J(p)$ and $J(p \circ bq) \subseteq J(bq)$, it is clear that $J(p \circ bq) \subseteq J(p) \cap J(bq)$. To show the reverse inclusion, suppose that $r \neq 0$ and that

$$\begin{aligned} r = x_1 \circ p \circ y_1 = x_2 \circ bq^{-1} \circ y_2 \ &\in \ J(p) \cap J(bq^{-1}) \\ &= J(p) \cap J((bq)^{-1}) = J(p) \cap J(bq) . \end{aligned}$$

Here, since $r \neq 0$, we must have $x_1 \in C(a, bb^{-1})$, $y_1 \in C(b,d)$, $x_2 \in C(a, bc)$, $y_2 \in C(b,d)$, for some a, d in B^*. Hence

$$\begin{aligned} r &= r \circ r^{-1} \circ r \\ &= x_1 \circ p \circ y_1 \circ y_2^{-1} \circ bq \circ bq^{-1} \circ bq \circ x_2^{-1} \circ r \\ &= x_1 \circ p \circ b\Big(b^{-1}(y_1 \circ y_2^{-1}) \circ (q \circ q^{-1})\Big) \circ bq \circ x_2^{-1} \circ r \ . \end{aligned}$$

Now $b^{-1}(y_1 \circ y_2^{-1}) \in C(b^{-1}b, b^{-1}b)$, and so by Lemma 2.1 is an idempotent in C_S. Commuting idempotents, we obtain

$$\begin{aligned} r &= x_1 \circ p \circ b\Big((q \circ q^{-1}) \circ b^{-1}(y_1 \circ y_2^{-1})\Big) \circ bq \circ x_2^{-1} \circ r \\ &= x_1 \circ (p \circ bq) \circ bq^{-1} \circ y_1 \circ y_2^{-1} \circ bq \circ x_2^{-1} \circ r \ , \end{aligned}$$

and so $r \in J(p \circ bq)$ as required. ∎

Since $e = ee^{-1}$ for every idempotent e in an inverse semigroup, we have the following easy consequence of Lemma 2.3:

Corollary 2.4. *Let $p \in C(e,e)$, $q \in C(f,f)$, where e, f are idempotents in B^*. Then $J(p \circ q) = J(p) \cap J(q)$. In particular, $J(p) \cap J(q) = \{0\}$ if $e \neq f$.* ∎

Let

$$\overline{C}_S = \Big\{(p,b)\colon\ p \in C(bb^{-1}, b),\ b \in B^*\Big\} \cup \{0\} \ ,$$

and define a multiplication on $\overline{C}_S$ by

$$(p,b)\,(q,c) = \begin{cases} (p \circ bq, bc) & \text{if } bc \neq 0, \\ 0 & \text{otherwise} \ , \end{cases}$$

$$(p,b)0 = 0(p,b) = 00 = 0 \ .$$

This operation is well-defined. If $p = (bb^{-1}, s, b)$, $q = (cc^{-1}, t, c)$ and $bc \neq 0$, then $bcc^{-1} = b$ by Theorem 1.1. Hence $(bc)\,(bc)^{-1} = bcc^{-1}b^{-1} = bb^{-1}$, and so

$$\begin{aligned} p \circ bq &= (bb^{-1}, s, b) \circ (b, t, bc) = (bb^{-1}, st, bc) \\ &= \Big((bc)(bc)^{-1}, st, bc\Big) \ \in \ C\Big((bc)(bc)^{-1}, bc\Big) \ , \end{aligned}$$

as required. The verification that the operation is associative is routine.

Now consider the map $\psi\colon\ S \to \overline{C}_S$ given by

$$\begin{aligned} s\psi &= \Big(([ss^{-1}], s, [s]), [s]\Big) \qquad (s \in S^*) \\ 0\psi &= 0 \ . \end{aligned}$$

Then ψ is clearly one-one. It is also onto, since for each $((bb^{-1}, s, b), b)$ in $\overline{C}^*_S$, we deduce from $bb^{-1}[s] = b$ that $[s] = b$ and hence that $((bb^{-1}, s, b), b) = s\psi$.

The map ψ is indeed even an isomorphism. Let $s, t \in S^*$, and suppose first that $[st] = 0$. Then $(st)\psi = 0$, and from the fact that $[s][t] = 0$ in B we deduce that

$$(s\psi)\,(t\psi) = \Big(([ss^{-1}], s, [s]), [s]\Big)\,\Big(([tt^{-1}], t, [t]), [t]\Big) = 0$$

in $\overline{C}_S$. Suppose now that $[st] \neq 0$. Then

$$\begin{aligned}(s\psi)\,(t\psi) &= \Big(([ss^{-1}], s, [s]), [s]\Big)\,\Big(([tt^{-1}], t, [t]), [t]\Big)\\ &= \Big(([ss^{-1}], st, [st]), [s][t]\Big)\\ &= \Big(([(st)(st)^{-1}], st, [st]), [st]\Big) \qquad (\text{since } [s] = [s]\,[t]\,[t]^{-1})\\ &= (st)\,\psi\ .\end{aligned}$$

We have shown

Lemma 2.5. *$\overline{C}_S$ is isomorphic to S.* ∎

In a sense we have in this section gone round in a circle, starting with S, moving to C_S, and returning to S *via* $\overline{C}_S$ and the isomorphism ψ. We shall see, however, that the set of principal ideals of C_S is the key to our main theorem.

3 – The main theorem

We begin with some observations concerning representations of Brandt semigroups. Let $\mathcal{X} = (\mathcal{X}, \leq)$ be a partially ordered set containing a least element 0, and let B be a Brandt semigroup. For each b in B, let λ_b be a partial order-isomorphism of $\mathcal{X}$, whose domain is an order-ideal of $\mathcal{X}$, and such that the map $b \mapsto \lambda_b$ is a faithful representation (see [3] for a definition of 'faithful') of B by partial one-one maps of $\mathcal{X}$. We shall find it convenient to regard each λ_b as acting on $\mathcal{X}$ *on the left*, writing $\lambda_b(X)$ rather than $X\lambda_b$. Notice that each $\operatorname{im}\lambda_b$ is an order-ideal also, since $\operatorname{im}\lambda_b = \operatorname{dom}\lambda_{b^{-1}}$.

Suppose that $\operatorname{dom}\lambda_0 = \operatorname{im}\lambda_0 = \{0\}$; then, by the faithful property we deduce that if $b \neq 0$ in B then $\operatorname{dom}\lambda_b$ and $\operatorname{im}\lambda_b$ are both non-zero order-ideals. The order-preserving property implies that $\lambda_b(0) = 0$ for every b in B. For each e in E^*_B, let $\Delta_e = \operatorname{dom}\lambda_e = \operatorname{im}\lambda_e$. Since λ_e is an idempotent in the symmetric inverse semigroup $\mathcal{I}_\mathcal{X}$, it is the identity map on its domain. If e, f are distinct idempotents in B^*,

$$\Delta_e \cap \Delta_f = \{0\} ,$$

since $ef = 0$ in B whenever $e \neq f$.

Let us suppose also that the representation is *effective*, by which we mean that every X in $\mathcal{X}$ lies in the domain of at least one λ_b. Equivalently, we have

$$\mathcal{X} = \bigcup \left\{ \Delta_e \colon e \in E_B^* \right\} .$$

Notice that (since we are writing mapping symbols on the left) for all b in B^*,

$$\operatorname{dom} \lambda_b = \operatorname{dom} \lambda_{b^{-1}b} = \Delta_{b^{-1}b} , \qquad \operatorname{im} \lambda_b = \operatorname{im} \lambda_{bb^{-1}} = \Delta_{bb^{-1}} .$$

Also, by Theorem 1.1, for all b, c in B^*,

$$bc \neq 0 \quad \text{if and only if} \quad \operatorname{im} \lambda_c = \operatorname{dom} \lambda_b ,$$
$$bc = 0 \quad \text{if and only if} \quad \operatorname{im} \lambda_c \cap \operatorname{dom} \lambda_b = \{0\} .$$

Now let $(\mathcal{X}, \leq)$ be a partially ordered set with a least element 0, and let $\mathcal{Y}$ be a subset of $\mathcal{X}$ such that

(**P1**) $\mathcal{Y}$ is a lower semilattice with respect to $\leq$, in the sense that for every J and K in $\mathcal{Y}$ there is a greatest lower bound $J \wedge K$, also in $\mathcal{Y}$;

(**P2**) $\mathcal{Y}$ is an order ideal, in the sense that for all A, X in $\mathcal{X}$,

$$A \in \mathcal{Y} \ \text{and} \ X \leq A \quad \Longrightarrow \quad X \in \mathcal{Y} .$$

Let B be a Brandt semigroup, and suppose that $b \mapsto \lambda_b$ is an effective, faithful representation of B, as described above. Thus each λ_b is a partial order-isomorphism of $\mathcal{X}$, acting on the left, and $\operatorname{dom} \lambda_b$ is an order-ideal of $\mathcal{X}$. In practice we shall write bX rather than $\lambda_b(X)$, and so in effect, for each b in B^*, we are supposing that there is a partial one-one map $X \mapsto bX$ ($X \in \mathcal{X}$), with the property that, for all X, Y in $\mathcal{X}$,

$$X \leq Y \implies bX \leq bY .$$

Suppose now that the triple $(B, \mathcal{X}, \mathcal{Y})$ has the following property:

(**P3**) For all $e \in E_B^*$, and for all $P, Q \in \Delta_e \cap \mathcal{Y}^*$, where $\Delta_e = \operatorname{dom} \lambda_e$,

$$P \wedge Q \neq 0 .$$

If $X, Y \in \mathcal{X}$ and if $X \wedge Y$ exists, then, for all b in B for which X and Y belong to $\operatorname{dom} b$, the element $bX \wedge bY$ exists, and

$$bX \wedge bY = b(X \wedge Y) \ .$$

To see this, notice first that $X \wedge Y \in \operatorname{dom} b$, since $\operatorname{dom} b$ is an order-ideal, and that $b(X \wedge Y) \leq bX$, $b(X \wedge Y) \leq bY$. Suppose next that $Z \leq bX$, $Z \leq bY$. Then $Z \in \operatorname{im} b$, since $\operatorname{im} b$ is an order-ideal, and so $Z = bT$ for some T in $\operatorname{dom} b$. Then

$$T = b^{-1}Z \leq b^{-1}(bX) = X \ ,$$

and similarly $T \leq Y$. Hence $T \leq X \wedge Y$, and so

$$Z = bT \leq b(X \wedge Y) \ ,$$

as required.

The triple $(B, \mathcal{X}, \mathcal{Y})$ is said to be a *Brandt triple* if it has the properties (P1), (P2) and (P3) together with the additional properties:

(**P4**) $B\mathcal{Y} = \mathcal{X}$;

(**P5**) for all b in B^*, $b\mathcal{Y}^* \cap \mathcal{Y}^* \neq \emptyset$.

Now let

$$S = \mathcal{M}(B, \mathcal{X}, \mathcal{Y}) = \left\{(P, b) \in \mathcal{Y}^* \times B^* \colon\ b^{-1}P \in \mathcal{Y}^*\right\} \cup \{0\} \ ,$$

where $(B, \mathcal{X}, \mathcal{Y})$ is a Brandt triple. We define multiplication on S by the rule that

$$(P, b)\,(Q, c) = \begin{cases} (P \wedge bQ, bc) & \text{if } bc \neq 0, \\ \qquad 0 & \text{otherwise} \ , \end{cases}$$

$$(P, b)0 = 0(P, b) = 00 = 0 \ .$$

To verify that S is closed with respect to this operation, notice first that bQ is defined, for the tacit assumption that $c^{-1}Q$ is defined and the assumption that $bc \neq 0$ in B implies that

$$Q \in \operatorname{dom} c^{-1} = \operatorname{im} c = \operatorname{dom} b \ .$$

Next, notice that $b^{-1}P \wedge Q$ exists, since both $b^{-1}P$ and Q are in $\mathcal{Y}$. Moreover, $b^{-1}P \wedge Q \in \mathcal{Y}^*$, since

$$b^{-1}P \in \operatorname{im}(b^{-1}) = \Delta_{b^{-1}b}\,, \qquad Q \in \operatorname{dom} b = \Delta_{b^{-1}b} \ ,$$

and so $b^{-1}P \wedge Q \neq 0$ by (P3). Also $b^{-1}P \cap Q \in \operatorname{dom} b$, since $Q \in \operatorname{dom} b$ and $\operatorname{dom} b$ is an order ideal. Hence $b(b^{-1}P \wedge Q) = P \wedge bQ$ exists, and is in $\mathcal{Y}^*$, since $P \wedge bQ \leq P \in \mathcal{Y}^*$. Moreover, if $bc \neq 0$, then

$$(bc)^{-1}(P \wedge bQ) = c^{-1}b^{-1}P \wedge c^{-1}Q \leq c^{-1}Q \ \in \ \mathcal{Y}^* \ ,$$

and so $(bc)^{-1}(P \wedge bQ) \in \mathcal{Y}^*$.

Next, the operation is associative. The Brandt semigroup B satisfies the 'categorical' condition

$$bcd = 0 \quad \Longrightarrow \quad bc = 0 \ \text{ or } \ cd = 0 \ ;$$

hence either both $[(P,b)(Q,c)](R,d)$ and $(P,b)[(Q,c)(R,d)]$ are zero, or both are equal to $(P \wedge bQ \wedge bcR,\, bcd)$.

Thus S is a semigroup with zero. It is even a regular semigroup, for if (P,b) is a non-zero element of S then $(b^{-1}P, b^{-1}) \in S$, and

$$(P,b)\,(b^{-1}P,b^{-1})\,(P,b) = (P,bb^{-1})\,(P,b) = (P,b) \ ,$$
$$(b^{-1}P,b^{-1})\,(P,b)\,(b^{-1}P,b^{-1}) = (b^{-1}P,b^{-1})\,(P,bb^{-1}) = (b^{-1}P,b^{-1}) \ .$$

It is, moreover, clear that a non-zero element (P,b) is idempotent if and only if b is idempotent in B and $bP = P$ (which is equivalent to saying that bP is defined). If (P,e), (Q,f) are idempotents in S, then either $e \neq f$, in which case $ef = 0$ and $(P,e)(Q,f) = (Q,f)(P,e) = 0$, or $e = f$, in which case

$$(P,e)\,(Q,e) = (Q,e)\,(P,e) = (P \wedge Q, e) \ .$$

Thus S is an inverse semigroup, and the unique inverse of (P,b) is $(b^{-1}P, b^{-1})$.

The natural order relation in S^* is given by

$$(P,b) \leq (Q,c) \iff bb^{-1}c \neq 0 \ \text{ and } \ (P,b) = (P,bb^{-1})\,(Q,c) = (P \wedge Q, bb^{-1}c) \ .$$

That is, since $bb^{-1}c = c$ in such a case,

$$(P,b) \leq (Q,c) \quad \iff \quad b = c \ \text{ and } \ P \leq Q \ . \tag{2}$$

It follows that S is E^*-unitary, for if $(P,e) \in E^*$ and $(Q,c) \in S^*$, then $(P,e) \leq (Q,c)$ if and only if $c = e$ and $P \leq Q$, and so in particular (Q,c) is idempotent.

Notice too that S is categorical, for the product $(P,b)(Q,c)(R,d)$ can equal zero only if $bcd = 0$, and the categorical property of B then implies that either $(P,b)(Q,c) = 0$ or $(Q,c)(R,d) = 0$. Indeed S is strongly categorical. That this is so follows by the work of Munn [8], for it is clear that the relation γ on S defined by

$$\gamma = \Big\{((P,b),(Q,c)) \in S \times S\colon\, b=c\Big\} \cup \{(0,0)\} \tag{3}$$

is a proper congruence on S and that S/γ is isomorphic to the Brandt semigroup B.

The congruence γ defined by (3) is in fact the minimum Brandt congruence on S. Suppose that $((P,b),(Q,b)) \in \gamma$. Then $b^{-1}P, b^{-1}Q \in \mathcal{Y}$, and so $bb^{-1}P = P$, $bb^{-1}Q = Q$. Hence $bb^{-1}(P \wedge Q) = P \wedge Q$, and $P \wedge Q \neq 0$ by (P3). Hence $(P \wedge Q, bb^{-1}) \in E_S^*$. It now follows that

$$(P \wedge Q, bb^{-1})\,(P,b) = (P \wedge Q, bb^{-1})\,(Q,b) = (P \wedge Q, b) \neq 0\ .$$

Hence, recalling Munn's characterization (1) of the minimum Brandt congruence, we conclude that $\gamma \subseteq \beta$, the minimum Brandt congruence on S. Since γ is, as observed before, a Brandt congruence, we deduce that $\gamma = \beta$.

It is useful also at this stage to note the following result:

Lemma 3.1. *The semilattice of idempotents of $\mathcal{M}(B,\mathcal{X},\mathcal{Y})$ is isomorphic to $\mathcal{Y}$.*

Proof: We have seen that the non-zero idempotents of $S = \mathcal{M}(B,\mathcal{X},\mathcal{Y})$ are of the form (P,e), where $P \in \mathcal{Y}^*$, $e \in E_B^*$ and $eP = P$. The statement that $eP = P$ is equivalent to saying in our previous notation that $P \in \mathcal{D}_e$, and since the order ideals Δ_e and Δ_f (with $e \neq f$) have zero intersection, there is for each P in $\mathcal{Y}^*$ at most one e such that $(P,e) \in E_S^*$.

In fact for each P in $\mathcal{Y}^*$ there is *exactly* one e in E_B^* such that $(P,e) \in E_S^*$; for by our assumption that the representation $b \mapsto \lambda_b$ is effective we can assert that $P \in \operatorname{dom} b$ for some b in B^*, and then $(P, b^{-1}b) \in E_S^*$. The conclusion is that for each P in $\mathcal{Y}^*$ there is a unique e_P in B^* such that $(P, e_P) \in E_S^*$. We have a bijection $P \mapsto (P, e_P)$ from $\mathcal{Y}^*$ onto E_S^*. If $P \wedge Q \neq 0$, then $e_P = e_Q = e$ (say), and

$$(P,e)\,(Q,e) = (P \wedge Q, e)\ .$$

If $P \wedge Q = 0$, then $e_P \neq e_Q$ by (P3), and so $(P,e_P)(Q,e_Q) = 0$. We deduce that the bijection $P \mapsto (P,e_P)$, $0 \mapsto 0$ is an isomorphism from $\mathcal{Y}$ onto E_S.

We have in fact proved half of the following theorem:

Theorem 3.2. *Let $(B,\mathcal{X},\mathcal{Y})$ be a Brandt triple. Then $\mathcal{M}(B,\mathcal{X},\mathcal{Y})$ is a strongly categorical E^*-unitary inverse semigroup. Conversely, every strongly categorical E^*-unitary inverse semigroup is isomorphic to one of this kind.*

Proof: To prove the converse part, let S be a strongly categorical E^*-unitary inverse semigroup. Let $\mathcal{X}$ be the set of principal two-sided ideals of the carrier semigroup C_S:

$$\mathcal{X} = \Big\{ J(p)\colon\ p \in C_S \Big\} \ .$$

The set $\mathcal{X}$ is partially ordered by inclusion, with a minimum element 0 (strictly the zero ideal $\{0\}$). Let $\mathcal{Y}$ be the subset of $\mathcal{X}$ consisting of 0 together with all principal ideals $J(p)$ for which $p \in C(e,e)$ for some idempotent e of B^*. Let $J(p), J(q) \in \mathcal{Y}$. Then by Corollary 2.4 we have either $J(p) \cap J(q) = 0 \in \mathcal{Y}$, or $e = f$, $p \circ q \in C(e,e)$ and

$$J(p) \cap J(q) = J(p \circ q) \in \mathcal{Y} \ .$$

Thus $\mathcal{Y}$ is a semilattice with respect to the inclusion order inherited from $\mathcal{X}$. This is the property (P1).

To show the property (P2), suppose that $J(p) \subseteq J(q)$, where $J(q) \in \mathcal{Y}^*$. Thus we may assume that $q = (e,i,e) \in C(e,e)$ for some idempotents e in B and i in S, such that $[i] = e$. We may suppose that p is idempotent in C_S. (If not we replace it by $p \circ p^{-1}$, observing that $J(p \circ p^{-1}) = J(p)$.) Hence there exist r, s in C_S such that

$$p = r \circ q \circ s \ .$$

Let $n = s \circ p \circ s^{-1}$. Then $n \in C(e,e)$, and clearly $J(n) \subseteq J(p)$. Also

$$\begin{aligned} p = p^3 &= (r \circ q \circ s) \circ p \circ (s^{-1} \circ q^{-1} \circ r^{-1}) \\ &= r \circ q \circ n \circ q^{-1} \circ r^{-1} \ \in \ J(n) \ , \end{aligned}$$

and so $J(p) = J(n) \in \mathcal{Y}$. Thus $\mathcal{Y}$ is an order ideal of $\mathcal{X}$.

Now we define a representation $b \mapsto \lambda_b$ of the Brandt semigroup $B = S/\beta$ by partial order-isomorphisms of $\mathcal{X}$. Let $\lambda_0 = \{(0,0)\}$. For each b in B^*, let

$$\lambda_b = \Big\{ (J(p), J(bp))\colon\ p, bp \neq 0 \Big\} \cup \{(0,0)\} \ .$$

That is to say, we define $\operatorname{dom} \lambda_b = \{J(p)\colon\ p,\ bp \neq 0\} \cup \{0\}$, and define $\lambda_b(J(p)) = J(bp)$, $\lambda_b(0) = 0$.

The domain of λ_b is in fact an order ideal of $\mathcal{X}$. For suppose that $0 \neq J(q) \subseteq J(p)$, where $p = (a,s,c)$ is such that $bp \neq 0$ and $q = (d,t,e)$. Then there exist elements (d,u,a), (c,v,e) in C_S^* such that

$$q = (d,u,a) \circ (a,s,c) \circ (c,v,e) = (d,usv,e) \ .$$

Now $d[u] = a$, and so if $bd = 0$ it follows that $ba = 0$, contrary to hypothesis. Hence $bq \neq 0$, and so $J(q) \in \operatorname{dom} \lambda_b$.

Notice now that

$$\operatorname{im}\lambda_b = \Big\{J(bp):\ p,\ bp \neq 0\Big\} = \Big\{J(q):\ q,\ b^{-1}q \neq 0\Big\} = \operatorname{dom}\lambda_{b^{-1}}\ ,$$

and that $\lambda_{b^{-1}}\,\lambda_b$ and $\lambda_b\,\lambda_{b^{-1}}$ are the identity maps of $\operatorname{dom}\lambda_b$, $\operatorname{im}\lambda_b$, respectively. Since $J(p) \subseteq J(q) \Rightarrow J(bp) \subseteq J(bq)$, each λ_b is a partial order-isomorphism of $\mathcal{X}$. Next, notice that if $bc = 0$ then $\lambda_b\lambda_c = \lambda_0$, the trivial map whose domain and image are both 0; for otherwise there exists $q \neq 0$ in C_S such that $J(q) \in \operatorname{dom}(\lambda_b\lambda_c)$, from which it follows that $(bc)q = b(cq) \neq 0$, a contradiction.

Suppose now that $bc \neq 0$. Then $\operatorname{dom}(\lambda_b\lambda_c) = \operatorname{dom}\lambda_{bc}$, since the conditions $p \neq 0$, $cp \neq 0$, $b(cp) \neq 0$ for $J(p)$ to be in $\operatorname{dom}(\lambda_b\lambda_c)$ are equivalent to the conditions $p \neq 0$, $(bc)p \neq 0$ for $J(p)$ to be in $\operatorname{dom}\lambda_{bc}$. Moreover, for all p in the common domain,

$$(\lambda_b\lambda_c)\,(J(p)) = \lambda_b(J(cp)) = J(b(cp)) = J((bc)p) = \lambda_{bc}(J(p))\ .$$

Thus $\lambda_b\lambda_c = \lambda_{bc}$ in all cases, and so $b \mapsto \lambda_b$ is a representation of B by partial order-isomorphisms of $\mathcal{X}$. We can regard B as acting on $\mathcal{X}$ on the left, and write $bJ(p)$ rather than $\lambda_b(J(p))$. Notice that $bJ(p) = J(bp)$ provided $bp \neq 0$.

To show that the representation is faithful, suppose that $\lambda_b = \lambda_c$, where $b, c \in B^*$, and let $p = (a, s, d)$ in C_S be such that $bp \neq 0$. Then $cp \neq 0$, and so

$$b = baa^{-1} = caa^{-1} = c\ .$$

To show that the representation is effective, let $p = (a, s, d)$ be an arbitrary element of C_S^*. Then $aa^{-1}p \neq 0$ and so $J(p) \in \operatorname{dom}\lambda_{aa^{-1}}$.

To verify (P3), let $e \in E_B^*$, and let $J(p), J(q) \in \mathcal{Y}^* \cap \Delta_e$. Thus $p = (f, i, f)$, $q = (g, j, g)$, where $f, g \in E_B^*$, $i, j \in E_S^*$ and $f[i] = f$, $g[j] = g$. Since ep and eq are non-zero, we must in fact have $f = g = e$. Thus $p \circ q = (e, ij, e) \neq 0$ and so, using Corollary 2.4, we see that

$$J(p) \cap J(q) = J(p \circ q) \neq 0\ .$$

To show the property (P4), consider a non-zero element $J(p)$ of $\mathcal{X}$, where $p = (a, s, b)$. Then $J(p) = J(p \circ p^{-1})$, with $p \circ p^{-1} = (a, ss^{-1}, a)$, and $a[ss^{-1}] = a$. Let q be the element $(a^{-1}a, ss^{-1}, a^{-1}a)$ of C_S. Then $J(q) \in \mathcal{Y}$, and

$$aq = (a, ss^{-1}, a) = p \circ p^{-1}\ .$$

It follows that $J(p) = aJ(q) \in a\mathcal{Y}$, and so $\mathcal{X} = B\mathcal{Y}$, as required.

To show (P5), let $a \in B^*$, let $q = (a, x, aa^{-1})$, where $[x] = a^{-1}$, and $p = (a^{-1}a, xx^{-1}, a^{-1}a)$. Then $J(p) \in \mathcal{Y}$. Also $ap = (a, xx^{-1}, a)$. If we define $r = (aa^{-1}, x^{-1}x, aa^{-1})$, then we easily verify that

$$q^{-1} \circ (ap) \circ q = r\,, \qquad q \circ r \circ q^{-1} = ap\ .$$

Hence $aJ(p) = J(q) \in a\mathcal{Y}^* \cap \mathcal{Y}^*$, as required.

It remains to show that $S \simeq \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$. We show in fact that

$$\overline{C}_S \simeq \mathcal{M}(B, \mathcal{X}, \mathcal{Y})\ ,$$

which by virtue of Lemma 2.5 is enough. Let ϕ: $\overline{C}_S \to \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$ be given by

$$\begin{aligned} (p, b)\phi &= (J(p), b)) \qquad (p \in C(bb^{-1}, b),\ b \in B^*) \\ 0\phi &= 0\ . \end{aligned}$$

Since $J(p) = J(p \circ p^{-1})$ and $p \circ p^{-1} \in C(bb^{-1}, bb^{-1})$, we deduce that $J(p) \in \mathcal{Y}^*$. Also, $b^{-1}p \neq 0$,

$$b^{-1}J(p) = J(b^{-1}p) = J\Big((b^{-1}p)^{-1} \circ (b^{-1}p)\Big)\ ,$$

and

$$(b^{-1}p)^{-1} \circ (b^{-1}p)\ \in\ C(bb^{-1}, b^{-1}) \circ C(b^{-1}, bb^{-1}) \subseteq C(bb^{-1}, bb^{-1})\ ;$$

hence $b^{-1}J(p) \in \mathcal{Y}^*$. Thus $(J(p), b) \in \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$.

To show that ϕ is one-one, suppose that $(J(p), b) = (J(q), c)$, where $p \in C(bb^{-1}, b)$, $q \in C(cc^{-1}, c)$. Then certainly $b = c$. If we now write $p = (bb^{-1}, s, b)$ and $q = (bb^{-1}, t, b)$, we have that

$$p \circ q^{-1} = (bb^{-1}, st^{-1}, bb^{-1})\ ,$$

and so $st^{-1} \in E_S^*$. Hence

$$st^{-1} = (st^{-1})^{-1}st^{-1} = ts^{-1}st^{-1}\ . \tag{4}$$

Next, since $p^{-1} \circ p \in J(q^{-1} \circ q)$, there exist elements (b, u, b), (b, v, b) in C_S^* such that $p^{-1} \circ p = (b, s^{-1}s, b) = (b, u, b)(b, t^{-1}t, b)(b, v, b)$; hence

$$s^{-1}s = u\,t^{-1}\,t\,v\ . \tag{5}$$

Now, from $b[u] = b$ we deduce that $[u]$ is idempotent in B, and hence (since β is idempotent-pure) that u is idempotent in S. The same argument applies to v, and so from (5) we conclude that $s^{-1}s \leq t^{-1}t$. The opposite inequality can be proved in just the same way, and so $s^{-1}s = t^{-1}t$.

It now easily follows from this and from (4) that

$$s = ss^{-1}s = st^{-1}t = ts^{-1}st^{-1}t = tt^{-1}tt^{-1}t = t \ .$$

Hence $(p, b) = (q, c)$ as required.

To show that ϕ is onto, suppose that $(J(p), b)$ is a non-zero element of $\mathcal{M}(B, \mathcal{X}, \mathcal{Y})$. Then we may assume that $p = (e, i, e)$, where $e \in E_B^*$, $i \in E_S^*$ and $[i] = e$. Also, $J(b^{-1}p) \in \mathcal{Y}^*$, and so, since $b^{-1}e \neq 0$, we deduce that $e = bb^{-1}$. We have $J(b^{-1}p) = J(q)$ for some $q = (f, j, f)$, with $f \in E_B^*$, $j \in E_S^*$ and $[j] = f$. Hence there exist (b^{-1}, u, f) and (f, v, b^{-1}) such that

$$b^{-1}p = (b^{-1}, i, b^{-1}) = (b^{-1}, u, f) \circ (f, j, f) \circ (f, v, b^{-1}) \ .$$

It follows that

$$p = b(b^{-1}p) = (bb^{-1}, i, bb^{-1}) = (bb^{-1}, u, bf) \circ (bf, j, bf) \circ (bf, v, bb^{-1}) \ . \tag{6}$$

Since $bf \neq 0$ we deduce that $bf = b$ and $f = b^{-1}b$.

Now, since $J(q) = J(b^{-1}p)$, there exist elements $(b^{-1}b, x, b^{-1})$ and $(b^{-1}, y, b^{-1}b)$ such that

$$q = (b^{-1}b, j, b^{-1}b) = (b^{-1}b, x, b^{-1}) \circ (b^{-1}, i, b^{-1}) \circ (b^{-1}, y, b^{-1}b) \ .$$

Hence

$$bq = (b, j, b) = (b, x, bb^{-1}) \circ p \circ (bb^{-1}, y, b) \ . \tag{7}$$

We may rewrite (6) as

$$p = (bb^{-1}, u, b) \circ (b, j, b) \circ (b, v, bb^{-1}) = r \circ (b, v, bb^{-1}) \ ,$$

where $r = (bb^{-1}, uj, b)$, and we immediately deduce that $J(p) \subseteq J(r)$. Also, from (7) it follows that

$$r = (bb^{-1}, u, b) \circ (b, j, b) \ \in \ J(p) \ ,$$

and so $J(r) = J(p)$. It now follows that $(r, b) \in \overline{C}_S$ and that $(J(p), b) = (r, b)\phi$. Thus ϕ is onto.

Finally, we show that ϕ is a homomorphism. Let $(p, b), (q, c) \in \overline{C}_S^*$. If $bc = 0$ in B then both $[(p, b)(q, c)]\phi$ and $[(p, b)\phi]\,[(q, c)\phi]$ are zero. Otherwise we use Lemma 2.3 and observe that

$$\begin{aligned}[(p, b)(q, c)]\phi &= (p \circ bq, bc)\phi = (J(p \circ bq), bc) \\ &= (J(p) \cap bJ(q), bc) = (J(p), b)\,(J(q), c) = [(p, b)\phi]\,[(q, c)\phi] \ .\end{aligned}$$

This completes the proof of Theorem 3.2. ∎

Example: Let $T = \mathcal{M}(G, \mathcal{X}, \mathcal{Y})$ be an E-unitary inverse semigroup (without zero), let I be a set, and let $S = (I \times T \times I) \cup \{0\}$, and define multiplication in S by

$$(i, a, j)\,(k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases},$$

$$0\,(i, a, j) = (i, a, j)\,0 = 0\,0 = 0\ .$$

Then it is not hard to check that S is a strongly categorical E^*-unitary inverse semigroup. Its maximum Brandt image is $B = (I \times G \times I) \cup \{0\}$, where G is the maximum group image of T.

For each i in I, let $\mathcal{X}_i$ be a copy of $\mathcal{X}$, and suppose that $X \mapsto X_i$ $(X \in \mathcal{X})$ is an order-isomorphism. Let $\mathcal{Y}_i$ correspond to $\mathcal{Y}$ in this isomorphism. Suppose that the sets $\mathcal{X}_i$ are pairwise disjoint, and form an ordered set $\mathcal{X}'$ as the union of all the sets $\mathcal{X}_i$ together with an extra minimum element 0. The order on $\mathcal{X}'$ coincides with the order on $\mathcal{X}_i$ within $\mathcal{X}_i$, and $0 \le X'$ for all X' in $\mathcal{X}'$. Define $\mathcal{Y}' = \bigcup\{\mathcal{Y}_i \colon i \in I\} \cup \{0\}$.

The action of B on $\mathcal{X}'$ is given as follows. If $b = (i, a, j) \in B$, then the domain of λ_b is $\mathcal{X}_j \cup \{0\}$, and the action of b on the elements of its domain is given by

$$\begin{aligned} (i, a, j)\,X_j &= (aX)_i \qquad (X \in \mathcal{X}) \\ (i, a, j)\,0 &= 0\ . \end{aligned}$$

(Trivially, if $b = 0$, then the domain of λ_0 is $\{0\}$, and the action of b simply sends 0 to 0.)

Then $(B, \mathcal{X}', \mathcal{Y}')$ is a Brandt triple, and $S \simeq \mathcal{M}(B, \mathcal{X}', \mathcal{Y}')$.

4 – An isomorphism theorem

Given two semigroups $S = \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$ and $S' = \mathcal{M}(B', \mathcal{X}', \mathcal{Y}')$, it is now important to describe the conditions under which $S' \simeq S$. In a sense it is clear from the last section that the 'building blocks' of S are intrinsic: B is the maximum Brandt homomorphic image of S, $\mathcal{X}$ is the partially ordered set of principal ideals of the carrier semigroup C_S, and $\mathcal{Y}$ is in effect the semilattice of idempotents of S. It is, however, conceivable that two non-isomorphic semigroups S and S' might have isomorphic maximum Brandt images, isomorphic semilattices of idempotents, and might be such that C_S and $C_{S'}$ have order-isomorphic sets of principal ideals, and so we must prove a formal isomorphism theorem.

Theorem 4.1. *Let $S = \mathcal{M}(B, \mathcal{X}, \mathcal{Y})$, $S' = \mathcal{M}(B', \mathcal{X}', \mathcal{Y}')$, and suppose that $\phi\colon S \to S'$ is an isomorphism. Then*

(**i**) *there exists an isomorphism $\omega\colon B \to B'$;*

(**ii**) *there exists an order isomorphism $\theta\colon \mathcal{X} \to \mathcal{X}'$ whose restriction to $\mathcal{Y}$ is a semilattice isomorphism from $\mathcal{Y}$ onto $\mathcal{Y}'$;*

(**iii**) *for all b in B and X in $\mathcal{X}$,*

(8) $$(bX)\,\theta = (b\omega)\,(X\theta)\ ;$$

(**iv**) *for all (P, b) in S^*,*

(9) $$(P, b)\,\phi = (P\theta, b\omega)\ .$$

Conversely, if there exist ω and θ with the properties (i), (ii) *and* (iii), *then* (9), *together with $0\phi = 0$, defines an isomorphism from S onto S'.*

Proof: Notice that (8) is to be interpreted as including the information that bX is defined if and only if $(b\omega)(X\theta)$ is defined.

We begin by proving the converse part. So, for each (P, b) in S^*, define $(P, b)\phi = (P\theta, b\omega)$, in accordance with (9). Notice first that this does define a map from S into S', for $P\theta \in (\mathcal{Y}')^*$ and by (8) we also have

$$(b\omega)^{-1}(P\theta) = (b^{-1}\omega)\,(P\theta) = (b^{-1}P)\,\theta \ \in\ (\mathcal{Y}')^*\ .$$

(The first equality follows from (i), and $(b')^{-1}P' \in (\mathcal{Y}')^*$ is a consequence of (ii).)

Next, the map ϕ defined by (9) is a bijection. If $(P', b') \in (S')^*$, then there exist a unique P in $\mathcal{Y}$ such that $P\theta = P'$ and a unique b in B such that $b\omega = b'$. Moreover,

$$(b^{-1}P)\,\theta = (b\omega)^{-1}(P\theta) = (b')^{-1}P' \ \in\ (\mathcal{Y}')^*\ .$$

Hence $(P, b) \in S$, and is the unique element of S mapping to (P', b') by ϕ.

Finally, ϕ is a homomorphism. Given $(P, b), (Q, c)$ in S^* such that $bc \neq 0$, we have that

$$[(P, b)(Q, c)]\phi = (P \wedge bQ, bc)\phi = \Big((P \wedge bQ)\theta, (bc)\omega\Big) =$$

$$
\begin{aligned}
&= \Big((b(b^{-1}P \wedge Q))\theta,\, (bc)\omega\Big)\,, \qquad \text{(where } b^{-1}P, Q \in \mathcal{Y}) \\
&= \Big((b\omega)((b^{-1}P \wedge Q)\theta),\, (bc)\omega\Big)\,, \qquad \text{(by (8)} \\
&= \Big((b\omega)((b^{-1}P)\theta \wedge Q\theta),\, (bc)\omega\Big)\,, \qquad \text{since } \theta|_{\mathcal{Y}} \text{ is a semilattice isomorphism,} \\
&= \Big((b\omega)((b\omega)^{-1}(P\theta) \wedge Q\theta),\, (bc)\omega\Big)\,, \qquad \text{by (8),} \\
&= \Big(P\theta \wedge (b\omega)(Q\theta),\, (b\omega)(c\omega)\Big) \\
&= (P\theta, b\omega)\,(Q\theta, c\omega) = [(P,b)\phi]\,[(Q,c)\phi]\ .
\end{aligned}
$$

If $bc = 0$ then $(b\omega)(c\omega) = 0$, and so both $[(P,b)(Q,c)]\phi$ and $[(P,b)\phi]\,[(Q,c)\phi]$ are equal to zero.

Conversely, suppose that $\phi\colon\ S \to S'$ is an isomorphism. Let β, β' be the minimum Brandt congruences on S, S', respectively. As we saw in the last section, $S/\beta \simeq B$ and $S'/\beta' \simeq B'$. In fact we have an isomorphism $\omega\colon\ B \to B'$ such that the diagram

$$
\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\gamma"'] & S' \arrow[d, "\gamma'"] \\
B \arrow[r, "\omega"'] & B'
\end{tikzcd}
$$

commutes. Here γ and γ' are the projections $(P,b) \mapsto b$, $(P',b') \mapsto b'$ respectively.

Now let (P',b') be the image under ϕ of (P,b). Then

$$b' = (P',b')\,\gamma' = (P,b)\,\phi\,\gamma' = (P,b)\,\gamma\,\omega = b\,\omega\ ,$$

and so $(P,b)\phi = (P', b\omega)$, where $P' \in \mathcal{Y}'$ and is such that $(b\omega)^{-1}P' \in \mathcal{Y}'$.

We now have a lemma

Lemma 4.2. *Let $(P,b),(P,c) \in S^*$, and suppose that $(P,b)\phi = (P',b\omega)$. Then $(P,c)\phi = (P',c\omega)$.*

Proof: Suppose that $(P,c)\phi = (P'',c\omega)$. Both $(P,bb^{-1}) = (P,b)(P,b)^{-1}$ and $(P,cc^{-1}) = (P,c)(P,c)^{-1}$ belong to S^*, and so, by the argument in the proof of Lemma 3.1, we must have $bb^{-1} = cc^{-1}$. Hence

$$
\begin{aligned}
(P',(bb^{-1})\omega) &= (P',b\omega)\,(P',b\omega)^{-1} = [(P,b)\phi]\,[(P,b)^{-1}\phi] \\
&= (P,bb^{-1})\phi = (P,cc^{-1})\phi = [(P,c)\phi]\,[(P,c)^{-1}\phi] \\
&= (P'',c\omega)(P'',c\omega)^{-1} = (P'',(cc^{-1})\omega)\ ,
\end{aligned}
$$

and so $P'' = P'$. ∎

From this lemma it follows that we can define a map $\theta\colon \mathcal{Y} \to \mathcal{Y}'$ such that, for all (P, b) in S,

$$(P, b)\,\phi = (P\theta, b\omega)\ .$$

The domain of θ is in fact the whole of $\mathcal{Y}$, since, by the effectiveness of the representation $b \mapsto \lambda_b$, there exists for every P in $\mathcal{Y}^*$ an element b in B^* such that $(P, b^{-1}b) \in S$.

Lemma 4.3. *The map $\theta\colon \mathcal{Y} \to \mathcal{Y}'$ is an order-isomorphism.*

Proof: That θ is a bijection follows from the observation that we can do for the inverse isomorphism $\phi^{-1}\colon S' \to S$ exactly what we have just done for ϕ, obtaining maps $\omega'\colon B' \to B$ and $\theta'\colon \mathcal{Y}' \to \mathcal{Y}$ such that $(P', b')\phi^{-1} = (P'\theta', b'\omega')$. Then, from the inverse property of ϕ^{-1}, we deduce that ω' and θ' are two-sided inverses of ω and θ respectively. Let $P \le Q$ in $\mathcal{Y}$, and let b be such that $Q \in \operatorname{dom} b$. Then, since $\operatorname{dom} b$ is an order-ideal, $P \in \operatorname{dom} b$ also, and so, by (2), $(P, b^{-1}b) \le (Q, b^{-1}b)$ in S. Applying ϕ, we deduce that $(P\theta, (b^{-1}b)\omega) \le (Q\theta, (b^{-1}b)\omega)$ in S', and so $P\theta \le Q\theta$. ∎

Lemma 4.4. *Let $P \in \mathcal{Y}^*$, and let b in B^* be such that $bP \in \mathcal{Y}^*$. Then $(bP)\theta = (b\omega)(P\theta)$.*

Proof: The elements (bP, b) and (P, b^{-1}) are both in S, and are mutually inverse. By applying ϕ to both sides of the equality

$$(bP, b)\,(P, b^{-1}) = (bP, bb^{-1})\ ,$$

we deduce that

$$((bP)\theta, b\omega)\,(P\theta, b^{-1}\omega) = ((bP)\theta, (b^{-1}b)\omega)\ ,$$

and hence that

$$(bP)\theta \wedge (b\omega)(P\theta) = (bP)\theta\ .$$

It follows that $(bP)\theta \le (b\omega)(P\theta)$.

Similarly, by applying ϕ to both sides of the equality

$$(P, b^{-1})\,(bP, b) = (P, b^{-1}b)\ ,$$

we obtain

$$P\theta \wedge (b^{-1}\omega)\,((bP)\theta) = P\theta\ ,$$

and from this it follows that $(b\omega)(P\theta) \le (bP)\theta$. ∎

To extend the map θ to $\mathcal{X}$ we use (P4) to express an arbitrary X in $\mathcal{X}^*$ in the form bP, where $b \in B^*$ and $P \in \mathcal{Y}^*$, and define $X\theta$ to be $(b\omega)(P\theta)$. To show that this defines $X\theta$ uniquely, we must show that $bP = cQ$ implies that $(b\omega)(P\theta) = (c\omega)(Q\theta)$. In fact we shall deduce this from the result that

$$bP \le cQ \;\Rightarrow\; (b\omega)(P\theta) \le (c\omega)(Q\theta) ,$$

and so obtain also the information that θ is order-preserving on $\mathcal{X}$. So suppose that $bP \le cQ$. Then $bP \in \mathrm{dom}(c^{-1})$, and so we may deduce that $c^{-1}bP \le Q$ in $\mathcal{Y}$. From Lemmas 4.3 and 4.4 we deduce that $((c^{-1}b)\omega)(P\theta) \le Q\theta$, which immediately gives the required inequality.

It is now easy to see that $\theta\colon \mathcal{X} \to \mathcal{X}'$ is a bijection. To show that it is one-one, suppose that $X\theta = Y\theta$, where $X = bP$ and $Y = cQ$, with b, c in B^* and P, Q in $\mathcal{Y}^*$. Then $(b\omega)(P\theta) = (c\omega)(Q\theta)$, and so, in $\mathcal{Y}'$,

$$(c^{-1}bP)\theta = \left((c\omega)^{-1}(b\omega)\right)(P\theta) = Q\theta .$$

Hence $c^{-1}bP = Q$, and from this it is immediate that $X = Y$. To show that θ is onto, consider an element $X' = b'P'$ in $\mathcal{X}'$, where $b' \in (B')^*$ and $P' \in (\mathcal{Y}')^*$. Then there exist b in B and P in $\mathcal{Y}$ such that $b\omega = b'$ and $P\theta = P'$, and so $(bP)\theta = b'P' = X'$.

Finally, we show that the equality (8) holds for all b in B^* and all X in $\mathcal{X}^*$. Let $X = cP$, where $c \in B^*$ and $P \in \mathcal{Y}^*$. Then

$$\begin{aligned}(bX)\theta = (b(cP))\theta = ((bc)P)\theta &= ((bc)\omega)(P\theta)\\ &= (b\omega)\left[(c\omega)(P\theta)\right] = (b\omega)(X\theta) .\end{aligned}$$

This completes the proof of Theorem 4.1. ∎

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